

Kepler's laws as properties of the kinematic equations of motion of a point along curves of the second order

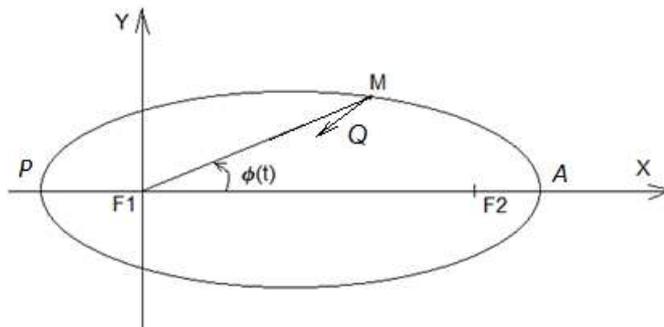
Abstract. Differential equations of motion on curves of the second order are inferred. Solutions to equations are made by computer programs. The results of the calculation are compared with Kepler's laws.

keywords: kinematic equations, Kepler's laws, ellipse.

If simple equations of speed and acceleration are sufficient to describe rectilinear motion: $V = S/t$, $a = S/t^2$, " then differential equations of motion are needed to solve problems on the curvilinear motion of material points and their systems. «The way we derive these equations doesn't matter»: [1, §11, п.3].

1. Differential equation of motion of a point along an ellipse relative to the left focus

To compose differential equations of motion, we represent the generalized force acting on a point in a fixed Cartesian coordinate system, Fig. 1.



M - material point, Q - the force acting on the point, $F1$ - left focus, $F2$ - right focus, $\varphi(t)$ - angle between X axis and the line connecting left focus and the point, A - aphelion, P - perihelion

Figure 1.

$$m\ddot{x} = -Q\cos(\varphi(t)) \quad (1)$$

$$m\ddot{y} = -Q\sin(\varphi(t)) \quad (2)$$

From equation (1) we can get

$$Q = \frac{-m\ddot{x}}{\cos(\varphi(t))} \quad (3)$$

Let us substitute equation (3) into equation (2)

$$\dot{y} = \frac{\ddot{x}}{\cos(\varphi(t))} \sin(\varphi(t)) \quad (4)$$

The point coordinates can be represented as the function of angle of deflection $\varphi(t)$ and radius $r(t)$.

$$x = r(\varphi(t)) \cdot \cos(\varphi(t)) \quad (5) \quad y =$$

$$r(\varphi(t)) \cdot \sin(\varphi(t)) \quad (6)$$

$$r(\varphi(t)) = \frac{p}{1-e \cdot \cos(\varphi(t))} \quad (7)$$

Let us calculate the first and second time derivative From equations (5), (6), (7).

$$\dot{x} = \frac{d}{dt} \left(\frac{p}{1-e \cdot \cos(\varphi(t))} \cos(\varphi(t)) \right) = - \frac{p \cdot \cos(\varphi(t)) \cdot e \cdot \sin(\varphi(t)) \cdot \frac{d}{dt} \varphi(t)}{(1-e \cdot \cos(\varphi(t)))^2} - \frac{p \cdot \sin(\varphi(t)) \cdot \frac{d}{dt} \varphi(t)}{1-e \cdot \cos(\varphi(t))} =$$

$$- \frac{b^2 \cdot \sin(\varphi(t)) \cdot \frac{d}{dt} \varphi(t)}{a \cdot (1-2 \cdot e \cdot \cos(\varphi(t)) + e^2 \cdot \cos(\varphi(t))^2)} \quad (8)$$

$$\dot{y} = \frac{d}{dt} \left(\frac{p}{1-e \cdot \cos(\varphi(t))} \sin(\varphi(t)) \right) = - \frac{p \cdot e \cdot \sin(\varphi(t))^2 \cdot \frac{d}{dt} \varphi(t)}{(1-e \cdot \cos(\varphi(t)))^2} + \frac{p \cdot \cos(\varphi(t)) \cdot \frac{d}{dt} \varphi(t)}{1-e \cdot \cos(\varphi(t))} =$$

$$\frac{b^2 \cdot \frac{d}{dt} \varphi(t) \cdot (-e + \cos(\varphi(t)))}{a \cdot (1-2 \cdot e \cdot \cos(\varphi(t)) + e^2 \cdot \cos(\varphi(t))^2)} \quad (9)$$

$$\ddot{x} = \frac{2 \cdot p \cdot e^2 \cdot \cos(\varphi(t)) \cdot \sin(\varphi(t))^2 \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{(1-e \cdot \cos(\varphi(t)))^3} + \frac{2 \cdot p \cdot e \cdot \sin(\varphi(t))^2 \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{(1-e \cdot \cos(\varphi(t)))^2} - \frac{p \cdot e \cdot \cos(\varphi(t))^2 \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{(1-e \cdot \cos(\varphi(t)))^2} -$$

$$\frac{p \cdot e \cdot \cos(\varphi(t)) \cdot \sin(\varphi(t)) \cdot \frac{d^2}{dt^2} \varphi(t)}{(1-e \cdot \cos(\varphi(t)))^2} - \frac{p \cdot \cos(\varphi(t)) \cdot \frac{d^2}{dt^2} \varphi(t)}{1-e \cdot \cos(\varphi(t))} - \frac{p \cdot \sin(\varphi(t)) \cdot \frac{d^2}{dt^2} \varphi(t)}{1-e \cdot \cos(\varphi(t))} \quad (10)$$

$$\ddot{y} = \frac{2 \cdot p \cdot e^2 \cdot \sin(\varphi(t))^3 \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{(1-e \cdot \cos(\varphi(t)))^3} - \frac{3 \cdot p \cdot e \cdot \sin(\varphi(t)) \cdot \cos(\varphi(t)) \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{(1-e \cdot \cos(\varphi(t)))^2} - \frac{p \cdot e \cdot \sin(\varphi(t))^2 \cdot \frac{d^2}{dt^2} \varphi(t)}{(1-e \cdot \cos(\varphi(t)))^2} -$$

$$\frac{p \cdot \sin(\varphi(t)) \cdot \left(\frac{d}{dt} \varphi(t) \right)^2}{1-e \cdot \cos(\varphi(t))} + \frac{p \cdot \cos(\varphi(t)) \cdot \frac{d^2}{dt^2} \varphi(t)}{1-e \cdot \cos(\varphi(t))} \quad (11)$$

Let the second time derivative be put in the equation (4) and move everything to the left side. A differential equation of second order curves with respect to left focus was obtained. Since (12) describes the motion of a point without considering the causes of motion, it is kinematic.

$$\ddot{\varphi} = \frac{2 \cdot e \cdot \sin(\varphi) \cdot \dot{\varphi}^2}{1-e \cdot \cos(\varphi)} \quad (12)$$

Different values of the eccentricity will lead into a different shape of the curve.

Equation (12) allows you to calculate the parameters of motion at the given points of the curve: angular and linear velocities and accelerations, sector velocity, directions of vectors of linear velocity and acceleration [2].

Let's set the semiaxes of the ellipse: a - semi-major axis, b - semi-minor axis.

Period $T = 1$. We split T into N time intervals (moments). At each moment, we will calculate the parameters of the movement. Let's use a suite of numerical analysis programs [3]. The subroutine for solving the differential equation calculates the first and second derivatives at a given point n_i , $i = 1, 2, \dots, N$.

Kepler's Second law

The subroutine for calculating a certain integral computes the sector area in a given time interval. The program TygeBraheKepler2_focal.exe [4] calculates the point motion parameters according to equation (12) and shows the equality of sector areas at equal time intervals, Fig. 2, 3, 4.

Figure 2 shows the program test. The area of the ellipse is πab . $3.14159 * 9 * 7 = 197.92017$.

```

C:\Users\wik\Documents\Force2\TygeBraheKepler2_focal\TygeBraheKepler2_focal.exe
4-angular velocity, 5-polar radius, 6-linear velocity
Enter char =
if char = "y" then the source data is specified:
y
a = 9.00 b = 7.00
dpi = 0.0000000 H = 1.00000005E-03
Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999
Set the start of the first sector (i0=1,..., 999) i0 =
1
Set the end of the first sector (i0<i1<1/H) i1 =
999
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
1
angle(i0) 0.00; angle(i1) 6.28
angle(i02)= 0.00; angle(i12) 6.28
Area of the first sector: 0.1975210E+03
IERR: 0
Area of the second sector: 0.1975210E+03
IERR: 0

```

Figure 2.

Figure 3 shows equal time intervals at different points in the period.

```

C:\Users\vik\Documents\Force2\TygeBraheKepler2_focus\TygeBraheKepler2_focus.exe
4-angular velocity, 5-polar radius, 6-linear velocity
Enter char =
if char = "y" then the source data is specified:
y
  a = 9.00 b = 7.00
  dpi = 0.0000000 H = 1.00000005E-03

Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999

Set the start of the first sector (i0=1,..., 999) i0 =
22
Set the end of the first sector (i0<i1<1/H) i1 =
333
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
555
angle(i0) 0.04; angle(i1) 0.81
angle(i02)= 4.57; angle(i12) 6.03
Area of the first sector: 0.6155315E+02
IERR: 0
Area of the second sector: 0.6155347E+02
IERR: 0

```

Figure 3.

In Figure 4, precession (dpi = 0.1) is added to the parameters of Figure 3.

```

C:\Users\vik\Documents\Force2\TygeBraheKepler2_focus\TygeBraheKepler2_focus.exe
7
  a = 9.00 b = 7.00
  (precession 0<=dpi<=pi/10) dpi=
0.1
  0.10000000
  (period = 1, 0 < H < 1) H =
0.001

Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999

Set the start of the first sector (i0=1,..., 999) i0 =
22
Set the end of the first sector (i0<i1<1/H) i1 =
333
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
555
angle(i0) 0.04; angle(i1) 0.89
angle(i02)= 4.94; angle(i12) 6.12
Area of the first sector: 0.6487499E+02
IERR: 0
Area of the second sector: 0.6487521E+02
IERR: 0

```

Figure 4.

Kepler's Third Law

Note that in perihelion and aphelion $\sin(\varphi) = 0$, therefore the acceleration at these points is zero, and the speed difference modulo is a constant:

$$v_1 - v_2 = \delta \quad (13)$$

v_1 is the linear velocity in perihelion, v_2 is the linear velocity in aphelion.

The sector speed according to the law of saving the amount of movement is a constant value:

$$v_s = 1/2 v r \quad (14)$$

Let's express the sector velocity modulo linear velocity.

Since $\sin(v_1 \wedge r_1) = \sin(v_2 \wedge r_2) = 1$, then

$$v_s = 1/2 v_1 r_1 = 1/2 r_1 (v_2 + \delta) \quad (15)$$

$$v_s = 1/2 v_2 r_2 \quad (16)$$

$$1/2 r_1 (v_2 + \delta) = 1/2 r_2 v_2 \quad (17)$$

$$v_2 = \frac{r_1 \delta}{r_2 - r_1} \quad (18)$$

$$\text{on the one hand } S_{\text{ellipse}} = \pi ab \quad (20)$$

$$\text{on the other } S_{\text{ellipse}} = v_s T = T \frac{\delta r_1 r_2}{2(r_2 - r_1)} \quad (21)$$

Therefore

$$T \frac{\delta r_1 r_2}{2(r_2 - r_1)} = \pi ab \quad (22)$$

For further transformations, use the geometric properties of the ellipse. We have the following ratios:

$$r_2 - r_1 = 2c, c = ae, r_1 r_2 = a^2 - c^2 = b^2$$

Substitute in (22):

$$T \frac{\delta b^2}{4ae} = \pi ab \quad (23)$$

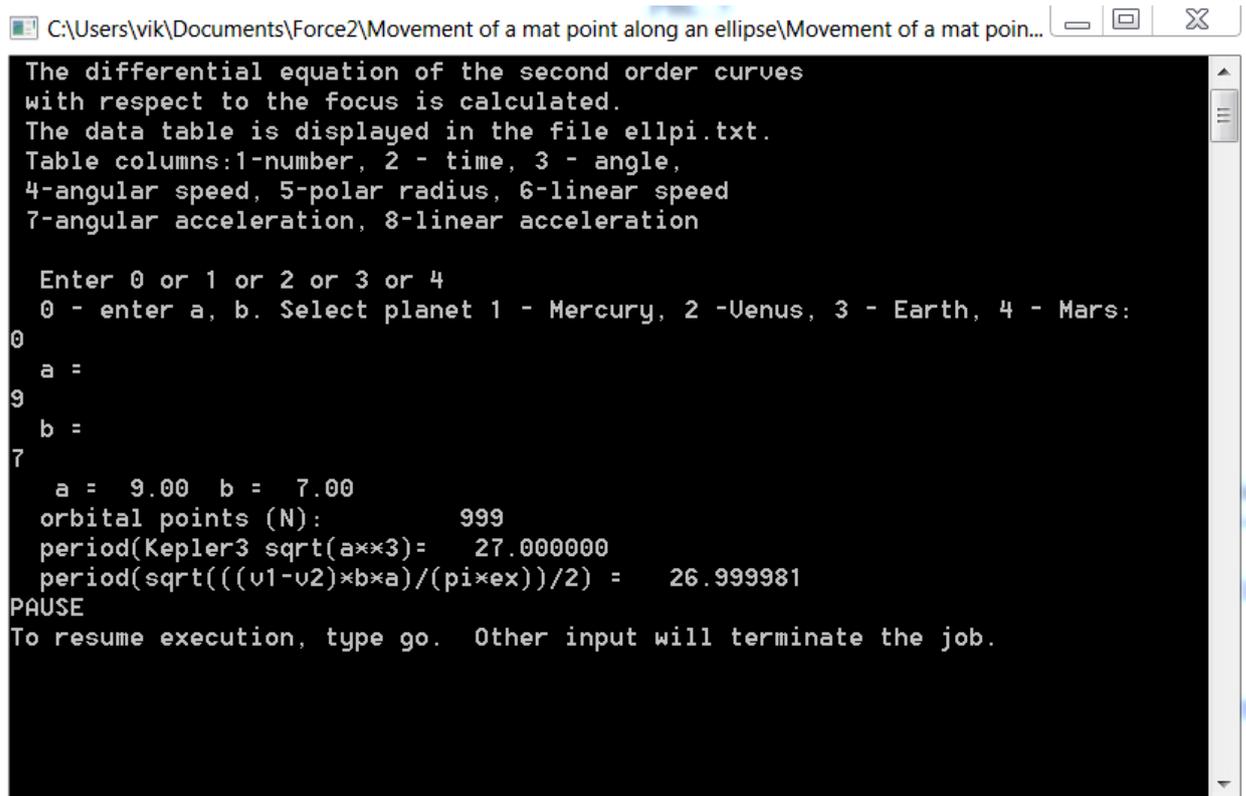
$$T \frac{\delta b}{a^2 e} = 4\pi ; \quad \text{где } T = 1; \quad (24)$$

$$\frac{\delta b}{4\pi a^2 e} = 1 \quad (25)$$

$$\text{Kepler's Third Law } \frac{T^2}{a^3} = 1 \quad (26)$$

$$\frac{\delta b}{4\pi a^2 e} = \frac{T^2}{a^3}; \frac{\delta b}{4\pi e} = \frac{T^2}{a}; T = \frac{1}{2} \sqrt{\frac{\delta b a}{\pi e}} = \frac{1}{2} \sqrt{\frac{(v_1 - v_2) b a}{\pi e}} \quad (27)$$

Program *Movement of a mat point along an ellipse.exe* [4] calculates periods using formulas (26) and (27). $\delta = v_1 - v_2$ [AU/(year of the planet)]



```

C:\Users\vik\Documents\Force2\Movement of a mat point along an ellipse\Movement of a mat poin...
The differential equation of the second order curves
with respect to the focus is calculated.
The data table is displayed in the file ellpi.txt.
Table columns: 1-number, 2 - time, 3 - angle,
4-angular speed, 5-polar radius, 6-linear speed
7-angular acceleration, 8-linear acceleration

Enter 0 or 1 or 2 or 3 or 4
0 - enter a, b. Select planet 1 - Mercury, 2 -Venus, 3 - Earth, 4 - Mars:
0
a =
9
b =
7
a = 9.00 b = 7.00
orbital points (N): 999
period(Kepler3 sqrt(a**3)= 27.000000
period(sqrt(((v1-v2)*b*a)/(pi*ex))/2) = 26.999981
PAUSE
To resume execution, type go. Other input will terminate the job.

```

Figure 5

```

C:\Users\vik\Documents\Force2\Movement of a mat point along an ellipse\Movement of a mat poin...
The differential equation of the second order curves
with respect to the focus is calculated.
The data table is displayed in the file ellpi.txt.
Table columns:1-number, 2 - time, 3 - angle,
4-angular speed, 5-polar radius, 6-linear speed
7-angular acceleration, 8-linear acceleration

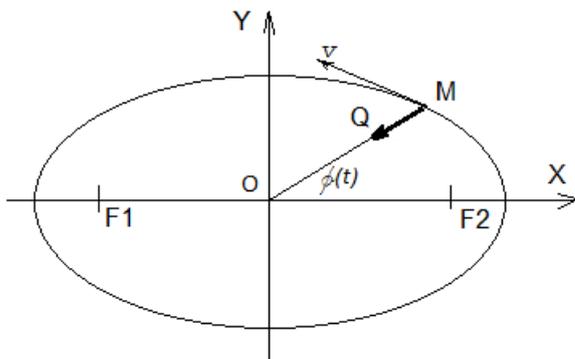
Enter 0 or 1 or 2 or 3 or 4
0 - enter a, b. Select planet 1 - Mercury, 2 -Uenus, 3 - Earth, 4 - Mars:
1
  a = 0.39 b = 0.38
orbital points (N):          999
period(Kepler3 sqrt(a**3))= 0.24084271
period(sqrt(((v1-v2)*b*a)/(pi*ex))/2) = 0.24084280
PAUSE
To resume execution, type go. Other input will terminate the job.

```

Figure 6

2. Differential equation of motion of a point along an ellipse relative to the center

Move the origin to the center of the ellipse, Fig. 7. The radius function (19) changes.



M - material point, Q – the force acting on the point, $F1$ - left focus, $F2$ - right focus, $\varphi(t)$ - angle between X axis and the line connecting left focus and the point, A - aphelion, P - perihelion

Figure 7

Repeat the reasoning of option 1.

$$m\ddot{x} = -Q\cos(\varphi(t)) \tag{28}$$

$$m\ddot{y} = -Q\sin(\varphi(t)) \tag{29}$$

From equation (27) we can get

$$Q = \frac{-m\ddot{x}}{\cos(\varphi(t))} \quad (30)$$

Let us substitute equation (29) into equation (30)

$$\dot{y} = \frac{\ddot{x}}{\cos(\varphi(t))} \sin(\varphi(t)) \quad (31)$$

The coordinates of the point can be represented as a function of the angle $\varphi(t)$ and a function of the radius $r(t)$.

$$x = r(\varphi(t)) \cdot \cos(\varphi(t)) \quad (32)$$

$$y = r(\varphi(t)) \cdot \sin(\varphi(t)) \quad (33)$$

$$r(\varphi(t)) = \frac{b}{\sqrt{1-e^2 \cos^2 \varphi(t)}} \quad (34)$$

Let us calculate the first and second time derivative From equations (32), (33), (34). Let's put the second derivatives into equation (31), transfer everything to the left and solve it.

$$\ddot{\varphi} = \frac{2 \cdot e^2 \cdot \cos(\varphi) \cdot \sin(\varphi) \cdot \dot{\varphi}^2}{1 - e^2 \cdot \cos(\varphi)^2} \quad (35)$$

We obtained a kinematic equation of the movement of a point along second-order curves relative to the center.

Kepler's Second Law

The subroutine for calculating a certain integral computes the sector area in a given time interval. The program TygeBraheKepler2_center.exe [4] calculates the point motion parameters according to equation (12) and shows the equality of sector areas at equal time intervals, Fig. 8, 9, 10.

Figure 8 shows the program test. The area of the ellipse is πab . $3.14159 \cdot 9 \cdot 7 = 197.92017$.

```

C:\Users\vik\Documents\Force2\TygeBraheKepler2_center\TygeBraheKepler2_center.exe
4-angular velocity, 5-polar radius, 6-linear velocity
Enter char =
if char = "y" then the source data is specified:
y
  a = 9.00 b = 7.00
  dpi = 0.0000000 H = 1.00000005E-03
Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999
Set the start of the first sector (i0=1,..., 999) i0 =
1
Set the end of the first sector (i0<i1<1/H) i1 =
999
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
1
angle(i0) 0.00; angle(i1) 6.28
angle(i02)= 0.00; angle(i12) 6.28
Area of the first sector: 0.1976214E+03
IERR: 0
Area of the second sector: 0.1976214E+03
IERR: 0

```

Figure 8

```

C:\Users\vik\Documents\Force2\TygeBraheKepler2_center\TygeBraheKepler2_center.exe
4-angular velocity, 5-polar radius, 6-linear velocity
Enter char =
if char = "y" then the source data is specified:
y
  a = 9.00 b = 7.00
  dpi = 0.0000000 H = 1.00000005E-03
Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999
Set the start of the first sector (i0=1,..., 999) i0 =
22
Set the end of the first sector (i0<i1<1/H) i1 =
333
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
555
angle(i0) 0.10; angle(i1) 2.20
angle(i02)= 3.41; angle(i12) 5.56
Area of the first sector: 0.6155317E+02
IERR: 0
Area of the second sector: 0.6155319E+02
IERR: 0

```

Figure 9

In Figure 10, precession (dpi = 0.1) is added to the parameters of Figure 9.

```

C:\Users\wik\Documents\Force2\TygeBraheKepler2_center\TygeBraheKepler2_center.exe
9
semiminor axis b =
7
a = 9.00 b = 7.00
(precession 0<=dpi<=pi/10) dpi=
0.1
0.10000000
(period = 1, 0 < H < 1 ) H =
0.001

Second law of Kepler H= 1.00000005E-03
The point bypasses the ellipse in 1/H time units (0 < H < 1), counterclockwise.
1/H = 999

Set the start of the first sector (i0=1,..., 999 ) i0 =
22
Set the end of the first sector (i0<i1<1/H) i1 =
333
Set the start of the second sector (0<i02<1/H-i1+i0) i02 =
555
angle(i0) 0.10; angle(i1) 2.25
angle(i02)= 3.47; angle(i12) 5.67
Area of the first sector: 0.6280998E+02
IERR: 0
Area of the second sector: 0.6280998E+02
IERR: 0

```

Figure 10

Kepler's Third Law

Program *Movement of a mat point along an ellipse center.exe* [4] calculates periods using formulas (26) and (27). $\delta = v_1 - v_2$ [AU/(year of the planet)].

Figures 11, 12, 13 show that the period difference increases as eccentricity increases.

```
C:\Users\vik\Documents\Force2\Movement of a mat point along an ellipse center\Movement of a m...
The differential equation of the second order curves
with respect to the focus is calculated.
The data table is displayed in the file ellpi.txt.
Table columns:1-number, 2 - time, 3 - angle,
4-angular speed, 5-polar radius, 6-linear speed
7-angular acceleration, 8-linear acceleration

Enter 0 or 1 or 2 or 3 or 4
0 - enter a, b. Select planet 1 - Mercury, 2 -Uenus, 3 - Earth, 4 - Mars:
0
a =
9
b =
7
a = 9.00 b = 7.00
orbital points (N):          999
period(Kepler3 sqrt(a**3)=   27.000000
period(sqrt(((v1-v2)*b*a)/(pi*ex))/2) =  21.000002
PAUSE
To resume execution, type go. Other input will terminate the job.
```

Figure 11

```
C:\Users\vik\Documents\Force2\Movement of a mat point along an ellipse center\Movement of a m...
The differential equation of the second order curves
with respect to the focus is calculated.
The data table is displayed in the file ellpi.txt.
Table columns:1-number, 2 - time, 3 - angle,
4-angular speed, 5-polar radius, 6-linear speed
7-angular acceleration, 8-linear acceleration

Enter 0 or 1 or 2 or 3 or 4
0 - enter a, b. Select planet 1 - Mercury, 2 -Uenus, 3 - Earth, 4 - Mars:
1
a = 0.39 b = 0.38
orbital points (N):          999
period(Kepler3 sqrt(a**3)=   0.24084271
period(sqrt(((v1-v2)*b*a)/(pi*ex))/2) =  0.23569536
PAUSE
To resume execution, type go. Other input will terminate the job.
```

Figure 12

```

C:\Users\vik\Documents\Force2\Movement of a mat point along an ellipse center\Movement of a m...
The differential equation of the second order curves
with respect to the focus is calculated.
The data table is displayed in the file ellpi.txt.
Table columns:1-number, 2 - time, 3 - angle,
4-angular speed, 5-polar radius, 6-linear speed
7-angular acceleration, 8-linear acceleration

Enter 0 or 1 or 2 or 3 or 4
0 - enter a, b. Select planet 1 - Mercury, 2 -Venus, 3 - Earth, 4 - Mars:
2
a = 0.73 b = 0.73
orbital points (N): 999
period(Kepler3 sqrt(a**3)= 0.62144679
period(sqrt(((v1-v2)*b*a)/(pi*ex))/2) = 0.62116992
PAUSE
To resume execution, type go. Other input will terminate the job.

```

Figure 13

Conclusion

Kinematic equation (12) accurately describes the motion along ideal second-order curves. Real orbits of cosmic bodies have deviations from the ideal curve: precession, periodic asymmetry of radius lengths [5] and other types of deviation.

If we use Kepler's third law refined by Newton, the law of conservation of angular momentum and equation (12), we can simulate the motion of three or more bodies along second-order curves. Example [7].

Kinematic equation (35) is applicable for modeling current lines of liquid and gas particles [6].

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