Magic Squares with Centrally Embedded Squares of Even Order: A Construction Method

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Abstract

We present a method which modifies a magic square of even order n and then adds two outer rows and two outer columns to produce a magic square of order $n + 2$. The modification of the original square will change half of its numbers and will preserve the equality of sums of the rows, columns, and main diagonals. This modified square will be centrally embedded in the magic square of order $n + 2$.

Definitions

For the purposes of this paper, a **magic square** of order n shall mean an $n \times n$ arrangement of the integers 1 through n^2 such that the sums of each row, each column and both main diagonals all equal the magic sum $S = \frac{n^2(n^2+1)}{2n} = \frac{n}{2}\,(n^2 + 1)$. An **embedded square** of order n shall mean an $n \times n$ arrangement of distinct positive integers such that each row, each column and both main diagonals have the same sum. For reasons which shall become obvious, the construction method presented here applies only to squares of even order. We will say that a magic square of even order n is **balanced** if each row, each column and each main diagonal contains exactly $\frac{n}{2}$ numbers which are greater than $\frac{n^2}{2}$. Figure 1 shows a balanced magic square of order 6, with a centrally embedded square of order 4.

9	15	17	19	26	25
27	4	35	34	1	10
24	29	6	7	32	13
23	5	30	31	8	14
16	36	3	2	33	21
12	22	20	18	11	28

Figure 1

9	15	17	47	54	53
55	4	63	62	1	10
52	57	6	7	60	13
51	5	58	59	8	14
16	64	3	2	61	49
12	50	48	18	11	56

Figure 2

Construction Method

Our method may be carried out with the aid of a calculator for the necessary arithmetic and a spreadsheet to check the results. Each stage of our construction starts with a magic square of even order n, modifies it and then surrounds it with two new rows and two new columns to arrive at a magic square of order $n + 2$. The original square will become an embedded square at the center of the larger one. We will signal those steps of this process which involve trial-and-error experimentation. One example will serve to illustrate our process.

Our example begins with the order-6 square shown in Figure 1, which will become the centrally embedded square within a magic square of order 8. The new square will contain integers 1 through 64. For this reason, we add 28 to each entry in the range 19 through 36 so that they become 47 through 64. The columns, rows and diagonals of this array (Figure 2) all sum to 195. The numbers in the two new rows and columns will be integers 19 through 46. They form 14 pairs, each summing to 65, as listed in Figure 3. So when any such pair is added at the ends of an existing row, column or diagonal, the new row, column or diagonal will consist of eight integers whose sum is 260. If the two new rows and two new columns also sum to 260, we will have a magic square of order 8.

pair		difference	
19	46	27	x
20	45	25	x
21	44	23	x
22	43	21	x
23	42	19	
24	41	17	x
25	40	15	x
26	39	13	
27	38	11	
28	37	9	x
29	36	7	x
30	35	5	
31	34	3	
32	33	1	

Figure 3

Here is where our trial and error experimentation begins. We first choose two of our 14 pairs to occupy the corners of our new order-8 square as shown in Figure 4. We illustrate by choosing the pairs (20,45) and (25,40).

20							40
	9	15	17	47	54	53	
	55	4	63	62	1	10	
	52	57	6	7	60	13	
	51	5	58	59	8	14	
	16	64	3	2	61	49	
	12	50	48	18	11	56	
25							45

Figure 4

The magic sum for an order-8 magic square is 260. Thus, we determine that the remaining six numbers in our top row must sum to 200 and the remaining numbers in the rightmost column must sum to 175. One choice for completion of the rightmost column is

$$175 = 44 + 37 + 19 + 22 + 24 + 29$$

Note that this choice keeps our square balanced.

At this point, 8 of our 14 pairs have been used. They are marked in Figure 3. Our attempt will succeed if we can achieve the needed sum of 200 for the remaining entries of the top row with the remaining pairs. If we add the larger numbers from each of these pairs, we have 221. We need numbers whose sum is 200 and $221 - 200 = 21$. We have succeeded if we can find three numbers in the difference column of the remaining six pairs whose sum is 21. Indeed, we find $13 + 5 + 3 = 21$, indicating that we should choose the smaller values of these pairs associated with these differences. Thus, we have $200 = 42 + 26 + 38 + 30 + 31 + 33$. We have completed our order-8 magic square as shown in Figure 5. This magic square is in fact balanced so that it can be used as the starting point to repeat our process and create a 10×10 magic square.

20	42	26	38	30	31	33	40
21	9	15	17	47	54	53	44
28	55	4	63	62	1	10	37
46	52	57	6	7	60	13	19
43	51	5	58	59	8	14	22
41	16	64	3	2	61	49	24
36	12	50	48	18	11	56	29
25	23	39	27	35	34	32	45

Figure 5

Observations and Suggestions for Further Investigation

The first choice in our example above was to pick two pairs to occupy the corners of the enlarged square. We now offer an example to illustrate what can go wrong with a bad choice at this initial stage. If we choose the pairs 20-45 and 24-41, then we have the following.

20							41
	9	15	17	47	54	53	
	55	4	63	62	1	10	
	52	57	6	7	60	13	
	51	5	58	59	8	14	
	16	64	3	2	61	49	
	12	50	48	18	11	56	
24							45

Figure 6

Now to complete the rightmost column we need six numbers totaling $260 - 86 = 174$. To maintain the balance of our square, exactly two of these six must be the larger number of their pairs. We may choose, for example, $174 = 42 + 37 + 19 + 22 + 25 + 29$. At this point the following six pairs remain available to complete the top and bottom row:

pair		difference
21	44	23
26	39	13
27	38	11
30	35	5
31	34	3
32	33	1

Figure 7

The sum of the six number to complete the top row must be $260 - 61 = 199$. To maintain balance, three of them must be from the first column (smaller numbers of their pairs) and three from the second column. The sum of the six numbers in the second column is 223 and we see that will need three numbers from the difference column whose total is $223 - 199 = 24$. This is impossible since all numbers in the difference column are odd.

The problem hinges on parity. If the numbers in the two left corners have the same parity then so do the two in the two right corners. This means that the sum S_1 of the remaining numbers to be placed in the rightmost column must be even and exactly two must be the larger number of their pairs. Finally, the sum S_2 of the remaining numbers to complete the top row must be odd and exactly half of them must be the larger number of their pairs. This is not achievable as shown in our example.

The above analysis applies when the embedded square has order $4n + 2$ and the completed magic square has order $4n$. In the opposite case (enlarging from order $4n$ to order $4n + 2$), the magic sum of the square under construction is odd. The parities in the above argument must be adjusted accordingly but the contradiction is the same. We have the following general result:

Theorem: When an order $2n$ balanced square is embedded in an order $2n + 2$ balanced magic square by the method presented in this paper, the smaller numbers of the two pairs occupying the four corners will be at opposite ends of one side (left or right) of the magic square and will be of opposite parity.

We choose not to provide a formal proof of this theorem as it is rather tedious and hardly more instructive than a well-chosen example. Moreover, it applies only to squares constructed by this method and therefore is of limited theoretical interest. Its value is primarily practical. That being said, a more elegant proof, not based so directly on details of our construction process, would be of interest and could possibly lead to further insights.

The balanced square concept can be used in another way. We begin with the order-6 magic square and embedded order-4 square shown in Figure 8. Now each number larger than 32 is increased by 288, resulting in the square shown in Figure 9. The largest number in this square is $324 = 18^2$. This is one of nine squares which will be arranged to form an order-18 magic square. If we increase the 32 smaller numbers in this square by 36 and decrease each larger number by 36, we arrive at the square shown in Figure 10. Repeating this process of increasing the smaller entries by 36 and decreasing the larger entries by 36 produces seven more squares, the last of which is shown in Figure 11. Now we have nine 6×6 squares with magic sum 975, each of which has an embedded 4×4 square with magic sum 650. These can be used to produce an order-18 magic square as shown in Figure 12. We leave it in this form in order to illustrate the construction method as clearly as possible. In fact, a very large number of variations are possible since the nine order-6 squares are interchangeable as are their embedded, order-4 subsquares. These interchanges, along with the eight symmetries of a square (rotations and reflections), could change the square in Figure 12 in ways which would make the details of our construction method much less discernable.

33	12	13	2	30	31
36	11	28	27	8	1
18	20	15	16	23	19
5	14	21	22	17	32
13	29	10	9	26	24
6	25	34	35	7	4

Figure 8

321	12	3	2	318	319
324	11	316	315	8	1
18	308	15	16	311	307
293	14	309	310	17	32
13	317	10	9	314	312
6	313	322	323	7	4

Figure 9

285	48	39	38	282	283
288	47	280	279	44	37
54	272	51	52	275	271
257	50	273	274	53	68
49	281	46	45	278	276
42	277	286	287	43	40

Figure 10

33	300	291	290	30	31
36	299	28	27	296	289
306	20	303	304	23	19
5	302	21	22	305	320
301	29	298	297	26	24
294	25	34	35	295	292

Figure 11

321	12	3	2	318	319	285	48	39	38	282	283	249	84	75	74	246	247
324	11	316	315	8	1	288	47	280	279	44	37	252	83	244	243	80	73
18	308	15	16	311	307	54	272	51	52	275	271	90	236	87	88	239	235
293	14	309	310	17	32	257	50	273	274	53	68	221	86	237	238	89	104
13	317	10	9	314	312	49	281	46	45	278	276	85	245	82	81	242	240
6	313	322	323	7	4	42	277	286	287	43	40	78	241	250	251	79	76
213	120	111	110	210	211	177	156	147	146	174	175	141	192	183	182	138	139
216	119	208	207	116	109	180	155	172	171	152	145	144	191	136	135	188	181
126	200	123	124	203	199	162	164	159	160	167	163	198	128	195	196	131	127
185	122	201	202	125	140	149	158	165	166	161	176	113	194	129	130	197	212
121	209	118	117	206	204	157	173	154	153	170	168	193	137	190	189	134	132
114	205	214	215	115	112	150	169	178	179	151	148	186	133	142	143	187	184
105	228	219	218	102	103	69	264	255	254	66	67	33	300	291	290	30	31
108	227	100	99	224	217	72	263	64	63	260	253	36	299	28	27	296	289
234	92	231	232	95	91	270	56	267	268	59	55	306	20	303	304	23	19
77	230	93	94	233	248	41	266	57	58	269	284	5	302	21	22	305	320
229	101	226	225	98	96	265	65	262	261	62	60	301	29	298	297	26	24
222	97	106	107	223	220	258	61	70	71	259	256	294	25	34	35	295	292

Figure 12

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