A refractive index of a kink in curved space

Miftachul Hadi[1, 2]

[1)] *Physics Research Centre, Badan Riset dan Inovasi Nasional (BRIN), Puspiptek, Serpong, Tangerang Selatan 15314, Banten, Indonesia. E-mail: itpm.id@gmail.com*

[2)] *Institute of Mathematical Sciences, Jalan Nuri I no. 68, Kb. Kopi, Pengasinan, Gn. Sindur Bogor 16340, Jawa Barat, Indonesia*

The refractive index and curvature relation is formulated using the Riemann-Christoffel curvature tensor. As a consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor, we found that the refractive index should be a second rank tensor. The second rank tensor of the refractive index describes a linear optics. It implies naturally that the Riemann-Christoffel curvature tensor is related to the linear optics. In case of a non-linear optics, the refractive index is a sixth rank tensor, if susceptibility is a fourth rank tensor. The Riemann-Christoffel curvature tensor can be formulated in the non-linear optics but with a reduction term. The relation between the (linear and non-linear) refractive index and a (linear and non-linear) mass in curved space are formulated. Related to the Riemann-Christoffel curvature tensor, we formulate "the (linear and non-linear) generalized Einstein field equations". Sine-Gordon model in curved space is shown, where the Lagrangian is the total energy. This total energy is the mass of a kink (anti-kink) associated with a topological charge (a winding number). We formulate the relation between the (linear and non-linear) refractive index of the kink (anti-kink) and the topological charge-the winding number. Deflection of light is discussed in brief where the (linear and non-linear) angle of light deflection are formulated in relation with the mass (the topological charge, the winding number) of the kink (anti-kink).

I. INTRODUCTION

What is really happened if light passes through a medium? This question becomes more interesting nowadays related to conceptual development and technological innovation. One of the very important idea to understand this question is the refractive index. The refractive index of a medium is an optical parameter, since it exhibits the optical properties of the material[1]. The refractive index is one of the physicochemical properties of optical medium[2]. It is a function that depends on various parameters, including the frequency of the applied electric field[3].

The refractive index, n, is defined as velocity of light of a given wavelength in empty space or vacuum (c) divided by its velocity in a substance, v,[2]

$$n = \frac{c}{v} \tag{1}$$

It[1] describes how matter affects light propagation, through the electric permittivity, ε, and the magnetic permeability, μ[4]

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}\ \frac{\mu}{\mu_0}} = \sqrt{\varepsilon_r\ \mu_r} \tag{2}$$

where ε_0 and μ_0 are the permittivity and the permeability of vacuum respectively, ε_r and μ_r is relative permittivity and relative permeability of non-vacuum medium respectively which the values are relative i.e. they depend on the characteristics of medium[4,5].

In the most substrates, the refractive index decreases by increasing temperature[2]. A denser material generally tends to have a larger refraction index[6]. The refractive index in an fibre optic can be changed due to external forces such as the tensile force, the bending force[7].

Mathematically, the refractive index is a zeroth rank tensor (scalar) and it can not be a first rank tensor (vector), but it can be a second rank tensor, a third rank tensor or a higher rank tensor (which is well known as nonlinear phenomena of second order, third order, etc)[8]. The refractive index is the zeroth rank tensor, if the medium or material is isotropic[2]. Generally, the refractive index is written as the second rank tensor, n_{ij}, a 3×3 matrix, if the material is linear[3]. It can be the third rank tensor or the fourth rank tensor if the material is non-linear[10].

The refractive index has a large number of applications. It is mostly applied to identify a particular substance, to confirm its purity or to measure its concentration. It also can be used in determination of drug concentration in pharmaceutical industry, to calculate a focusing power of lenses and a dispersive power of prisms. Also, it can be applied to estimate a thermophysical properties of hydrocarbons and petroleum mixtures[2].

[1] The sign of the refractive index is often taken as positive, but in 1968 Veselago shows that there are substrates with negative permitivity and negative permeability. In these substrates, the refractive index has a negative value[2].

[2] Isotropy comes from the Greek words *isos* (equal) and *tropos* (way): uniform in all directions[9]. An isotropic material is a material that has the same optical properties, regardless of the direction in which light propagates through the material[3,9].

[3] Linear material is a material that when exposed to light at a certain frequency will generate light with the same frequency[5].

II. THE REFRACTIVE INDEX AND THE RIEMANN-CHRISTOFFEL CURVATURE TENSOR

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. ϕ defines a differentiable scalar field). Then the gradient of ϕ is defined by[11], p.57

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \tag{3}$$

In the tensorial form[11]

$$\vec{\nabla}\phi = \text{grad } \phi = \phi_{,j} = \frac{\partial \phi}{\partial x^j} \tag{4}$$

where $\phi_{,j}$ is the covariant derivative of ϕ with respect to x^j. Here, $\vec{\nabla}\phi$ defines a vector field i.e. the gradient of a scalar field is a vector field[11,12].

Let us analyse the equation below[13–15]

$$\frac{1}{R} = \hat{N} \; . \; \vec{\nabla} \; \ln \; n(r) \tag{5}$$

We write N_k as a tensorial representation, i.e. a first rank tensor, of an unit vector, $\hat{N}$. By using relation in eq.(4) and tensorial notation N_k for $\hat{N}$, the right hand side of eq.(5) can be written as

$$N_k \; \frac{\partial}{\partial x^j}[\ln \; n(r)] \tag{6}$$

Because of the Riemann-Christoffel curvature tensor is the fourth rank tensor, so *the refractive index in eq.(6) should be written as the second rank tensor*. We obtain the relation between the curvature tensor and the refractive index tensor as below[16]

$$\frac{R_{mijk}}{g} = N_k \; \frac{\partial \ln \; n_{mi}}{\partial x^j} \tag{7}$$

Eq.(7) implies that the curved space which is described by the Riemann-Christoffel curvature tensor related naturally to linear medium of optics.

How about the form of the non-linear refractive index i.e. the refractive index related to the non-linear optics? In optics, non-linear properties of materials are usually described by non-linear susceptibilities[17]. Mathematically, the optical non-linear response can be expressed as a relationship between the polarization[4], $\vec{P}(\omega)$, and the electric field, $\vec{E}(\omega)$. Both, $\vec{P}(\omega)$ and $\vec{E}(\omega)$, are the function of angular frequency, ω.

In the linear case, a relation between the polarization and the electric field is simply expressed[19]

$$\vec{P}(\omega) = \chi^{(1)}(\omega) \; \vec{E}(\omega) \tag{8}$$

[4] Light is an electromagnetic wave, and the electric field of this wave oscillates perpendicularly to the direction of light propagation. If the direction of the electric field of light is well defined, it is polarized light. The most common source of polarized light is a laser[18].

where $\chi^{(1)}(\omega)$ is the first order susceptibility and it is a scalar, whereas the polarization and the electric field are vectors.

In the non-linear case[5], the polarization can be modelled as a power series of the electric field, $\vec{E}(\omega)$, as below

$$\begin{aligned}\vec{P}(\omega) = \chi^{(1)}(\omega) \; \vec{E}(\omega) + \chi^{(2)}(\omega) \; \vec{E}^2(\omega) \\ + \; \chi^{(3)}(\omega) \; \vec{E}^3(\omega) + ...\end{aligned} \tag{9}$$

where $\vec{E}^2(\omega) = \vec{E}(\omega) \; \vec{E}(\omega)$, $\vec{E}^3(\omega) = \vec{E}(\omega) \; \vec{E}(\omega) \; \vec{E}(\omega)$, etc. The quantities $\chi^{(2)}(\omega)$ and $\chi^{(3)}(\omega)$ are known as the second order and third order susceptibilities, respectively. In general, $\chi^{(1)}(\omega)$, $\chi^{(2)}(\omega)$ and $\chi^{(3)}(\omega)$ are the second, third and fourth rank tensors, respectively[19]. In optical Kerr effect, the third order susceptibility, $\chi^{(3)}(\omega)$, related to the non-linear refractive index[3], p.34.

Now, we have a question: if the non-linear refractive index related to the third order susceptibility and the third order susceptibility is the fourth rank tensor[19] then how to define the non-linear refractive index related to the fourth rank tensor of the third order susceptibility? For linearly polarized monochromatic light in an isotropic medium or a cubic crystal, the non-linear refractive index, n_2, can be expressed by[22]

$$n_2 = \frac{12\pi}{n_0} \; \text{Re} \; \chi^{(3)}(\omega) \tag{10}$$

or

$$n_0 = \frac{12\pi}{n_2} \; \text{Re} \; \chi^{(3)}(\omega) = 12\pi \; (n_2)^{-1} \; \text{Re} \; \chi^{(3)}(\omega) \tag{11}$$

where n_0 is linear refractive index and Re $\chi^{(3)}(\omega)$ is a real part of the third order non-linear susceptibility. Eq.(10) shows that the non-linear refractive index is a function of the linear refractive index.

We see from eq.(7), the linear refractive index is the second rank tensor and refer to Jatirian, et al.[19] the third order susceptibility is the fourth rank tensor, so we can write eq.(11) as below

$$n_{mi} = 12\pi \; n_{mi}^{pqrs} \; \chi_{pqrs}^{(3)} \tag{12}$$

where $\chi_{pqrs}^{(3)}$ is the fourth rank tensor of the real part of third order susceptibility. It means that *the non-linear refractive index should be the sixth rank tensor*, n_{mi}^{pqrs}.

As we see from eqs.(11), (12), the non-linear refractive index is the sixth rank tensor as follows

$$(n_2)^{-1} = n_{mi}^{pqrs} \rightarrow n_2 = n_{pqrs}^{mi} \tag{13}$$

Here, what we call the non-linear refractive index is n_{mi}^{pqrs}. We see from eq.(12), the non-linear refractive index is

[5] A non-linear system is a system in which the change of the output is not proportional to the change of the input[20]. In optics, the non-linearity is typically observed only at very high intensities (field strength) of light such as those provided by lasers[21].

a mixed tensor of fourth rank contravariant and second rank covariant.

If we assume that the third order non-linear susceptibility, $\chi^{(3)}_{pqrs}$, has a positive value then the value of the non-linear refractive index is smaller than the linear refractive index. It implies that the speed of light in the non-linear optics is larger than the speed of light in the linear optics. It also implies that the light energy of the non-linear optics is larger than the light energy of the linear optics.

Substituting (12) into (7), we obtain

$$\frac{R_{mijk}}{g} = N_k \frac{\partial}{\partial x^j} \left\{ \ln \left(12\pi \ n^{pqrs}_{mi} \ \chi^{(3)}_{pqrs} \right) \right\} \tag{14}$$

where π is a constant, so the gradient of π gives zero. Using relation $\ln \ (ABC) = \ln A \ + \ \ln B \ + \ \ln C$, then eq.(14) becomes

$$\frac{R_{mijk}}{g} - N_k \frac{\partial \ \ln \ \chi^{(3)}_{pqrs}}{\partial x^j} = N_k \frac{\partial \ \ln \ n^{pqrs}_{mi}}{\partial x^j} \tag{15}$$

Eq.(15) shows that the non-linear optics relates the sixth rank tensor of the refractive index with the curvature tensor.

The main difference between the linear optics, eq.(7), and the non-linear optics, (15), is that there exist the second term in the left hand side of eq.(15): $N_k \frac{\partial \ \ln \ \chi^{(3)}_{pqrs}}{\partial x^j}$. If we assume that this term has a positive value, then the curvature will be reduced by such term. It means that the curvature in the non-linear optics is smaller compared to the curvature in the linear optics.

III. THE REFRACTIVE INDEX AND A MASS IN CURVED SPACE

Let us consider the Schwarzschild metric[14,23] and assume that the space is isotropic and spherically symmetric. Then, the line element is[14]

$$\begin{aligned} ds^2 &= g_{00}(r) \ c^2 \ dt^2 - g_{rr}(r) \ dr^2 \\ &= \left(1 - \frac{2GM}{c^2 r}\right) c^2 \ dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 \end{aligned} \tag{16}$$

where r is the spatial coordinate and M is a mass of an object in curved space. The world line corresponding to the propagation of light is defined as null geodesic as follows

$$ds^2 = 0 \tag{17}$$

Substitute this eq.(17) into (16), we obtain

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \frac{dr}{dt} = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} c \tag{18}$$

If we substitute $dr/dt = v$ into (18) and rearrange the terms, then we obtain

$$\left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} = \frac{c}{v} \tag{19}$$

where $c/v = n(r)$ as eq.(1), so we have the space dependent refractive index, $n(r)$, related to the mass of an object, M, as below[14,15]

$$n(r) = \left(1 - \frac{2G}{c^2 r} \ M\right)^{-1} \tag{20}$$

where G is the gravitational constant and c is the speed of light in vacuum.

How to formulate the space dependent linear (the second rank tensor) and non-linear (the sixth rank tensor) refractive indices related to the mass of an object as expressed in eq.(20)? In order to answer this question, we need to understand the quantities G, c in eq.(20). The simple understanding of G is coming from the Einstein field equation as follows[24–26]

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} \ T_{\mu\nu} \tag{21}$$

We are informed from eq.(21) that the gravitational constant, G, is a scalar (because the speed of light, c, is a scalar).

As previously stated, in case of the linear optics, we take the space dependent refractive index as the second rank tensor. Because of gravitational constant, G, the speed of light, c, the spatial coordinate (distance), r, are scalars, then eq.(20) can be written as

$$n_{mi} = \left(1 - \frac{2G}{c^2 r} \ M^{mi}\right)^{-1} \tag{22}$$

where M^{mi} is the second rank tensor of mass[27,28].

In case of the non-linear optics, we substitute eq.(12) into (22), then we obtain

$$n^{pqrs}_{mi} = \frac{1}{12\pi \ \chi^{(3)}_{pqrs}} \left(1 - \frac{2G}{c^2 r} \ M^{mi}\right)^{-1} \tag{23}$$

or, in analogy with (22)

$$n^{pqrs}_{mi} = \left(1 - \frac{2G}{c^2 r} \ M^{mi}_{pqrs}\right)^{-1} \tag{24}$$

We obtain from (23), (24), that

$$M^{mi}_{pqrs} = \frac{c^2 r}{2G} \left\{ 1 - 12\pi \ \chi^{(3)}_{pqrs} \left(1 - \frac{2G}{c^2 r} \ M^{mi}\right) \right\} \tag{25}$$

What is the value of M^{mi}_{pqrs} compared to M^{mi}?

Let us return to eq.(5). If we substitute eq.(20) into (5), we obtain "the Einstein field equation" for one spatial dimension as below

$$\frac{1}{R} = \hat{N} \ . \ \vec{\nabla} \ \ln \ \left(1 - \frac{2G}{c^2 r} \ M\right)^{-1} \tag{26}$$

How about "the generalized Einstein field equation" for the second rank tensor of the linear refractive index and the sixth rank tensor of the non-linear refractive index?

If we substitute eq.(22) into (7), we obtain

$$\frac{R_{mijk}}{g} = N_k \frac{\partial}{\partial x^j}\left\{\ln\left(1-\frac{2G}{c^2 r}M^{mi}\right)^{-1}\right\} \quad (27)$$

We call this equation (27) as "the linear generalized Einstein field equation".

If we substitute eq.(24) into (15), we obtain

$$\frac{R_{mijk}}{g} - N_k \frac{\partial \ln \chi^{(3)}_{pqrs}}{\partial x^j} = N_k \frac{\partial}{\partial x^j}\left\{\ln\left(1-\frac{2G}{c^2 r}M^{mi}_{pqrs}\right)^{-1}\right\} \quad (28)$$

This equation (28) is called "the non-linear generalized Einstein field equation".

We see from eqs.(27), (28), the curvature of the non-linear optics is smaller than the curvature of the linear optics by factor $N_k \frac{\partial \ln \chi^{(3)}_{pqrs}}{\partial x^j}$ (we assume that this factor has a positive value). This factor will affect the angle of deflection of light. This will be discussed more detail later.

IV. SINE-GORDON MODEL IN A CURVED SPACE

The Lagrangian density of the sine-Gordon model defined on the curve has the following form[29]

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\mathcal{F}(\partial_s\phi)^2 - V(\phi) \quad (29)$$

where the function $\mathcal{F}(s)$ contains information about the curvature of the considered space

$$\mathcal{F}(s) = \frac{1}{a\ K(s)}\ln\left[\frac{2+a\ K(s)}{2-a\ K(s)}\right] \quad (30)$$

Here, $K(s)$ is the curvature of the central curve of the isolating layer/junction and a is thickness of the dielectric layer[29,30]. In a natural way, the above Lagrangian is[29]

$$L = \int_{-\infty}^{+\infty} ds\ \mathcal{L} = E_k - E_p \quad (31)$$

In this case, we treat L as the total energy, E. So,

$$E = L \quad (32)$$

Using the natural unit, $c = 1$, we can treat the total energy as mass, M, as below

$$E \sim M \quad (33)$$

Topological charge is related to total energy[31]. The total energy or mass[6] of the kink (anti-kink)[7] in the sine-Gordon model is[32]

$$M = 8|Q| \quad (34)$$

or

$$M = 8|N| \quad (35)$$

if we treat the topological charge, Q, is equal to the winding number, N, in harmony with the Skyrme's idea. Here, we should remember that the mass of kink (anti-kink), eq.(34) or (35), is the mass which is formulated in the curved space.

V. THE REFRACTIVE INDEX OF THE KINK IN CURVED SPACE

A kink is a topological soliton in one-dimensional space, $\phi(x)$[33]. Its energy density, at any given location, does not vanish with time in the long time limit. By definition, the kink is a map[33]

$$\phi : Z_2 \to Z_2 \quad (36)$$

where Z_2 denotes the group of integer with size or modulo 2. In general, Z_p is group of integer with size or modulo p. The elements of Z_p are $0, \pm1, .., \pm(p-1)$. So, the subscript 2 in Z_2 of eq.(36) indicates the modulo of the group of integer where the elements or members are 0 and ±1[34–36].

In the sine-Gordon model[8], the topological charge is given by[32]

$$Q = \frac{\phi_{(x\ =\ +\infty)} - \phi_{(x\ =\ -\infty)}}{2\pi} \quad (37)$$

Refer to Skyrme[38], the field configuration with boundary conditions for the sine-Gordon model[9] is given by

$$\phi(x = +\infty) - \phi(x = -\infty) = 2N\pi \quad (38)$$

where N is the winding number.

If we accommodate the idea of Skyrme in eq.(38), by substituting eq.(38) into (37), we obtain

$$Q = \frac{2N\pi}{2\pi} = N \quad (39)$$

It means that the topological charge, Q, is equal to the winding number[10], N.

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[6] Here we use natural unit $c = 1$, so $E = Mc^2$ gives $E = M$.

[7] In this case, the kink (anti-kink) is a solution of the sine-Gordon equation.

[8] If we compare the sine-Gordon and ϕ^4 models, the formulation of the topological charge for both models looked different. Does it mean that the formulation of the topological charge in the ϕ^4 model is more general than the sine-Gordon (e.g. if we take $m = \pi$)? Here, m is a arbitrary normalisation parameter for the topological charge. For the ϕ^4 model, it is convenient to fix it to 1 so that the topological charge is in units of 1 $(0, +1, -1)$. For the sine-Gordon model, we can replace 2π by m so that the topological charge will be in units of π (if our fields go from 0 to π)[37].

[9] Actually, Skyrme uses $\alpha(x)$ as a notation for describing a single angle-type field variable instead of $\phi(x)$.

[10] In order to be classically stable, soliton should have energy with special lower bound. The bound usually involves the topological index: $\varepsilon_{\phi\in Q_n} \geq C|N|$, where C is a constant and N is topological index (which is similar with the winding number[39])[40] p.103.

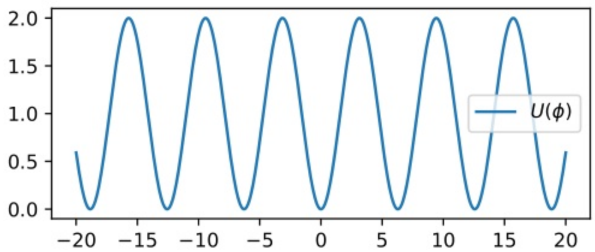

Fig. 4. Plot of the potential $U(\phi) = 1 - \cos\phi$ in the sine-Gordon model.

The vacua set has an infinite number of components[32].

As previously mentioned, there exist the relationship between the refractive index and the mass, as written in eq.(20). By substituting eqs.(34), (35), into (20) we obtain the relationship between the refractive index and the mass of the kink (anti-kink) in the sine-Gordon model, as below

$$n(r) = \left\{1 - \frac{2G}{c^2 r}\ (8|Q|)\right\}^{-1} = \left\{1 - \frac{2G}{c^2 r}\ (8|N|)\right\}^{-1} \quad (40)$$

Eq.(40) show that the refractive index can be formulated related to the topological charge and it also shows that the refractive index is also a function of the winding number.

How about the linear and non-linear refractive indices formulations related to the topological charge in the sine-Gordon model? In case of the linear optics for sine-Gordon model, we treat the refractive index as the second rank tensor, and because G, c, r, m, λ are scalars, so the topological charge is the second rank tensor, Q^{mi}, then eq.(40) becomes

$$n_{mi} = \left[1 - \frac{2G}{c^2 r}\ (8|Q^{mi}|)\right]^{-1} = \left[1 - \frac{2G}{c^2 r}\ (8|N^{mi}|)\right]^{-1} \quad (41)$$

The non-linear refractive index for the sine-Gordon model is

$$\begin{aligned} n_{mi}^{pqrs} &= \left(12\pi\ \chi^{(3)}_{pqrs}\right)^{-1} \left[1 - \frac{2G}{c^2 r}\ (8|Q^{mi}|)\right]^{-1} \\ &= \left(12\pi\ \chi^{(3)}_{pqrs}\right)^{-1} \left[1 - \frac{2G}{c^2 r}\ (8|N^{mi}|)\right]^{-1} \quad (42) \end{aligned}$$

Here, in eqs.(41), (42), we treat the winding number as the second rank tensor, as a consequence of the second rank tensor of the topological charge.

VI. DEFLECTION OF LIGHT BY THE KINK

Gravitational lensing is a direct consequence of general relativity. If light passes near an object of massive mass, M, at an impact parameter, D, (i.e. its shortest distance to the object), the curvature of space-time (due to such the object of the massive mass) will cause light to be deflected by an angle of deflection, Φ, as below[41]

$$\Phi = \frac{4G}{c^2 D}\ M \quad (43)$$

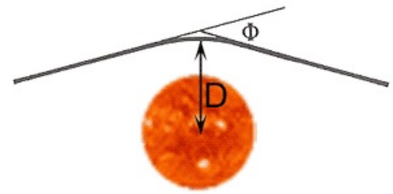

Fig. 5. Deflection of light by the massive mass of the object[42].

It means that the angle of deflection, Φ, by which light is deflected depends on the impact parameter, D, and the massive mass of the object, M[43].

The linear mass of the kink (anti-kink) can be expressed in the sine-Gordon model, as below

$$M^{mi} = 8|Q^{mi}| \quad (44)$$

or

$$M^{mi} = 8|N^{mi}| \quad (45)$$

By substituting eq.(45) into (25), we obtain the non-linear mass of the kink (anti-kink) in the sine-Gordon model, respectively, as below

$$M^{mi}_{pqrs} = U\left[1 - 12\pi\ \chi^{(3)}_{pqrs}\left(1 - U^{-1}\ 8|Q^{mi}|\right)\right] \quad (46)$$

or

$$M^{mi}_{pqrs} = U\left[1 - 12\pi\ \chi^{(3)}_{pqrs}\left(1 - U^{-1}\ 8|N^{mi}|\right)\right] \quad (47)$$

where $U = c^2 r/2G$ so $U^{-1} = 2G/c^2 r$.

How about the angle of deflection in case of the linear and non-linear optics? By substituting eq.(45), into eq.(43), we obtain the angle of deflection by the linear mass of the kink (anti-kink) in the sine-Gordon model as follows

$$\Phi^{mi} = V(8|Q^{mi}|) \quad (48)$$

or

$$\Phi^{mi} = V(8|N^{mi}|) \quad (49)$$

where $V = 4G/c^2 D$. Here, we note Φ^{mi} as the second rank tensor (because we treat G, c, D, m, λ as scalars). It is the consequence of the second rank tensor of the topological charge.

By substituting eq.(47), into eq.(43), we obtain the angle of deflection by the non-linear mass of the kink (anti-kink) in the sine-Gordon model as follows

$$\Phi^{mi}_{pqrs} = V\left\{U\left[1 - 12\pi\ \chi^{(3)}_{pqrs}\left(1 - U^{-1}\ 8|Q^{mi}|\right)\right]\right\} \quad (50)$$

or

$$\Phi^{mi}_{pqrs} = V\left\{U\left[1 - 12\pi\ \chi^{(3)}_{pqrs}\left(1 - U^{-1}\ 8|N^{mi}|\right)\right]\right\} \quad (51)$$

where Φ^{mi}_{pqrs} is the sixth rank tensor. Eqs.(48), (50), show that the angle of the light deflection can be linked to the topological charge. Eqs.(49), (51), show that the angle of the light deflection is a function of the winding number.

We see from eqs.(27) and (28), the curvature in the non-linear optics is smaller than the curvature in the linear optics by a factor $N_k \frac{\partial \ln \chi^{(3)}_{pqrs}}{\partial x^j}$ (we assume that this factor has a positive value). Because the curvature is related to the mass, then eqs.(27), (28) imply that the mass of the non-linear optics, M^{mi}_{pqrs} is smaller than the mass of the linear optics, M^{mi}. So, the angle of deflection in the non-linear medium is smaller compared to the angle of deflection in the linear medium.

VII. DISCUSSION AND CONCLUSION

In one and two dimensional space, the curvatures are given by $1/R$ and $1/R^2$ respectively. Georg Friedrich Bernhard Riemann, a student of Johann Carl Friedrich Gauss, generalize the Gauss curvature of space for more than two dimensions. The result is the Riemann-Christoffel curvature tensor where the Christoffel symbol is used in the formulation of the generalized curvature.

Because the refractive index is related to the curvature of space for one and two dimensions, and this curvature of space can be generalized to more than two dimensions, then the refractive index should be able to be formulated in more than two dimensional curved space. It gives the refractive index as the second rank tensor as the consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor.

The second rank tensor of the refractive index describes the linear optics. It implies that the Riemann-Christoffel curvature tensor is related naturally to the linear medium or the linear optics. Because the non-linear refractive index can be expressed as a function of the linear refractive index and the third order of the susceptibility, where the linear refractive index is the second rank tensor and the susceptibility is the fourth rank tensor then the non-linear refractive index should be the sixth rank tensor. It means that the Riemann-Christoffel curvature tensor can be related to the non-linear optics.

The relation between the refractive index and the mass, especially for the linear and non-linear optics are formulated. We found "the Einstein field equations" for the linear and non-linear optics. The mass of the kink (anti-kink) in the sine-Gordon models is shown, where the mass of the kink (anti-kink) is associated with its topological charge. We found the relation between the linear and non-linear refractive index of the kink (anti-kink) and the topological charge in the sine-Gordon model. In this model, the linear and non-linear refractive indices are also a function of the winding number.

Why is space curved? Refer to the general relativity, geometry of space-time is related to the energy-mass of the object. The massive mass of the object bends the surrounding space-time to form curvature of space-time near the object. Because the curvature of space is related to the mass of the object and the mass of the object can be related to the refractive index, then the curvature of space is related to the refractive index. Because the mass of the object, i.e. the mass of the kink (anti-kink), is related to its topological charge and the mass of the kink (anti-kink) is also related to the refractive index then the refractive index is related to the topological charge of the kink (anti-kink).

Why do we use the kink (anti-kink) as the object with mass? Because, the kink (anti-kink) lives in (1+1) dimensional space-time, i.e. one dimensional space and one dimensional time, so the formulation of the kink (anti-kink) is suitable if the mass of the kink (anti-kink) is related to the curvature of one dimensional space. Another topological objects which have mass, e.g. vortex, which lives in two dimensional space, domain walls (as the extension of the kink (anti-kink)) and Skyrmion which live in three dimensional space, can be formulated in relations with the curvature of two and three dimensional space, respectively. Because the Riemann-Christoffel curvature tensor can accommodate the topological object in higher dimensional space, so how are the topological objects in the higher dimensional space looked like?

If light passes through the space near the object with mass, i.e. the kink (anti-kink), then the light path will be deflected. The angle of deflection is proportional to the mass. In the sine-Gordon model, we formulate the angle of deflection related to the topological charge of the kink (anti-kink). In this model, the angle of deflection is also a function of the winding number. What is happened if light passes through the space near the massive collapsing kink (anti-kink) to black hole?

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