Series for Particular Values of the Gamma function

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Abstract. We give some series for the number $\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$, where $\Gamma(x)$ is the Gamma function.

Introduction

Recall that

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \int_0^{4\sqrt{7}} \frac{1}{\sqrt{x(112+21x+x^2)}}dx = \frac{2}{\sqrt[4]{7}}\int_0^1 \frac{1}{\sqrt{4+3\sqrt{7}x^2+4x^4}}dx$$

$$= \frac{2}{\sqrt[4]{7}}\int_1^\infty \frac{1}{\sqrt{4+3\sqrt{7}x^2+4x^4}}dx = \frac{1}{\sqrt[4]{7}}\int_0^\infty \frac{1}{\sqrt{4+3\sqrt{7}x^2+4x^4}}dx$$

$$= \frac{1}{4\sqrt{2}\,\sqrt[4]{7}}\int_0^1 \sqrt{\frac{\sqrt{64-x^2}-3\sqrt{7}x}{x}}dx = \frac{\sqrt{3}}{2}\int_0^\infty \frac{1}{\sqrt{x(252+63x+4x^2)}}dx$$

$$= \frac{\sqrt{3}}{2}\int_0^\infty \frac{1}{\sqrt{x(4+63x+252x^2)}}dx = \sqrt{3}\int_0^\infty \frac{1}{\sqrt{252+63x^2+4x^4}}dx$$

$$= \sqrt{3}\int_0^\infty \frac{1}{\sqrt{4+63x^2+252x^4}}dx = \frac{1}{\sqrt[4]{7}}\int_1^\infty \frac{1}{\sqrt{\left(3\sqrt{7}+8x\right)(x^2-1)}}dx$$

$$= \frac{1}{\sqrt[4]{7}}\int_0^1 \frac{1}{\sqrt{x\left(8+3\sqrt{7}x\right)(1-x^2)}}dx = \frac{1}{\sqrt[4]{7}}\int_0^\infty \frac{1}{\sqrt{3\sqrt{7}+8\cosh x}}dx$$

$$= \frac{1}{\sqrt[4]{7}}\int_0^{(3-\sqrt{7})/\sqrt{2}} \cosh^{-1}\left(\frac{1-3\sqrt{7}\,x^2}{8x^2}\right)dx = 2\int_0^{2\sqrt[4]{7}} \frac{1}{\sqrt{112+21x^2+x^4}}dx$$

$$= \frac{2}{\sqrt[4]{7}}\int_0^\infty \frac{1}{\sqrt{8+3\sqrt{7}+16(\sinh x)^2}}dx = \frac{2}{\sqrt[4]{7}}\int_0^\infty \frac{1}{\sqrt{16(\cosh x)^2-8+3\sqrt{7}}}dx$$

$$= \frac{2}{\sqrt[4]{7}}\int_0^\infty \frac{1}{\sqrt{\left(8+3\sqrt{7}+16x^2\right)(x^2+1)}}dx = \frac{2}{\sqrt[4]{7}}\int_1^\infty \frac{1}{\sqrt{\left(16x^2-8+3\sqrt{7}\right)(x^2-1)}}dx$$

$$= \frac{1}{\sqrt[4]{7}}\int_0^1 \frac{1}{\sqrt{x(1-x)(2-x)\left(8+3\sqrt{7}-3\sqrt{7}x\right)}}dx$$

Notations: $i=\sqrt{-1}$,

The number Pi is defined by

$$\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

The Gauss hypergeometric function is defined by

$$F(a,b,c,x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n\ , \qquad |x|<1$$

where $(a)_n=a(a+1)(a+2)\ldots(a+n-1)\ ,(a)_0=1$

The Appell hypergeometric function is defined by

$$F_1(a,b,c,d,x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}m!\,n!}\ x^m y^n\ \ ,|x|<1,|y|<1$$

The Gamma function is defined by

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}\ \ ,Re\ z>0$$

Other integrals

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}=2\sqrt{6}\int_0^{\tan^{-1}\sqrt{8/(3\sqrt{7})}}\frac{1}{\sqrt{63+(\cos x)^4}}dx=2\sqrt{6}\int_{\tan^{-1}\sqrt{(3\sqrt{7})/8}}^{\pi/2}\frac{1}{\sqrt{63+(\sin x)^4}}dx$$

$$=2\sqrt{6}\int_0^{\sinh^{-1}\sqrt{8/(3\sqrt{7})}}\frac{\cosh x}{\sqrt{1+63(\cosh x)^4}}dx=\frac{1}{\sqrt[4]{7}}\int_0^{\infty}\frac{1}{\sqrt{x(2+x)\left(8+3\sqrt{7}+8x\right)}}dx$$

$$=\frac{1}{2\sqrt[4]{7}}\int_0^{\pi/2}\frac{1}{\sqrt{1-\left(\frac{1}{2}-\frac{3\sqrt{7}}{16}\right)(\sin x)^2}}dx=\frac{\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\int_0^{\pi/2}\frac{1}{\sqrt{1+\left(8-3\sqrt{7}\right)^2(\cos x)^2}}dx$$

$$=\frac{2\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\int_0^{1}\frac{1}{\sqrt{1+\left(510-192\sqrt{7}\right)x^2+x^4}}dx$$

$$=\frac{1}{\sqrt[4]{7}}\int_0^{1/2}\frac{1}{\sqrt{x(1-x)\left(4-\left(8-3\sqrt{7}\right)x(1-x)\right)}}dx$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{2\sqrt{21}}\, F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{8}{63+3\sqrt{7}\,i},-\frac{8}{63-3\sqrt{7}\,i}\right) + \sqrt{3}\,\int_0^1 \frac{1}{\sqrt{4+63x^2+252x^4}}\,dx$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{\sqrt[4]{7}} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} P_n\left(\frac{3\sqrt{7}}{8}\right) + \frac{2}{\sqrt[4]{7}} \int_a^1 \frac{1}{\sqrt{4+3\sqrt{7}x^2+4x^4}}\,dx$$

where $0 \le a \le 1$ and $P_n(x)$ is the Legendre polynomials.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi\left(3-\sqrt{7}\right)}{\sqrt{2}\,\sqrt[4]{7}} - \frac{1}{2\sqrt[4]{7}} \int_1^{6\sqrt{2}-2\sqrt{14}} \sin^{-1}\left(\left(6\sqrt{2}+2\sqrt{14}\right)\sqrt{1-\frac{1}{x^2}}\right) dx$$

$$\begin{aligned}\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} \\ &= -\frac{\sqrt{2}\left(3-\sqrt{7}\right)\left(-1+\ln\left(6-2\sqrt{7}\right)\right)}{\sqrt[4]{7}} \\ &\quad + \frac{1}{\sqrt[4]{7}} \int_0^{(3-\sqrt{7})/\sqrt{2}} \ln\left(1-3\sqrt{7}\,x^2+\sqrt{1-6\sqrt{7}\,x^2-x^4}\right) dx\end{aligned}$$

$$\begin{aligned}\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} \\ &= \frac{1}{\sqrt{21+8\sqrt{7}}} \\ &\quad + \int_s^{\infty} \left(-7 + 35\sqrt[3]{2}\left(2646+\sqrt{-4630500+\left(2646+\frac{27}{x^2}\right)^2}+\frac{27}{x^2}\right)^{-\frac{1}{3}}\right. \\ &\quad \left. + \frac{1}{3\sqrt[3]{2}}\left(2646+\sqrt{-4630500+\left(2646+\frac{27}{x^2}\right)^2}+\frac{27}{x^2}\right)^{1/3}\right) dx\end{aligned}$$

where $s = 1/\sqrt{2352+896\sqrt{7}}$.

$$\begin{aligned}\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} &= \frac{\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}} \int_0^{\infty} \frac{(\operatorname{sech} x)^2}{\sqrt{1-(\operatorname{sech} x)^2+4\left(8-3\sqrt{7}\right)(\operatorname{sech} x)^4}}\,dx \\ &= \frac{2\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}} \int_0^{\infty} \frac{e^{-x}}{\sqrt{1+\left(510-192\sqrt{7}\right)e^{-2x}+e^{-4x}}}\,dx\end{aligned}$$

Series for $\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$

Entry 1.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt[4]{7}}{\sqrt{168+42\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\frac{4+3\sqrt{7}}{12+3\sqrt{7}}\right)^{n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\left(\frac{6\sqrt{7}}{4+3\sqrt{7}}\right)^{k}\sum_{m=0}^{k}\binom{k}{m}\frac{\left(4\sqrt{7}/21\right)^{m}}{2k+2m+1}$$

Entry 2.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(2-\frac{3\sqrt{7}}{4}\right)^{n}F\left(2n+1,n+\frac{1}{2},n+\frac{3}{2},-1\right)$$

Entry 3.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2\sqrt[4]{7}}{\sqrt{3}}\sum_{n=0}^{\infty}\binom{2n}{n}(-1)^{n}2^{-2n}\left(\frac{\sqrt{3}}{5+2\sqrt{7}}\right)^{2n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{\left(\sqrt{7}/3\right)^{k}}{2k+1}F\left(2n+1,1,k+\frac{3}{2},\frac{2\sqrt{7}}{5+2\sqrt{7}}\right)$$

Entry 4.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{8}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\binom{2n}{n}(-1)^{n}2^{-2n}\left(8-3\sqrt{7}\right)^{2n+1}F\left(1,2n+1,\frac{3}{2},\frac{8}{8+3\sqrt{7}}\right)$$

Entry 5.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt{21+4\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^{n}2^{-2n}}{4n+1}\left(\frac{4}{4+3\sqrt{7}}\right)^{n}F\left(1,n+\frac{1}{2},2n+\frac{3}{2},\frac{3\sqrt{7}}{4+3\sqrt{7}}\right)$$

Entry 6.

$$\frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}{8\pi\sqrt{7}} = \frac{2\sqrt{2}\,\sqrt[4]{7}\,\sqrt{14\sqrt{7}-37}}{3}\times$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}(-1)^{n}2^{-2n}\left(\frac{14\sqrt{7}-37}{9}\right)^{n}F_{1}\left(1,n+\frac{1}{2},n+\frac{1}{2},\frac{3}{2},\frac{2\sqrt{7}}{5+2\sqrt{7}},\frac{4\sqrt{7}}{11+4\sqrt{7}}\right)$$

Entry 7.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt[4]{7}\left(3+\sqrt{7}\right)}\sum_{n=0}^{\infty}\binom{2n}{n}^{2}2^{-4n}\left(24\sqrt{7}-63\right)^{n}F\left(\frac{1}{2},n+\frac{1}{2},n+1,\frac{1}{2}\right)$$

Entry 8.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt[4]{7}\,(3+\sqrt{7})}\ F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{3\sqrt{7}}{8+3\sqrt{7}}\right)$$

Entry 9.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{\sqrt{2}\sqrt{3}\sqrt{7}}\sum_{n=0}^{\infty}\binom{2n}{n}\binom{4n}{2n}(-1)^n 2^{-6n}63^{-n}$$

Entry 10.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi}{4\sqrt[4]{7}}\sum_{n=0}^{\infty}\binom{2n}{n}^2 2^{-4n}\left(\frac{8-3\sqrt{7}}{16}\right)^n$$

Entry 11.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = 2\sqrt{\frac{2}{21}}\sum_{n=0}^{\infty}\binom{2n}{n}(-1)^n 2^{-2n}\left(\frac{8}{3\sqrt{7}}-1\right)^n\sum_{k=0}^{n}\binom{n}{k}\frac{1}{4k+1}F\left(2n+1,2k+\frac{1}{2},2k+\frac{3}{2},-1\right)$$

Entry 12.

$$\begin{aligned}\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} &= \frac{1}{2\sqrt{21}}\ F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{8}{63+3\sqrt{7}\,i},-\frac{8}{63-3\sqrt{7}\,i}\right)\\ &+\frac{4\sqrt{3}}{71}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n}{4n+1}\left(\frac{63}{20164}\right)^n F\left(2n+1,1,2n+\frac{3}{2},\frac{63}{71}\right)\end{aligned}$$

Entry 13.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\frac{1}{\binom{2n+2}{n+1}(n+1)}\sum_{k=0}^{[n/2]}(-1)^k\binom{2n-2k}{n-k}\binom{n-k}{k}\left(16\left(8-3\sqrt{7}\right)\right)^k$$

Entry 14.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\frac{2^{-3n}}{\binom{2n+2}{n+1}(n+1)}\sum_{k=0}^{[n/2]}(-1)^k\binom{2n-4k}{n-2k}^2\binom{2k}{k}\binom{n}{n-2k}^{-1}\left(16\left(8-3\sqrt{7}\right)\right)^{2k}$$

Entry 15. For $a = 255+180\sqrt{2}-96\sqrt{7}-68\sqrt{14}$, we have

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}\,\sqrt{a}}{\sqrt[4]{7}\,(3+\sqrt{7})(1+a)}\ F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{1+a},\frac{a}{1+a}\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}}{\sqrt[4]{7}\,(3+\sqrt{7})} \sqrt{\frac{a}{1+a}}\, F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{1+a},1-a\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}}{\sqrt[4]{7}\,(3+\sqrt{7})\sqrt{1+a}}\, F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},1-\frac{1}{a},\frac{a}{1+a}\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{4\sqrt{2}}{\sqrt[4]{7}\,(3+\sqrt{7})\sqrt{a}}\, F_1\left(1,\frac{1}{2},\frac{1}{2},\frac{3}{2},-\frac{1}{a},1-\frac{1}{a}\right)$$

Entry 16.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$
$$= \frac{1}{\sqrt{42}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-4n} 3^{-2n} 7^{-n}}{4n+1}\, F\left(1,2n+1,2n+\frac{3}{2},\frac{1}{2}\right)$$
$$+ \frac{\sqrt{6}}{8+3\sqrt{7}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2n+1} \left(\frac{6072\sqrt{7}-16065}{2}\right)^n F\left(1,2n+1,n+\frac{3}{2},\frac{3\sqrt{7}}{8+3\sqrt{7}}\right)$$

Entry 17.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}\,(3-\sqrt{7})}{\sqrt[4]{7}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{2n+1} \left(\frac{8-3\sqrt{7}}{2}\right)^{2n} F\left(1,2n+1,n+\frac{3}{2},\frac{1}{2}\right)$$

Entry 18.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$
$$= \frac{\sqrt{3}}{\sqrt{91}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-2n}}{4n+1} \left(\frac{36}{91}\right)^n F\left(1,n+\frac{1}{2},2n+\frac{3}{2},\frac{9}{13}\right)$$
$$+ \frac{1}{\sqrt{33}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n 2^{-2n}}{4n+1} \left(\frac{28}{99}\right)^n F\left(1,n+\frac{1}{2},2n+\frac{3}{2},\frac{7}{11}\right)$$

Entry 19. For $a = 1-\sqrt{3\sqrt{7}(8-3\sqrt{7})}$, we have

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$
$$= \frac{4\sqrt{a}}{\sqrt{21}} \sum_{n=0}^{\infty} (-1)^n\, 2^{-2n} 63^{-n} \sum_{k=0}^{n} (-1)^k \binom{2n-2k}{n-k} \binom{2k}{k} \left(\frac{63a}{2}\right)^k \sum_{m=0}^{4n-4k} \binom{4n-4k}{m} \frac{(-1)^m a^m}{2k+2m+1}$$

Entry 20.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{3\sqrt[4]{7}} \sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n} 2^{-2k}\, 3^{-k} (8-3\sqrt{7})^k \binom{n+k}{n-k} \binom{2k}{k} \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{(-2)^m}{2k+2m+1}$$

Entry 21.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt{3}\,\sqrt[4]{7}\,\sqrt{1+\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 12^{-n}}{\left(1+\sqrt{7}\right)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{2^{2k}}{4k+1}F\left(n+\frac{1}{2},1,2k+\frac{3}{2},\frac{7-\sqrt{7}}{6}\right)$$

Entry 22.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2\sqrt{2}}{\sqrt[4]{7}\,\sqrt{10+3\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\frac{18\sqrt{7}-23}{37}\right)^n\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{2}{4+3\sqrt{7}}\right)^k\sum_{m=0}^{n+k}\binom{n+k}{m}\frac{(-2)^m}{2m+1}$$

Entry 23.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt[4]{7}\,\sqrt{3+3\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-2n}}{\left(3+3\sqrt{7}\right)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{2^{2k}}{2n+2k+1}F\left(n+\frac{1}{2},1,n+k+\frac{3}{2},\frac{3\sqrt{7}-1}{3+3\sqrt{7}}\right)$$

Entry 24.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{2}{\sqrt[4]{7}\,\sqrt{4+3\sqrt{7}}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n}{4n+1}\left(4+3\sqrt{7}\right)^{-n}F\left(n+\frac{1}{2},1,2n+\frac{3}{2},\frac{3\sqrt{7}}{4+3\sqrt{7}}\right)$$

Entry 25.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\frac{1}{2n+1}F\left(-n,n+1,n+\frac{3}{2},4\left(8-3\sqrt{7}\right)\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\frac{\left(12\sqrt{7}-31\right)^n}{2n+1}F\left(-n,\frac{1}{2},n+\frac{3}{2},-\frac{12\sqrt{7}-16}{47}\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\sqrt{2}\left(3-\sqrt{7}\right)\sqrt{12\sqrt{7}-31}}{\sqrt[4]{7}}\sum_{n=0}^{\infty}\frac{\left(12\sqrt{7}-31\right)^n}{2n+1}F\left(2n+\frac{3}{2},\frac{1}{2},n+\frac{3}{2},4\left(8-3\sqrt{7}\right)\right)$$

Entry 26.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7}\sin\left(\frac{3\pi}{7}\right)}\left(7\cdot 2^{-1/7}F\left(\frac{5}{7},\frac{1}{7},\frac{8}{7},\frac{1}{2}\right)+\frac{7}{2}\cdot 2^{-2/7}F\left(\frac{6}{7},\frac{2}{7},\frac{9}{7},\frac{1}{2}\right)\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7}\sin\left(\frac{5\pi}{7}\right)}\left(7\cdot 2^{-1/7}F\left(\frac{3}{7},\frac{1}{7},\frac{8}{7},\frac{1}{2}\right)+\frac{7}{4}\cdot 2^{-4/7}F\left(\frac{6}{7},\frac{4}{7},\frac{11}{7},\frac{1}{2}\right)\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{1}{8\sqrt{7}\sin\left(\frac{6\pi}{7}\right)}\left(\frac{7}{2}\cdot 2^{-2/7}F\left(\frac{3}{7},\frac{2}{7},\frac{9}{7},\frac{1}{2}\right)+\frac{7}{4}\cdot 2^{-4/7}F\left(\frac{5}{7},\frac{4}{7},\frac{11}{7},\frac{1}{2}\right)\right)$$

Entry 27. For $a=\sqrt{3\sqrt{7}\left(8-3\sqrt{7}\right)}$, we have

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$

$$=\sqrt{\frac{2}{21}}\,\pi\,F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,-\frac{i}{3\sqrt{7}},\frac{i}{3\sqrt{7}}\right)$$

$$-2\sqrt{\frac{2}{21}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}(-63)^{-k}\sum_{m=0}^{k}\binom{k}{m}\frac{(-1)^m a^{2n+2k+2m+1}}{2n+2k+2m+1}$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$

$$=\sqrt{\frac{2}{21}}\,\pi\,F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,-\frac{i}{3\sqrt{7}},\frac{i}{3\sqrt{7}}\right)-2\sqrt{\frac{2}{21}}\sum_{n=0}^{\infty}\frac{a^{2n+1}2^{-2n}}{2n+1}\sum_{k=0}^{[n/2]}\binom{2n-4k}{n-2k}\binom{2k}{k}\left(-\frac{4}{63}\right)^k$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$

$$=\sqrt{\frac{2}{21}}\,\pi\,F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,-\frac{i}{3\sqrt{7}},\frac{i}{3\sqrt{7}}\right)$$

$$-2\sqrt{\frac{2}{21}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}a^{2n+1}}{2n+1}\,F\left(\frac{1}{2},\frac{2n+1}{4},\frac{2n+5}{4},-\frac{a^4}{63}\right)$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$

$$=\sqrt{\frac{2}{21}}\,\pi\,F_1\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,-\frac{i}{3\sqrt{7}},\frac{i}{3\sqrt{7}}\right)$$

$$-2\sqrt{\frac{2}{21}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}(-1)^n 63^{-n}a^{4n+1}}{4n+1}\,F\left(\frac{1}{2},2n+\frac{1}{2},2n+\frac{3}{2},a^2\right)$$

Entry 28.

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}}$$

$$=\frac{4\sqrt{2}\left(3-\sqrt{7}\right)}{\sqrt{21}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}(-63)^{-k}\sum_{m=0}^{3k}\binom{3k}{m}\frac{(-1)^m\left(8\left(8-3\sqrt{7}\right)\right)^{n-k+m}}{2n-2k+2m+1}$$

Endnote

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{\pi^2}{7\Gamma(3/7)\Gamma(5/7)\Gamma(6/7)}$$

$$\frac{\Gamma(1/7)\Gamma(2/7)\Gamma(4/7)}{8\pi\sqrt{7}} = \frac{B(1/7,2/7)}{8\sqrt{7}\sin\left(\frac{3\pi}{7}\right)} = \frac{B(1/7,4/7)}{8\sqrt{7}\sin\left(\frac{5\pi}{7}\right)} = \frac{B(2/7,4/7)}{8\sqrt{7}\sin\left(\frac{6\pi}{7}\right)}$$

where $B(x,y)$ is the Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad ,x > 0, y > 0$$

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