A REVISIT TO LEMOINE'S CONJECTURE

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Abstract. In this paper we prove Lemoine's conjecture. By exploiting the language of circles of partition, we show that for all sufficiently large $n \in 2\mathbb{N}+1$

$$\#\{p+2q=n|\ p,q \in \mathbb{P}\} > 0.$$

This proves that every sufficiently large odd number can be written as the sum of a prime and a double of a prime.

1. Introduction

The binary Goldbach conjecture may seem to be one of the hardest unsolved problems in additive number theory. The difficulty arises in part due to the inefficiency of the current available tools. However powerful they may seem to be, one often ran into complete deadlock in tackling binary problems relating to questions concerning partitions. Perhaps one of the closely related problem is Lemoine's conjecture (see [2]) known to have been first conjectured by Émile Lemoine. Despite it's difficulty and given that no apparent progress has been made, the conjecture has been verified to hold up to 10^9. The numerical threshold has been pushed a little further to 10^{10}. In this paper by extending greatly the method of circle of partitions [1], we show that every sufficiently large odd number can be written as the sum of a prime number and double of a prime number.

2. The Circle of Partition

In this section we introduce the concept of the circle of partition. We study some elementary properties of this structure in the following sequel.

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subset \mathbb{N}$. We denote with

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x,y \in \mathbb{M}, n = x+y\}$$

the Circle of Partition generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n,\mathbb{M})$ as points and denote them by $[x]$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ if and only if $x+y=n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the same axis. The point $[x] \in \mathcal{C}(n,\mathbb{M})$ such that $2x=n$ is the **center** of the CoP. If it exists then it is their only point which is not an axis point. The line joining any two arbitrary point which are not axes partners on the CoP will be referred to as a **chord** of the CoP.

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We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\}$$

the **sequence** of the first n natural numbers. Further we will denote

$$||[x]|| := x$$

as the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $||\mathcal{C}(n,\mathbb{M})||$.

Proposition 2.3. *Each axis is uniquely determined by points* $[x] \in \mathcal{C}(n,\mathbb{M})$.

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. □

Corollary 2.4. *Each point of a CoP* $\mathcal{C}(n,\mathbb{M})$ *except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n,\mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$||[x]|| + ||[y]|| \neq n.$$

This contradiction to the Definition 2.1. Due to Proposition 2.3 the case of more than one axis partners is impossible. This completes the proof. □

Let us denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x + y = n$$

and the number of axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M}) \mid x < y\}. \tag{2.1}$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

3. Main result

In this section we present the main result including preparatory lemmas. We exploit the notion of the signature in a subtle manner to establish the main result.

Lemma 3.1. *Let* $\mathbb{P}$ *denotes the set of all prime numbers. Then*

$$\#\left\{p < x \mid 2p < x,\ p \in \mathbb{P}\right\} = \pi(\frac{x}{2})$$

where $\pi(n)$ *is the prime counting function.*

Proof. The quantity

$$\begin{aligned}\#\left\{p < x \mid 2p < x,\ p \in \mathbb{P}\right\} &= \sum_{\substack{p<x \\ 2p<x}} 1 \\ &= \sum_{p<\frac{x}{2}} 1 = \pi(\frac{x}{2}).\end{aligned}$$

□

Lemma 3.2. *Let $\mathbb{P} \cup 2\mathbb{P}$ denotes the set of all prime and double prime numbers. If $\nu(n, \mathbb{P} \cup 2\mathbb{P}) = 0$ for infinitely many $n \in 2\mathbb{N}+1$ then*

$$\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ \{x,y\} \cap (\mathbb{P} \cup 2\mathbb{P}) \neq \emptyset\} \geq (1+o(1))\frac{3n}{2\log n}$$

holds for those $n \in 2\mathbb{N}+1$.

Proof. By the uniqueness of the axes of CoPs we can write

$$\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ \{x,y\} \cap (\mathbb{P} \cup 2\mathbb{P}) \neq \emptyset\} = \nu(n, \mathbb{P} \cup 2\mathbb{P}) + \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ x \in \mathbb{P} \cup 2\mathbb{P}, \\ y \in \mathbb{N} \setminus \mathbb{P} \cup 2\mathbb{P}\}$$

so that under the assumption $\nu(n, \mathbb{P} \cup 2\mathbb{P}) = 0$ for infinitely many $n \in 2\mathbb{N}+1$ we obtain

$$\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ x \in \mathbb{P} \cup 2\mathbb{P}, \ y \in \mathbb{N} \setminus \mathbb{P} \cup 2\mathbb{P}\} = \pi(n) + \pi(\frac{n}{2}) \\ \geq (1+o(1))\frac{3n}{2\log n}$$

by virtue of the prime number theorem. □

Remark 3.3. Crucially the lower bound of the quantity

$$\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ x \in \mathbb{P} \cup 2\mathbb{P}, \ y \in \mathbb{N} \setminus (\mathbb{P} \cup 2\mathbb{P})\}$$

in Lemma 3.2 cannot be lowered any further down in the case $\nu(n, \mathbb{P} \cup 2\mathbb{P}) = 0$. This fact will be exploited in establishing the main result of the paper.

Definition 3.4. Let $\mathbb{H} \subset \mathbb{N}$ with $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a CoP. Then by the signature of the CoP $\mathcal{C}(n)$ with the **pen** $\mathbb{H}$ on the CoP $\mathcal{C}(n, \mathbb{M})$ we mean the ratio

$$\text{Sign}[\mathcal{C}(n)_{\mathbb{H}} \mid \mathcal{C}(n, \mathbb{M})] = \frac{\#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ \{x,y\} \cap \mathbb{H} \neq \emptyset\}}{\left\lfloor \frac{|\mathbb{M} \cap \mathbb{N}_n| - 1}{2} \right\rfloor}$$

Proposition 3.5. *Let $\mathbb{H} \subset \mathbb{N}$ with $\mathbb{M}, \mathbb{K} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M}), \mathcal{C}(n, \mathbb{K})$ be CoPs. Then the following properties hold*

(i) $\text{Sign}[\mathcal{C}(n)_{\mathbb{H}} \mid \mathcal{C}(n, \mathbb{M})] \geq \text{Sign}[\mathcal{C}(n)_{\mathbb{H}} \mid \mathcal{C}(n, \mathbb{K})]$ *if and only if* $\mathbb{M} \subseteq \mathbb{K}$.

(ii)

$$\text{Sign}[\mathcal{C}(n)_{\mathbb{H}} \mid \mathcal{C}(n)] \geq \frac{\left\lfloor \frac{|\mathbb{H} \cap \mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{n-1}{2} \right\rfloor}$$

and $\lim_{n \longrightarrow \infty} \text{Sign}[\mathcal{C}(n)_{\mathbb{H}} \mid \mathcal{C}(n)] \geq \mathcal{D}(\mathbb{H})$.

Proof. These properties are easy consequences of Definition 3.4 and the notion of density of CoPs. □

Remark 3.6. The intuitive essence of the notion of the signature allows to count the number of axes in a typical CoP. This notion counts the number of axes with specified character of the weight of the corresponding points as the pen that appends the required signature.

It is important to notice that the signature of the CoP $\mathcal{C}(n)$ with the pen $\mathbb{H}$ on itself the CoP $\mathcal{C}(n)$ mimicks the structure of the underlying CoP, by recovering the local density of the overlap of the first n positive integers with the set $\mathbb{H}$. This perspective is somewhat different when we deal with other CoPs that lack a somewhat uniformity as opposed to the CoP $\mathcal{C}(n)$. If we let $\mathbb{P}$ denotes the set of all prime numbers then the signature of the CoP $\mathcal{C}(n)$ with the pen $\mathbb{P}$ on the CoP $\mathcal{C}(n,\mathbb{P})$ no longer generates the local density of the first few prime numbers. Instead it generates the local density of the number of axes within the CoP $\mathcal{C}(n,\mathbb{P})$.

Proposition 3.7. *Let $\mathbb{P}\cup 2\mathbb{P}$ be the set of all prime and double prime numbers and $\mathcal{C}(n,\mathbb{P}\cup 2\mathbb{P})$ be a CoP. Then for all $n\in 2\mathbb{N}+1$ with $n>5$ the inequality holds*

$$\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ \{x,y\}\cap(\mathbb{P}\cup 2\mathbb{P})\neq\emptyset\}\geq(1+o(1))\frac{9}{8}\frac{n}{\log^2 n}$$

Proof. Since $\mathbb{P}\cup 2\mathbb{P}\subset\mathbb{N}$ it follows by appealing to Proposition 3.5

$$\mathrm{Sign}[\mathcal{C}(n)_{\mathbb{P}\cup 2\mathbb{P}}\mid\mathcal{C}(n)]\leq\mathrm{Sign}[\mathcal{C}(n)_{\mathbb{P}\cup 2\mathbb{P}}\mid\mathcal{C}(n,\mathbb{P}\cup 2\mathbb{P})]$$

so that we have

$$\mathrm{Sign}[\mathcal{C}(n)_{\mathbb{P}\cup 2\mathbb{P}}\mid\mathcal{C}(n,\mathbb{P}\cup 2\mathbb{P})]\geq\frac{\left\lfloor\frac{|(\mathbb{P}\cup 2\mathbb{P})\cap\mathbb{N}_n|}{2}\right\rfloor}{\left\lfloor\frac{n-1}{2}\right\rfloor}$$
$$=(1+o(1))\frac{3}{2\log n}$$

by appealing to the prime number theorem. It follows that

$$\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ \{x,y\}\cap(\mathbb{P}\cup 2\mathbb{P})\neq\emptyset\}\geq(1+o(1))\frac{3}{2\log n}\left\lfloor\frac{\pi(n)+\pi(\frac{n}{2})-1}{2}\right\rfloor$$

and the inequality follows by appealing one more time to the prime number theorem. □

Remark 3.8. We are now ready to state the main result of this paper.

Theorem 3.9. *For all sufficiently large $n\in 2\mathbb{N}+1$ holds*

$$\nu(n,\mathbb{P}\cup 2\mathbb{P})>0.$$

Proof. By exploiting the uniqueness of the axes of CoPs, we can write

$$\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ \{x,y\}\cap(\mathbb{P}\cup 2\mathbb{P})\neq\emptyset\}=\nu(n,\mathbb{P}\cup 2\mathbb{P})+\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ x\in\mathbb{P}\cup 2\mathbb{P},\ y\in\mathbb{N}\setminus(\mathbb{P}\cup 2\mathbb{P})\}.$$

Let us suppose to the contrary that there are infinitely many $n\in 2\mathbb{N}+1$ such that $\nu(n,\mathbb{P}\cup 2\mathbb{P})=0$, then it follows that

$$\#\{\mathbb{L}_{[x],[y]}\hat{\in}\mathcal{C}(n)|\ \{x,y\}\cap(\mathbb{P}\cup 2\mathbb{P})\neq\emptyset\}\geq(1+o(1))\frac{3}{8}\frac{n}{\log^2 n}$$

for those values of $n\in 2\mathbb{N}+1$, thereby contradicting the lower bound in Lemma 3.7. □

References

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