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Assuming $c < R^{1+0.63}$, or $c < R^2 \Longrightarrow$ Implies The abc Conjecture Is False

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Abstract In this paper about the *abc* conjecture, assuming the conjecture $c < R^{1.63}$ or $c < R^2$ is true, we give the proof that the *abc* conjecture is false and it is true if we consider only for $\epsilon \geq 0.63$ or $\epsilon \geq 2$.

Keywords Elementary number theory · real functions of one variable.

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To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1 (***abc* Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

where K is a constant depending only of ϵ.

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The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2] in November 2018. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. Since 2018, I have studied the conjecture and tried some methods to resolve it.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [3]. A conjecture was proposed that $c < rad^2(abc)$ [4]. It follows obtaining one proof, the *abc* conjecture can be resolved. In my paper, I assume that $c < rad^{1.63}(abc)$ or $c < rad^2 abc$ holds, then I give the proof that the *abc* conjecture is false for $R < c$ and $\forall \epsilon$, $0 < \epsilon < \epsilon_0$ with $\epsilon_0 = 0.63$ or $\epsilon_0 = 2$. If $c < R$, the proof is trivial and the abc conjecture holds.

2 Preliminaries

Let a, b, c (respectively a, c) positive integers relatively prime with $c = a + b, a > b, b \geq 2$ (respectively $c = a + 1$, $a \geq 2$). We denote ϵ_0 one of the two values 0.63, 2 and $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$.

As cited above, we know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [3]. It concerned the best example given by E. Reyssat [3]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{4}$$

In 2012, A. Nitaj [5] proposed the following conjecture:

Conjecture 2 Let a, b, c be positive integers relatively prime with $c = a + b$, then:

$$c < rad^{1.63}(abc) \tag{5}$$

$$abc < rad^{4.42}(abc) \tag{6}$$

We assume in the following that (5) holds or $c < R^2$. We recall the following proposition [5]:

Proposition 1 *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{7}$$

After studying the *abc* conjecture using different choices of the constant $K(\epsilon)$ and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that $c < rad^2(abc)$ or $c < rad^{1.63}$ is true, the *abc* conjecture does not hold when $0 < \epsilon < 1$ or $0 < \epsilon < 0.63$, it follows that the *abc* conjecture as it was defined is false.

3 The Proof of the *abc* Conjecture is false

Proof - We recall the definition of the *abc* conjecture:

For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{8}$$

where K is a constant depending only of ϵ.

We choose one $\epsilon > 0$, it exists one function $K(\epsilon)$. From the equation (8) above, $K(\epsilon) > 0$. Let a, b, c positive integers relatively prime with $c = a+b$ and we assume that:

$$c < R^{1+\epsilon_0} \tag{9}$$

is true.

A1 - We can write the equation (9) for all $\epsilon \geq \epsilon_0$ as $c < K(\epsilon).R^{1+\epsilon}$ and taking $K(\epsilon)$ any positive function ≥ 1 for $\epsilon \geq \epsilon_0$ and in this case the *abc* conjecture is verified.

A2 - We choose one ϵ so that $0 < \epsilon < \epsilon_0$. We suppose that the *abc* conjecture is true, it exists one positive function $K(\epsilon) > 0$. We can write:

$$c < K(\epsilon)R^{1+\epsilon}$$

As we have assumed that $c < R^{1+\epsilon_0}$ is true, we have the three following cases:

A2-1- $R^{1+\epsilon_0} = K(\epsilon)R^{1+\epsilon} \Longrightarrow LogR = \dfrac{LogK(\epsilon)}{\epsilon_0 - \epsilon}$, then the contradiction because the choice of $K(\epsilon)$ is independent of a, b, c and $R = rad(abc)$.

A2-2- $R^{1+\epsilon_0} < K(\epsilon)R^{1+\epsilon} \Longrightarrow R^{\epsilon_0-\epsilon} < K(\epsilon)$. As $\epsilon_0 - \epsilon > 0$ and the constant $K(\epsilon)$ is bounded : $K(\epsilon) < +\infty$. If R becomes very large, the inequality $R^{\epsilon_0-\epsilon} < K(\epsilon)$ gives a contradiction.

A2-3- $K(\epsilon)R^{1+\epsilon} < R^{1+\epsilon_0} \Longrightarrow K(\epsilon) < R^{\epsilon_0-\epsilon}$. If we choose $\epsilon \ll 1$: $\epsilon \to 0^+$ and as R is bounded, from the proposition (7) above, $K(\epsilon)$ becomes very large, then the inequality $K(\epsilon) < R^{\epsilon_0-\epsilon}$ gives a contradiction.

It follows that the hypothesis supposed in paragraph A2 that the *abc* conjecture is true, is false. Hence, the *abc* conjecture is false and the proof is finished.

4 Conclusion

Assuming one of the two conjectures $c < R^{1.63}$ or $c < R^2$ holds, we have given an elementary proof that the *abc* conjecture is false. We can announce the theorem:

Theorem 1 *Assuming one of the two conjectures $c < R^{1.63}$ or $c < R^2$ holds, then the abc conjecture is false.*

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