# ON ODD PERFECT NUMBERS

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Abstract. In this note, we introduce the notion of the disc induced by an arithmetic function and apply this notion to the odd perfect number problem. We show that under certain special local condition an odd perfect number exists by exploiting this concept.

## 1. Introduction

Let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ denotes the sum-of-divisor function, defined as

$$\sigma(N) := \sum_{n|N} n$$

for a fixed $N \in \mathbb{N}$. We say $N$ is a perfect number if and only if $\sigma(N) = 2N$. If $N$ is perfect and is odd then we say it is an odd perfect number. It is still unknown if there exist any odd perfect numbers and the problem for asserting their existence or non-existence still remains an active area of research. Much work has already been done in this area and most subtle and basic properties about odd perfect - if they exist - are now known. The eighteenth century mathematician Leonard Euler was the first to show that if any odd perfect number $N$ exists then it must be of the form

$$N := q^{\beta} \prod_{i=1}^{n} p_i^{\alpha_i}$$

where $q, \beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$. It is also know that, if an odd perfect number $N$ exists then it must satisfy the inequality $N > 10^{1500}$ [1]. It is also known that (see [2]) an odd perfect number must not be divisible by 105 and must satisfy the congruence conditions (see [3])

$$N \equiv 1 \pmod{12} \quad \textbf{and} \quad N \equiv 117 \pmod{468} \quad N \equiv 81 \pmod{324}.$$

If there are $k$ of the exponents $\alpha_i$ in the prime factorization of $N$ with $\alpha_i \equiv 0 \pmod 2$, then it is known that the smallest prime factor of $N$ is at most $\frac{k-1}{2}$ [4]. In this case, it has been shown that (see [5])

$$N < 2^{4^{k+1}-2^{k+1}}$$

and with $q\prod_{i=1}^{k} p_i < 2N^{\frac{17}{26}}$ [6]. The scale of the largest and the second largest prime factor of an odd perfect number - if they exist - has also been studied quite extensively in a series of papers by several authors. It is now known that the largest prime factor of $N$ is greater than $10^{18}$ (see [7]) and less than $(3N)^{\frac{1}{3}}$ [8]. It has also

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been shown that the second largest prime factor of an odd perfect number $N$ must be greater than $10^4$ and less than $(2N)^{\frac{1}{5}}$ [9]. The third largest prime factor is now known to be greater than 100. All of these result could conceivably be synthesized in a nice way to study the main question of the existence or non-existence of an odd perfect number.
In this paper, by using the notion of the disc induced by arithmetic functions, we show conditionally that there exists an odd perfect number.

## 2. **Preliminary results**

In this section we launch some fundamentally well-known traditional results concerning odd perfect numbers.

**Lemma 2.1** (Euler)**.** *Let $N$ be an odd perfect number, then $N$ has the unique representation*

$$N = q^{\beta}\prod_{i=1}^{n}p_i^{\alpha_i}$$

*where $q,\beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$.*

**Theorem 2.2.** *If $N$ an odd perfect number then*

$$\varphi(N) \leq \lfloor \frac{N}{2} \rfloor$$

*where $\varphi$ denotes the Euler totient function.*

*Proof.* Let us assume there exists an odd perfect number $N$. It is clear that $N$ must be composite so that by the fundamental theorem of arithmetic and Lemma 3.4 the representation holds for $N$

$$N := q^{\beta}\prod_{i=1}^{n}p_i^{\alpha_i}$$

where $q,\beta \equiv 1 \pmod 4$ and $\alpha_i \equiv 0 \pmod 2$ for each $1 \leq i \leq n$. Next let us apply the sum-of-divisor function $\sigma$ on $N$ and study their internal structure

$$\sigma(N) := \sigma\bigg(q^{\beta} \times \prod_{i=1}^{n}p_i^{\alpha_i}\bigg).$$

Since the sum-of-divisor function $\sigma$ is multiplicative, we obtain further the decomposition

$$\sigma(N) = \bigg(\sum_{j=0}^{\beta}q^{\beta-j}\bigg) \times \bigg(\prod_{i=1}^{n}\sum_{j=0}^{\alpha_i}p_i^{\alpha_i-j}\bigg)$$

where we have used the elementary identity $x^n-1 = (x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$. Since we have assumed that $N$ is an odd perfect number, it follows that $\sigma(N) = 2N$ so that

$$2q^{\beta}\prod_{i=1}^{n}p_i^{\alpha_i} = \bigg(\sum_{j=0}^{\beta}q^{\beta-j}\bigg) \times \bigg(\prod_{i=1}^{n}\sum_{j=0}^{\alpha_i}p_i^{\alpha_i-j}\bigg).$$

By rearranging terms the following representation holds

$$\left(\sum_{j=0}^{\beta}\frac{1}{q^j}\right)\times\left(\prod_{\substack{i=1\\p_i^{\alpha_i}||N}}^{n}\sum_{j=0}^{\alpha_i}\frac{1}{p_i^j}\right)=2.$$

Then we have the inequality

$$\begin{aligned}2&=\left(\sum_{j=0}^{\beta}\frac{1}{q^j}\right)\times\left(\prod_{\substack{i=1\\p_i^{\alpha_i}||N}}^{n}\sum_{j=0}^{\alpha_i}\frac{1}{p_i^j}\right)\\&\leq\prod_{p|N}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\right)\\&=\frac{N}{\varphi(N)}\end{aligned}$$

and the claimed inequality follows immediately. □

[1]

## 3. The notion of the disc induced by arithmetic functions and application to the odd perfect number problem

In this section we introduce and study the notion of the disc induced by arithmetic functions. We find this notion suitable for verifying the non-existence of odd perfect numbers. We launch the following language.

**Definition 3.1.** Let $f:\mathbb{N}\longrightarrow\mathbb{N}$ and let $a,r\in\mathbb{N}$ be fixed. Then by the disc induced by $f$ with center $a$ and radius $r$, denoted $\mathcal{D}_f(a,r)$, we mean

$$\mathcal{D}_f(a,r):=|f(m)-a|\leq r$$

for $m\in\mathbb{N}$. We say $s\in\mathcal{D}_f(a,r)$ if and only if $|f(s)-a|\leq r$. We say the disc induced is **degenerative** if there exists some $t\in\mathcal{D}_f(a,0)$ and we call $\mathcal{D}_f(a,0)$ the degenerated disc. Otherwise we say the disc induced is non-degenerative. We say the disc induced is **uniformly** degenerative if it is degenerative for all $a\in\mathbb{N}$.

**Proposition 3.2.** The following properties hold

(i) Let $g:\mathbb{N}\longrightarrow\mathbb{N}$ be multiplicative and $s=uv$ with $(u,v)=1$ with $u,v>1$. If $s\in\mathcal{D}_g(a,r)$ for a fixed $r,a\in\mathbb{N}$ and

$$a<\frac{g(u)+g(s)}{2}\quad a<\frac{g(v)+g(s)}{2}$$

then $u\in\mathcal{D}_g(a,r-\epsilon)$ and $v\in\mathcal{D}_g(a,r-\delta)$ for some $\epsilon,\delta>0$.

(ii) Let $g:\mathbb{N}\longrightarrow\mathbb{N}$ be additive and $s=uv$ with $(u,v)=1$ with $u,v>1$. If $s\in\mathcal{D}_g(a,r)$ for a fixed $r,a\in\mathbb{N}$ and

$$a<\frac{g(u)+g(s)}{2}\quad a<\frac{g(v)+g(s)}{2}$$

then $u\in\mathcal{D}_g(a,r-\epsilon)$ and $v\in\mathcal{D}_g(a,r-\delta)$ for some $\epsilon,\delta>0$.

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*Proof.* We only prove property $(i)$ since the same approach could be adapted for property $(ii)$. Let $s \in \mathcal{D}_g(a,r)$ and write $s = uv$ such that $(u,v) = 1$ with $u, v > 1$. Then since $g$ is multiplicative and

$$a < \frac{g(u)+g(s)}{2} \quad a < \frac{g(v)+g(s)}{2}$$

we can write

$$|g(u)-a| < |g(s)-a| = |g(u)g(v)-a| \leq r$$

so that there exists some $\epsilon > 0$ such that $|g(s)-a| = |g(u)-a| + \epsilon \leq r$ and it follows that $u \in \mathcal{D}_g(a, r-\epsilon)$. It follows similarly that there exists some $\delta > 0$ such that $v \in \mathcal{D}_g(a, r-\delta)$. □

*Remark* 3.3. Now we verify an important but yet trivial preparatory observation for asserting the truth of our main result. It conveys the principal notion that no degenerated disc induced by an arithmetic function will ever contain a composite.

**Proposition 3.4.** Let $g : \mathbb{N} \longrightarrow \mathbb{N}$ be multiplicative (resp. additive). If $s = uv$ with $(u,v) = 1$ and $u, v \geq 3$ such that at least one of the following holds

$$a < \frac{g(u)+g(s)}{2} \quad a < \frac{g(v)+g(s)}{2}$$

then $s \notin \mathcal{D}_g(a, 0)$.

*Proof.* Let $s = uv$ such that $(u,v) = 1$ with $u, v \geq 3$ and assume to the contrary that $s \in \mathcal{D}_g(a, 0)$. Since $g$ is multiplicative, let us assume at least one of the following holds

$$a < \frac{g(u)+g(s)}{2} \quad a < \frac{g(v)+g(s)}{2}.$$

Then it follows from Proposition 3.2 that at least one of the following holds

$$u \in \mathcal{D}_g(a, -\epsilon) \quad v \in \mathcal{D}_g(a, -\delta)$$

for some $\epsilon, \delta > 0$. This is impossible since the radius of each of the degenerated disc is negative, thereby ending the proof. □

**Theorem 3.5** (Main theorem)**.** *If there exists an $l \in \mathcal{D}_\sigma(2N, \epsilon)$ such that $l = Nd$ with $(N,d) = 1$ and*

$$2N < \frac{\sigma(N)+\sigma(l)}{2}$$

*for any $\epsilon > 0$, then there must exist an odd perfect number.*

*Proof.* It suffices to show that some odd composite $N$ must satisfy $N \in \mathcal{D}_\sigma(2N, 0)$. Suppose to the contrary that there exists no odd perfect number. Let $r$ be fixed, then for any odd composite $N$ we choose $P = Ns$ with $(s,N) = 1$ such that $P \in \mathcal{D}_\sigma(2N, r)$ and

$$2N < \frac{\sigma(N)+\sigma(P)}{2}.$$

By appealing to Proposition 3.2, we have

$$N \in \mathcal{D}_\sigma(2N, r-\epsilon)$$

for some $\epsilon > 0$. Again we choose $Q \in \mathcal{D}_{\sigma}(2N, r-\epsilon)$ such that $Q = Nt$ with $(N,t) = 1$ and

$$2N < \frac{\sigma(N) + \sigma(Q)}{2}$$

then by appealing one more time to Proposition 3.2 and the assumption that there exists no odd perfect number, we obtain the containment

$$N \in \mathcal{D}_{\sigma}(2N, r-\epsilon-\delta).$$

Under the assumption that there exists no odd perfect number and the additional local condition, we obtain the following infinite descending sequence of the radius of each smaller disc

$$r > r-\epsilon > r-\epsilon-\delta > \cdots$$

and yet the disc will never degenerate, which is an impossible phenomenon. This completes the proof of the theorem. □

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