Proof of Legendre's conjecture

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Abstract

By a consideration of this research, since we found that at least one prime number exists between n^2 and $n(n+1)$ when $n \geqq 3$ holds, we have obtained a conclusion that Legendre's conjecture is true.

Contents

1. Introduction

This is the conjecture that there is a prime number always between n^2 and $(n+1)^2$ for arbitrary positive integer n. It was set up by the French mathematician Adrien-Marie Legendre. (Quoted from Wikipedia)

2. Proof

Let n and p be positive integers. If Legendre's conjecture is true, there is at least one prime number p satisfying the following inequalities.

$$n^2 < p < (n+1)^2 \ (n \geqq 1) \ \dots (1)$$

Ⅰ. When $n < 3$

There are prime numbers 2 and 3 between 1 and 4. In the same way, there are prime numbers 5 and 7. Therefore, Legendre's conjecture is true when $n < 3$ holds.

Ⅱ. When $n \geqq 3$

Suppose that p satisfies the following inequalities,

$$n^2 < p < n(n+1) \ \dots (2)$$

Let q be a positive integer. If any p is a composite number and has prime factors, the largest possible factor of p in the range of the inequalities (2) is $n(n+1)/2-1$ and p must be divided by q which is from $n+2$ to $(n^2+n-2)/2$ since n and $n+1$ cannot divide p and the product of two factors between 2 and $n-1$ cannot be p.

$$n+1 < q < n(n+1)/2 \ \dots (3)$$

We will consider the case where p is divisible by q satisfying the inequalities above. Let r be a positive integer and the quotient when p is divided. r must be satisfied the following inequalities.

$$1 < r < n \ \dots (4)$$

If it is assumed that p are all composite numbers in the range of the inequalities (2), p must be divided by q in the inequalities (3). When p is a composite number, one p corresponds to some combinations of q and r. p has a one-to-one correspondence with q since the maximum value of p in the inequalities (2) is less than twice the minimum value of p and the maximum p is equal to or more than double the other p if p has a one-to-many correspondence with q. And p has a one-to-many correspondence with r.

We will apply a rule to select the relations from p to r and consider the case when $n = 9$ holds.

When $p = 82$, $(q, r) = (41,2)$

When $p = 84$, $(q, r) = (42,2), (28,3), (21,4), (14,6), (12,7)$

When $p = 85$, $(q, r) = (17,5)$

When $p = 86$, $(q, r) = (43,2)$

When $p = 87$, $(q, r) = (29,3)$

When $p = 88$, $(q, r) = (44,2), (22,4), (11,8)$

Define $[p, r]$ as a relation from p to r. We will select the relations between p and r so that there are all one-to-one correspondences. At first the relations are selected by r which are multiples of 2 for each p. [82,2] is sorted out when $p = 82$ holds. Then [84,4] is sorted out since $r = 2$ has been selected. When $p = 86$ holds, there is one combination $(q, r) = (43,2)$ and $r = 2$ has been taken from. In this case, we consider to use the factor 2 of 6 and think that there is a relation [86,6]. Then [88,8] is sorted out. Next, we select the relation by 3 multiples r and [87,3] is sorted out.

When r is a composite number, we skip the number since we have already taken from the relations by a multiple of the prime factor of r. Next, we select the relations by multiples of prime numbers greater than or equal to 5.

Let $a(n, r)$ and $b(n, r)$ be integers and $a(n, r)$ be the number of r multiples in the range of the inequalities (2) and $b(n, r)$ be that in the range of the inequalities (4). The following inequalities hold.

$$a(n, r) \leqq b(n, r) + 1$$

When $n = 8$ holds, $a(8,5) = 2$, $b(8,5) = 1$ and $a(8,5) > b(8,5)$ hold.
When $p = 65$, $(q, r) = (13,5)$
When $p = 66$, $(q, r) = (33,2), (22,3), (11,6)$
When $p = 68$, $(q, r) = (34,2), (17,4)$
When $p = 69$, $(q, r) = (23,3)$
When $p = 70$, $(q, r) = (35,2), (14,5), (10,7)$
Let s be a positive integer. Starting with the smallest prime number 2, for the sth p that is a multiple of the prime number, we select a relation with r as s multiples of the prime number. In this case, we select the relations [66,2], [68,4], [70,6] when $r = 2$ holds, [69,3] when $r = 3$ holds and [65,5] when $r = 5$ holds. Let t be a prime number less than r. In the case of $a(n, r) > b(n, r)$, the actual increase in the number of relations between p and r at the time of making the selection is less than or equal to $b(n, r)$ because one of the t adjacent multiples of r is a multiple of t and the relations have already been selected by t multiples. The value of t can be considered 2 or 3 since $a(n, t) = b(n, t)$ holds as follows.

Let m be an integer.
・When $n = 2m$ and $m > 1$
$a(2m, 2) = \text{floor}(((2m)^2 + 2m - 1)/2) - \text{floor}((2m)^2/2) = m - 1$
$b(2m, 2) = \text{floor}((2m - 1)/2) = m - 1$
・When $n = 2m + 1$ and $m > 0$
$a(2m + 1,2) = \text{floor}(((2m + 1)^2 + 2m + 1 - 1)/2) - \text{floor}((2m + 1)^2/2) = m$
$b(2m + 1,2) = \text{floor}((2m + 1 - 1)/2) = m$
Therefore, $a(n, 2) = b(n, 2)$ holds when $n \geqq 3$ holds.

・When $n = 3m$ and $m > 0$
$a(3m, 3) = \text{floor}(((3m)^2 + 3m - 1)/3) - \text{floor}((3m)^2/3) = m - 1$
$b(3m, 3) = \text{floor}((3m - 1)/3) = m - 1$
・When $n = 3m + 1$ and $m > 0$
$a(3m + 1,3) = \text{floor}(((3m + 1)^2 + 3m + 1 - 1)/3) - \text{floor}((3m + 1)^2/3) = m$
$b(3m + 1,3) = \text{floor}((3m + 1 - 1)/3) = m$
・When $n = 3m + 2$ and $m > 0$
$a(3m + 2,3) = \text{floor}(((3m + 2)^2 + 3m + 2 - 1)/3) - \text{floor}((3m + 2)^2/3) = m$
$b(3m + 2,3) = \text{floor}((3m + 2 - 1)/3) = m$
Therefore, $a(n, 3) = b(n, 3)$ holds when $n \geqq 3$ holds.
From the above, the prime number r with $a(n, r) > b(n, r)$ satisfies $r \geqq 5$.

We will consider the case when $n = 17$ holds.

When $n = 17$ holds, $a(17,5) = 4$, $b(17,5) = 3$ and $a(17,5) > b(17,5)$ hold.

When $p = 290$, $(q,r) = (145,2),(58,5),(29,10)$

When $p = 291$, $(q,r) = (97,3)$

When $p = 292$, $(q,r) = (146,2),(73,4)$

When $p = 294$, $(q,r) = (147,2),(98,3),(49,6),(42,7),(21,14)$

When $p = 295$, $(q,r) = (59,5)$

When $p = 296$, $(q,r) = (148,2),(74,4),(37,8)$

When $p = 297$, $(q,r) = (99,3),(33,9),(27,11)$

When $p = 298$, $(q,r) = (149,2)$

When $p = 299$, $(q,r) = (23,13)$

When $p = 300$, $(q,r) = (150,2),(100,3),(75,4),(60,5),(50,6),(30,10),(25,12),(20,15)$

When $p = 301$, $(q,r) = (43,7)$

When $p = 302$, $(q,r) = (151,2)$

When $p = 303$, $(q,r) = (101,3)$

When $p = 304$, $(q,r) = (152,2),(76,4),(38,8),(19,16)$

When $p = 305$, $(q,r) = (61,5)$

In the beginning, we select the relations [290,2], [292,4], [294,6], [296,8], [298,10], [300,12], [302,14] and [304,16] when $r = 2$ holds. Then we select [291,3], [297,9] and [303,15] when $r = 3$ holds. The numbers of r, 6 and 12 are skipped since these have already been selected when $r = 2$. When $r = 5$ holds, we should select the relations in the case of $p = 295$ and $p = 305$. However, there is only 5 for r which corresponds to p since 10 and 15 have already been taken from. With this method, we cannot select the one-to-one relations between p and r.

And so, we will change the rules as follows. We select relations by the prime numbers in descending order. When $a(n,r) > b(n,r)$ holds, a composite number p can be skipped since one of the relations can later be selected by multiples of a prime number less than r, which is 2 or 3. When $n = 17$ holds, the relations are selected as follows.

When $r = 13$, [299,13]

When $r = 11$, [297,11]

When $r = 7$, [294,7],[301,14]

When $r = 5$, [290,5],[295,10],[305,15]

When $r = 3$, [291,3],[300,6],[303,9]

When $r = 2$, [292,2],[296,4],[298,8],[302,12],[304,16]

The minimum n when there exists r with $a(n, r) > b(n, r)$ is 7 and the minimum r where $a(n, r) > b(n, r)$ holds is 5. Therefore, if we select the relations this way, one-to-one correspondence with all composite numbers p can be set for r for all n where $n \geqq 3$ holds.

However, it becomes a contradiction since the number of p in the inequalities (2), $n - 1$ is greater than the number of r in the inequalities (4), $n - 2$ and it does not become a one-to-one correspondence between p and r. Therefore, the assumption that p are all composite numbers in the range is false and there is at least one prime number in the range of the inequalities (2) when $n \geqq 3$ holds. From the above Ⅰ and Ⅱ, it is proved that Legendre's conjecture is true. (Q.E.D.)

3. Complement

Oppermann's conjecture states that, for every integer $x > 1$, there is at least one prime number between $x(x - 1)$ and x^2, and at least another prime between x^2 and $x(x + 1)$. It is named after Danish mathematician Ludvig Oppermann, who announced it in an unpublished lecture in March 1877. (Quoted from Wikipedia)

$$x(x + 1) < p < x(x + 2)$$

Considering an integer p satisfying these inequalities, because $a(x, r) \leqq b(x, r) + 1$ holds and the minimum x where there exists r with $a(x, r) > b(x, r)$ is 9 and the minimum r where $a(x, r) > b(x, r)$ holds is 7, we found that at least one prime number between $x(x + 1)$ and $(x + 1)^2$ when $x \geqq 3$ holds in the same way as this proof. Therefore, we proved that Oppermann's conjecture is true.

4. Acknowledgement

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5. References

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