Fibonacci Sequence, Golden Ratio and Generalized Additive Sequences

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Abstract. In this article, we recall the Fibonacci sequence, the golden ratio, their properties and applications, and some early generalizations of the golden ratio. The Fibonacci sequence is a 2-sequence because it is generated by the sum of two previous terms, $f_{n+2} = f_{n+1} + f_n$. As a natural extension of this, we introduce several typical p-sequences where every term is the sum of p *previous* terms given p initial values called *seeds*. In particular, we introduce the notion of 1-sequence. We then discuss generating functions and limiting ratio values of p-sequences. Furthermore, inspired by Fibonacci's rabbit pair problem, we consider a general problem whose particular cases lead to nontrivial additive sequences.

1 Introduction

We are familiar with the celebrated Fibonacci sequence [1–3]: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... In this sequence, each number is the sum of the previous two, starting from 1 and 1. The ratio of consecutive Fibonacci numbers approaches the unique number 1.618. That is, $f_n = f_{n-1} + f_{n-2}$ with $f_1 = 1$ and $f_2 = 1$ [1], and $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = 1.618$ (upto three decimal places). This sequence arose from the Fibonacci's famous rabbit pair problem. A version of this problem is: *A man puts a male-female pair of adult rabbits in a field. Rabbits take a month to mature before mating. One month after mating, females give birth to one male-female pair and then mate again. It is assumed that no rabbits die but continue breeding. How many rabbit pairs are there after one year?* See Table 1 for the answer.

Another problem, a modified version of Pingala's (c. 200 BC) [4–9], which yields the Fibonacci sequence is: *Suppose* $\{s_k \equiv k\}_{k=1}^{p}$ *is the set of syllable elements, and a*

[1] One can also start with $f_0 = 0$ and $f_1 = 1$. In that case the Fibonacci sequence will be $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$.

n (month)	adult pair (a_n)	baby pair (b_n)	total pair (t_n)
0	1	0	1
1	1	0	1
2	1+0	1	2
3	1+1	1	3
4	2+1	2	5
5	3+2	3	8
6	5+3	5	13
7	8+5	8	21
8	13+8	13	34
9	21+13	21	55
10	34+21	34	89
11	55+34	55	144
12	89+55	89	233
13	144+89	144	377
14	233+144	233	610
15	377+233	377	987
16	610+377	610	1597
17	987+610	987	2584
18	1597+987	1597	4181

Table 1: The Fibonacci sequence resulting from Fibonacci's famous rabbit pair problem. In the beginning ($n = 0$), there is one male-female adult pair. At the start of the first month, there is one adult pair (they mate) and zero juvenile pair so there is only 1 rabbit pair. At the start of the second month they produce a new pair, so there are 2 pairs in the field. At the start of the third month, the original pair produce a new pair, but the second pair only mate without breeding, so there are 3 pairs in all. And so on. Note that $a_{n\geq 2} = a_{n-1} + b_{n-1}$, $b_{n\geq 2} = a_{n-1}$, and $t_{n\geq 2} = a_n + a_{n-1} = t_{n-1} + t_{n-2}$.

n	possible arrangements	total
1	$s_1^1 s_2^0(1)$ [1]	**1**
2	$s_1^2 s_2^0(1)$ [11], $s_1^0 s_2^1(1)$ [2]	**2**
3	$s_1^3 s_2^0(1)$ [111], $s_1^1 s_2^1(2)$ [12, 21]	3
4	$s_1^4 s_2^0(1)$ [1111], $s_1^2 s_2^1(3)$ [112, 121, 211], $s_1^0 s_2^2(1)$ [22]	5
5	$s_1^5 s_2^0(1)$ [11111], $s_1^3 s_2^1(4)$ [1112, 1121, 1211, 2111], $s_1^1 s_2^2(3)$ [122, 212, 221]	8

Table 2: Possible arrangements of occupation of n-syllable room with 1-syllable and 2-syllable elements. Each occupation is of the form $s_1^{n_1} s_2^{n_2}(m)$, where $s_k^{n_k}$ means that s_k occurs n_k times and number m in the parenthesis denotes the possible arrangements or *multiplicity* of $s_1^{n_1} s_2^{n_2}$. The numbers in the last column build up a sequence. We call this *syllable* 2-sequence. This can be straightforward generalized for any number of syllable elements: $\{s_k \equiv k\}_{k=1}^{p}$. Note that $n = \sum_{k=1}^{p} k\, n_k$ and multiplicity $m = \frac{(n_1+n_2+\cdots+n_p)!}{n_1!\; n_2! \cdots n_p!}$. And the last column (total number of ways in which n-syllable room can be occupied by these syllable elements): numbers in the first p rows will be $2^0,\ 2^1,\ \cdots, 2^{p-1}$, and number in the n^{th} row $(n > p)$ will be the sum of p-previous terms. Numbers $2^0,\ 2^1,\ \cdots, 2^{p-1}$ serve as the *seeds* for the syllable p-sequence.

room of n syllables is available. In how many ways this n-syllable room can be occupied by these syllable elements? See Table 2 for the answer. Indeed, there are many ways to obtain the Fibonacci sequence.

1.1 Historical background

It is acknowledged that the notions of binomial coefficients via the *Mount Meru* and the Fibonacci sequence were well known to Indian mathematicians–Pingala (c. 200 BC), Varahamihira (505-587), Kedara (7^{th} century), Virahanka (7^{th} century), Halayudha (10^{th} century), Gopala (c. 1135) and Hemachandra (1089-1172) [4–9], and Persian mathematicians–Al-Karaji (953-1029) and Omar Hayyam (1048-1131) (see [10]) before Fibonacci who had introduced it to the Western world in his book *Liber Abaci* (1202) [2]. The shallow diagonals of the Mount Meru sum to the Fibonacci numbers (see Fig. 1), and the Mount Meru is today popularly called the Pascal's triangle [10–13] after Blaise Pascal (1623-1662) who introduced this triangle in his treatise *Traité du triangle arithmétique* (1653) [3]. The notion of Pascal's triangle and its properties were also known to the Chinese–Jia Xian (1010-1070) and Yang Hui (1238-1298), the Germans–Petrus Apianus (1495-1552) and Michael Stifel (1487-1567), and the Italian mathematicians

[2] We, therefore, advocate that the Fibonacci sequence be called the Pingala sequence or the Pingala-Fibonacci sequence.

[3] Keeping up with the tradition of giving due credit to the original propounder, the Pascal's triangle should be called the Pingala's triangle or the Pingala-Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

Figure 1: Pingala's Mount Meru (Pascal's triangle).

Niccolo Fontana Tartaglia (1499-1557) and Gerolamo Cardano (1501-1576).

1.2 Other additive sequences

Lucas sequence, like Fibonacci sequence, is given by $l_n = l_{n-1} + l_{n-2}$ with $l_1 = 2$ and $l_2 = 1$ [2,3]. In general, starting with $g_1 = a$ and $g_2 = b$, one can construct the following sequence: $a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ \cdots,\ \mathbf{g_{n\geq 3} = f_{n-2}a + f_{n-1}b},\ \cdots$. This general sequence is customarily called the Gopala-Hemchandra sequence [5, 6]. Furthermore, Narayana Pandita in his book *Ganita Kaumudi* (1356) [14] studies additive sequences where each term is the sum of the p-previous terms. He states the problem as: *A cow gives birth to a calf every year. The calves become young and they begin giving birth to calves when they are three years old. Tell me, O learned man, the number of progeny produced during twenty years by one cow.*

1.3 Golden ratio

The golden ratio [15–23] [4] as defined by Euclid in his book *The Elements* [22] is: *A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less.*

[4]The golden ratio is also called *golden proportion*, *golden number*, *golden section*, *golden mean*, *divine proportion*, and *extreme and mean ratio*.

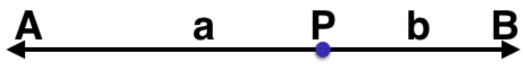

Figure 2: Division of a line into 2 segments.

That is, the *golden ratio* arises when we consider division of a line segment AB with a point P such that $\frac{AP}{BP} = \frac{AB}{AP}$, where $AP > BP$ (see Fig. 2). Given $AP = a$ and $BP = b$ are two positive numbers, the above problem translates as

$$\frac{a}{b} = \frac{a+b}{a}. \tag{1}$$

Taking $\frac{a}{b} = x$, the above equation can be rewritten as $x = 1 + \frac{1}{x}$. This reduces to the characteristic equation [5]

$$X(x) = x^2 - x - 1 = 0, \tag{2}$$

whose positive solution is

$$\Phi = \frac{\sqrt{5}+1}{2} = 1.618. \tag{3}$$

1.4 Relation between Fibonacci sequence and golden ratio

The Fibonacci sequence is closely related to the Golden Ratio in the sense that the limiting ratio value of the Fibonacci sequence, i.e., the ratio of successive numbers of the Fibonacci sequence tends to the golden ratio,

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \Phi. \tag{4}$$

1.5 Properties of Fibonacci numbers and golden ratio

The golden ratio has a number of interesting properties. They are listed below:

1. Some relations between Fibonacci numbers $f_0 = 0$, $f_1 = 1$, $f_{n\geq 2} = f_{n-1} + f_{n-2}$, and Lucas numbers $l_0 = 2$, $l_1 = 1$, $l_{n\geq 2} = l_{n-1} + l_{n-2}$.

   (a) $l_n = f_{n+1} + f_{n-1} = 2f_{n+1} - f_n$.

   (b) $f_n + f_{n+2} = l_{n+1}$.

   (c) $l_n + l_{n+2} = 5f_{n+1}$.

[5] The *characteristic equation* is the *minimal polynomial* from which all the algebraic properties of an algebraic number (here Φ) can be drawn.

2. $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ (Cassini's identity).

3. $\Phi^2 = \Phi + 1$.

4. $\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$.

5. $\Phi = \frac{1+\sqrt{5}}{2}$ and $\phi = -\frac{1-\sqrt{5}}{2}$.

6. $\phi = \frac{1}{\Phi} = \Phi - 1$.

7. $\Phi = 1 + \frac{1}{\Phi}$.

8. The golden ratio, continued fractions and its convergents.

   (a) The continued fraction [6] of the golden ratio:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}} \equiv [1; \overline{1}]. \tag{5}$$

   (b) The convergents [7] of the golden ratio:

$$[\Phi]_n = [1; \overline{1}]_n = \frac{f_{n+1}}{f_n}. \tag{6}$$

   (c) The continued fraction of powers of the golden ratio:

$$[\Phi^n] = \begin{cases} [l_n; \overline{l_n}] & (n \text{ odd}), \\ [l_n - 1; \overline{1, l_n - 2}] & (n \text{ even}). \end{cases} \tag{7}$$

   (d) The convergents of powers of the golden ratio:

$$\frac{f_{a(n+1)}}{f_{an}} = \begin{cases} [\Phi^a]_n & (a \text{ odd}), \\ [\Phi^a]_{2n} & (a \text{ even}). \end{cases} \tag{8}$$

[6] A *continued fraction* is a form of representing a number by nested fractions, all of whose numerators are 1. The continued fraction of a rational number x is finite and is represented as $x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} \equiv [a_0; a_1, a_2, \cdots, a_n]$, where $a_1, a_2, \cdots, a_n$ are positive integers and a_0 is any integer. For example, $\frac{5}{3} = 1 + \frac{1}{1+\frac{1}{2}} \equiv [1; 1, 2]$ and $\frac{10}{7} = 1 + \frac{1}{2+\frac{1}{3}} \equiv [1; 2, 3]$. Note that the first term is followed by a semicolon, while other terms are followed by commas. If x is irrational, then $n \to \infty$.

[7] A *convergent* is the truncation of a continued fraction. For example, the second convergent of $[1; 2, 3]$ is $[1; 2]$ and the m^{th} convergent of $[a_0; a_1, a_2, \cdots, a_n]$ is $[a_0; a_1, a_2, \cdots, a_{m-1}]$. That is, $[a_0; a_1, a_2, \cdots, a_n]_m := [a_0; a_1, a_2, \cdots, a_{m-1}]$.

9. $\Phi = \lim_{n\to\infty} \frac{f_{n+1}}{f_n}$.

10. $\Phi^n = \Phi^{n-1} + \Phi^{n-2} = \Phi f_n + f_{n-1}$.

11. $f_n = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}}$ (Binet's formula).

12. $\Phi^n = \frac{l_n + f_n\sqrt{5}}{2}$.

13. Φ as an infinite series: $\Phi = \frac{13}{8} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+1)!}{n!(n+2)!4^{2n+3}}$.

14. Φ as trigonometric functions: $\Phi = 1 + 2\sin 18^\circ = 2\sin 54^\circ = \frac{1}{2}\csc 18^\circ$.

1.6 Applications of golden ratio

The golden ratio is certainly a famous number, and also a *divine* one as considered by some [15, 21]. It allegedly appears everywhere.

- *In geometry, maths and science* [21, 24–47]. The golden ratio appears, by construction, in geometrical objects such as the golden polygons (triangle, rectangle, pentagon, etc.) and golden spirals. It also appears in science, physical theories and problems.

- *In nature* [21, 48, 49]. It exhibits in natural flora and fauna in the form of golden shapes such as spirals and pentagon. A tantalizing connection appears between the Fibonacci numbers (and hence the golden golden) and phyllotaxis (i.e., the arrangement of leaves on a stem, scales on a pine cone, florets on a sunflower, infloresences on a cauliflower, etc.). The plant tendrils get twisted by spirals, the helical motions are seen in the growth of roots and sprouts, and the sunflower seeds are arranged along the spirals. The spirals in sunflowers appear to rotate both clockwise (21 spirals) and counterclockwise (34 spirals). Remarkably, the numbers 21 and 34 are consecutive Fibonacci numbers. The golden ratio often shows in horns of rams, goats and antelopes. The pentagonal symmetry in the form of star fish, five-petal flowers, certain cactus plants, etc. are widespread in nature.

- *In human body* [21, 50–53]. It is believed that human body and its parts appear in the golden ratio. A ratio of feet-to-head height to feet-to-navel height (and also, ratio of feet-to-navel height to navel-to-head height) is called the navel ratio. A perfect human body is divided by the navel into the golden section. The human hand and face are also based on the golden ratio.

- *In architecture* [21, 54, 55]. It appears that the golden ratio has been used significantly in architecture: in Parthenon, in Great Pyramids of Egypt, in Indian meditation symbol *Sri Yantra*, in Taj Mahal and several ancient Indian temples such as Tanjavur Brihadeeshwara temple.

- *In art, painting and music* [8, 9, 21, 56–62]. The golden ratio is also prevalent in art, music and painting. For example, in the works of Da Vinci (*The Annunciation, Madonna with Child and Saints, The Mona Lisa, St. Jerome, An Old Man, and The Vitruvian Man*), in *The Holy Family* by Michelangelo, *The Crucifixion* by Raphael and *The Sacrament of the Last Supper* by Salvador Dali. It is illustrated in prolific number in portraits, paintings of Christian God and sculptures during the renaissance epoch. In music, it is present in works of Beethoven, Mozart, Wilson's *Meru 1* etc.

As asserted by many, it exists in any place where life and beauty are present.

1.7 Are Fibonacci sequence and golden ratio sacred?

Despite all-round great appearance of the golden ratio, many hold skeptical views on this [63–68]. The reasons are multifold.

- Application of the golden ratio to aesthetics is, by its nature, subjective and controversial. In order to find the golden ratio in our everyday life, we consider the following either separately or in combination [68]: (i) arbitrary placement of points, lines, rectangles and spirals, (ii) arbitrary thickness of points and lines used as basis for measurements, and (iii) measurements of monuments eroded by time and of objects in photographs distorted by perspective.

- Not all spirals in the nature are the golden ones. The nautilus shell, a prime pedagogical example, corresponds to a spiral with the value $\Phi' = 1.33$ (< 1.618).

2 Early generalizations of golden ratio

There have been sincere attempts to extend or generalize the notion of golden ratio from various perspectives such as generalizations of Euclid's problem, limits of recurrence relations, and the characteristic equations [8, 9, 17, 18, 20, 69–82]. Fowler [69] revisited the Euclid's problem *the line divided in extreme and mean ratio* and explored the propositions not investigated and proved in Euclid's *Elements*. Here, we review briefly some early generalizations.

n	0	1	2	3	4	5	6	7	8	9
$f_n(p=0)$	1	2	4	8	16	32	64	128	256	512
$f_n(p=1)$	1	1	2	3	5	8	13	21	34	55
$f_n(p=2)$	1	1	1	2	3	4	6	9	13	19
$f_n(p=3)$	1	1	1	1	2	3	4	5	7	10

Table 3: The Fibonacci p-numbers $f_n(p)$ for different p values. $f_n(p=0)=2^n$ are the binary numbers, $f_n(p=1)$ are the Fibonacci numbers, and so on.

2.1 Golden p-proportions of Alexey Stakhov

Recall the Euclid's division problem of a line segment AB into two segments $AP(=a)$ and $BP(=b)$ where $AP>BP$ (see Fig. 2). Alexey Stakhov, a Russian mathematician, considered the following generalization in his book [20]

$$\frac{AP}{BP}=(\frac{AB}{AP})^p \Rightarrow \frac{a}{b}=(\frac{a+b}{a})^p, \tag{9}$$

where p is a non-negative integer. From Eq. (9), with $\frac{AB}{AP}=x$, we obtain the following algebraic equation

$$x^{p+1}=x^p+1, \tag{10}$$

whose the only positive solution χ_p is called the *golden p-proportion*. The *Fibonacci p-numbers* are obtained with the recurrence relation(s),

$$\begin{aligned} f_n(p) &= t_{n-1}(p)+t_{n-(p+1)}(p), \quad (n\geq p+1) \\ f_{n+1}(p) &= t_n(p)+t_{n-p}(p), \quad (n\geq p) \end{aligned} \tag{11}$$

where $f_k(p)=1,\ k=0,1,\cdots,p$. These numbers are related to the concept of *"deformed" Pascal's p-triangles* via the binomial coefficients as

$$f_{n+1}(p)=\sum_{k=0}^{\infty}\binom{n-kp}{k}. \tag{12}$$

Note the following observations:

1. $f_n(p=0)=2^n$ are the binary numbers, $f_n(p=1)$ are the Fibonacci numbers, and so on (see Table 3).
2. $\chi_0=2$, $\chi_1=\frac{1+\sqrt{5}}{2}=\Phi$, $\chi_\infty=1$, and $1\leq\chi_p\leq 2$.
3. $\chi_p^n=\chi_p^{n-1}+\chi_p^{n-(p+1)}=\chi_p\times\chi_p^{n-1}$.
4. Binomial coefficients, Fibonacci p-numbers, and golden p-proportions. $f_{n+1}(p)=\sum_{k=0}^{\infty}\binom{n-kp}{k}$ and $\chi_p=\lim_{n\to\infty}\frac{f_{n+1}(p)}{f_n(p)}$.

The notions of the golden p-proportions and Fibonacci p-numbers generalized the original mathematical concepts, and led to several interesting applications including in the different fields of mathematics and computer science [20, 28, 80].

2.2 Metallic means family of Vera Spinadel

Vera Spinadel, an Argentinean mathematician, considered an interesting generalization of the Fibonacci recurrence relation, $t_{n+1} = t_n + t_{n-1}$, in the following form

$$\begin{aligned} t_{n+1} &= pt_n + qt_{n-1}, \\ \Rightarrow \frac{t_{n+1}}{t_n} &= p + q\frac{t_{n-1}}{t_n}, \end{aligned} \tag{13}$$

where p and q are non-negative integers. Assuming that $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = x$ exists, we have

$$x = p + \frac{q}{x} \Rightarrow x^2 = px + q. \tag{14}$$

The algebraic equation (14) has a solution

$$\chi_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}. \tag{15}$$

Positive solutions in Eq. (15) form a *metallic means family* (MMF), and Vera Spinadel gave a number of applications of the metallic means in her works [72–76]. Note the following observations:

1. $x^2 = px + q$ implies $x = \sqrt{q + p\sqrt{q + p\sqrt{q + p\sqrt{q + \cdots}}}}$.
2. $x = p + \frac{q}{x}$ implies $x = p + \cfrac{q}{p + \cfrac{q}{p + \cfrac{q}{p + \ddots}}}$.
3. $\chi_{p,q} = \frac{p+\sqrt{p^2+4q}}{2} = \lim_{n\to\infty} \frac{t_{n+1}}{t_n}$.
4. $\chi_{1,1} = \frac{1+\sqrt{5}}{2} = [1; \overline{1}] = \Phi$.
5. $\chi_{p,1} = \frac{p+\sqrt{p^2+4}}{2} = [p; \overline{p}]$.
6. $\chi_{4,1} = 2 + \sqrt{5} = [4; \overline{4}] = \Phi^3$.

meru	recurrence relation	characteristic equation	convergence limit
Meru 1	$A_n = A_{n-1} + A_{n-2}$	$x^2 = x + 1$	1.61803
Meru 2	$B_n = B_{n-1} + B_{n-3}$	$x^3 = x^2 + 1$	1.46557
Meru 3	$C_n = C_{n-2} + C_{n-3}$	$x^3 = x + 1$	1.32472
Meru 4	$D_n = D_{n-1} + D_{n-4}$	$x^4 = x^3 + 1$	1.38028
Meru 5	$E_n = E_{n-3} + E_{n-4}$	$x^4 = x + 1$	1.22074
Meru 6	$F_n = F_{n-1} + F_{n-5}$	$x^5 = x^4 + 1$	1.32472
Meru 7	$G_n = G_{n-2} + G_{n-5}$	$x^5 = x^3 + 1$	1.23651
Meru 8	$H_n = H_{n-3} + H_{n-5}$	$x^5 = x^2 + 1$	1.19386
Meru 9	$I_n = I_{n-4} + I_{n-5}$	$x^5 = x + 1$	1.16730

Table 4: Wilson's *Meru 1* through *Meru 9*, each with its recurrence relation, the characteristic equation, and the convergence limit.

2.3 Mount merus of Erwin Wilson

Recall Pingala's Mount Meru (Pascal's Triangle) in Fig. 1. It was illustrated in 1968 by Thomas Green [83] that the sum of the simplest diagonals of Mount Meru yields the Fibonacci sequence, and that the sum of other diagonals similarly generate other recurrence relations, each with its own limit. Ervin Wilson, a Mexican/American music theorist, investigated these other diagonals and their recurrence relations in music [8,9]. He considered recurrence relations that he called *Meru 1* through *Meru 9*. See Table 4.

2.4 Lower and upper golden ratios of Vedran Krcadinac

We saw earlier that Stakhov considered generalization of the form $\frac{a}{b} = \left(\frac{a+b}{a}\right)^p$ leading to the algebraic equation $x^{p+1} - x^p - 1 = 0$ when $\frac{a+b}{a} = x$. A similar generalization, proposed by Krcadinac [79], for non-negative integer p, is

$$\left(\frac{a}{b}\right)^p = \frac{a+b}{a}. \tag{16}$$

This relation, in general, leads to two algebraic equations:

$$X_1(x) = x^{p+1} - x - 1 = 0, \quad (\text{when } \frac{a}{b} = x) \tag{17}$$

$$X_2(x) = x(x-1)^p - 1 = 0, \quad (\text{when } \frac{a+b}{a} = x) \tag{18}$$

Let φ_p be the positive root of the polynomial $X_1(x)$ and ϕ_p be that of the polynomial $X_2(x)$. Then, φ_p and ϕ_p are respectively called the p^{th} *lower* and *upper* goden ratio. Note the following observations:

1. $\varphi_0 =$ *undefined* and $\phi_0 = 1$.

2. $\lim_{p\to\infty} \varphi_p = 1$ and $\lim_{p\to\infty} \phi_p = 2$.

3. Evidently, $(\varphi_p)^p = \phi_p$.

4. Recurrence relation for $X_1(x)$: $f_n(p) = f_{n-p}(p) + f_{n-(p+1)}(p)$.

5. Recurrence relation for $X_2(x)$: $F_n(p) = \sum_{k=1}^{p} \binom{p}{k} (-1)^{k+1} F_{n-k}(p) + F_{n-(p+1)}(p)$.

6. $\lim_{n\to\infty} \frac{f_{n+1}(p)}{f_n(p)} = \varphi_p$ and $\lim_{n\to\infty} \frac{F_{n+1}(p)}{F_n(p)} = \phi_p$.

7. $\lim_{n\to\infty} \frac{f_{n+p}(p)}{f_n(p)} = (\varphi_p)^p = \phi_p$.

3 p-sequences

We call the Fibonacci sequence a 2-sequence because it is generated by the sum of two previous terms. In a similar spirit, we introduce the p-sequence [8].

To construct a p-sequence, we begin with p seeds $(s_0, s_1, \cdots, s_{p-1})$ such that $t_0 = s_0,\ t_1 = s_1, \cdots, t_{p-1} = s_{p-1}$, and the n^{th} term is the sum of its p *previous* terms [9]:

$$t_n(p) := t_{n-1}(p) + t_{n-2}(p) + \cdots + t_{n-p}(p) = \sum_{k=n-p}^{n-1} t_k(p). \tag{19}$$

By definition of $t_n(p)$, we have

$$t_{n+1}(p) > t_n(p), \tag{20}$$
$$t_{n+1}(p) = 2t_n(p) - t_{n-p}(p) < 2t_n(p). \tag{21}$$

Depending on the values of seeds, one can construct an infinite number of p-sequences. A few typical p-sequences are:

(i) *General* p-sequence whose seeds are arbitrary.

$$S_G(p) \equiv \{(s_0, s_1, \cdots, s_{p-1}),\ t_n(p)\}. \tag{22}$$

(ii) *k* p-sequence whose k^{th} seed is unity and other seeds are zero.

$$S_k(p) \equiv \{(s_i = \delta_{ik},\ 0 \le i \le p-1),\ t_n(p)\}. \tag{23}$$

[8] p in the p-sequence is for *Pingala*, *Phi(Φ)*, and *previous*.

[9] This can be equivalently rewritten as $t_{n+p}(p) := t_{n+p-1}(p) + t_{n+p-2}(p) + \cdots + t_n(p)$.

For example, $S_0(p) \equiv \{(1, 0, \cdots, 0),\ t_n(p)\}$, $S_1(p) \equiv \{(0, 1, 0, \cdots, 0),\ t_n(p)\}$, and $S_{p-1}(p) \equiv \{(0, 0, \cdots, 1),\ t_n(p)\}$. Interestingly, we can rewrite $t_n[S_G(p)]$ in terms of seeds using these *k* p-sequences,

$$t_n[S_G(p)] = \sum_{k=0}^{p-1} t_n[S_k(p)] s_k \ \ (n \geq 0). \tag{24}$$

For example, $t_1[S_G(p)] = 0.s_0 + 1.s_1 + \cdots + 0.s_{p-1} = s_1$.

(iii) *Coefficient* p-sequence whose all seeds are unity.

$$S_C(p) \equiv \{(s_k = 1,\ 0 \leq k \leq p-1),\ t_n(p)\}. \tag{25}$$

There is an important relation between the terms of *coefficient* p-sequence and those of *k* p-sequences: $S_C(p) \equiv \sum_{k=1}^{p} S_k(p)$. Put differently,

$$t_n[S_C(p)] = \sum_{k=0}^{p-1} t_n[S_k(p)]. \tag{26}$$

(iv) *Exponent* p-sequence whose seeds are $(0, 1, \cdots, p-1)$.

$$S_X(p) \equiv \{(s_k = k,\ 0 \leq k \leq p-1),\ t_n(p)\}. \tag{27}$$

(v) *Syllable* p-sequence whose seeds are $(1, 2, \cdots, 2^{p-1})$.

$$S_S(p) \equiv \{(s_k = 2^k,\ 0 \leq k \leq p-1),\ t_n(p)\}. \tag{28}$$

We will learn the significance of these particular sequences in the forthcoming articles. For illustrations of and getting familiarized with these sequences, see Tables 5, 6, 7 and 8. Henceforth, $S_G(p) \equiv S(Gp)$, $S_k(p) \equiv S(kp)$, $S_C(p) \equiv S(Cp)$, $S_X(p) \equiv S(Xp)$, and $S_S(p) \equiv S(Sp)$ will be used interchangeably. We will denote the n^{th}-term of an arbitrary p-sequence by $t_n(p)$, and that of a particular p-sequence, viz. *exponent* sequence by $t_n(Xp)$, *syllable* sequence by $t_n(Sp)$, *k* sequence by $t_n(kp)$, and so on. Furthermore, in case of no ambiguity, we will not mention p explicitly in the sequence names and their terms.

3.1 1-sequence

What is 1-sequence? We construct a 1-sequence by choosing a seed $s_0 \geq 0$ and a constant $a \geq 0$ such that $t_0 = s_0$, and for $n \geq 1$

$$\begin{aligned} t_1 &= t_0 + a = s_0 + a, \\ t_2 &= t_1 + a = s_0 + 2a, \\ t_n &= t_{n-1} + a = s_0 + na. \end{aligned} \tag{29}$$

Thus, an additive 1-sequence is essentially an arithmetic progression. With $s_0 = 0$ and $a = 1$, 1-sequence is the set of whole numbers

$$S(1) = \{0, 1, 2, \cdots, 99, 100, \cdots\}. \tag{30}$$

When $a = 0$, the 1-sequence is a *constant* sequence: $\{s_0, s_0, s_0, \cdots\}$.

4 Generating functions of p-sequences

The generating function for p-sequences can be given by the power series

$$f_p(x) = \sum_{n=0}^{\infty} t_n(p) x^n, \tag{31}$$

where $t_n(p)$ is the n^{th} term of a given p-sequence. If we assume that the power series converges, we can show that $f_p(x)$ is given by

$$\big(1 - \sum_{k=1}^{p} x^k\big) f_p(x) = \sum_{k=0}^{p-1} \big[t_k(p) - \sum_{j=0}^{k-1} t_j(p)\big] x^k. \tag{32}$$

For example, for the *exponent* p-sequence $S(Xp)$, the generating functions are

$$\begin{aligned}
f_{X2}(x) &= \frac{x}{1 - x - x^2}, \\
f_{X3}(x) &= \frac{x + x^2}{1 - x - x^2 - x^3}, \\
f_{X4}(x) &= \frac{x + x^2}{1 - x - x^2 - x^3 - x^4}, \\
f_{X5}(x) &= \frac{x + x^2 - 2x^4}{1 - x - x^2 - x^3 - x^4 - x^5}.
\end{aligned}$$

5 Limiting ratio value of p-sequences

For an arbitrary p-sequence whose subsequent terms are the sum of p-previous terms $[t_n(p) := \sum_{k=n-p}^{n-1} t_k]$, we see from Tables 5, 6, 7 and 8 that the limiting ratio value of every p-sequence approaches a constant, say Φ_p. That is,

$$\Phi_p = \lim_{n \to \infty} \frac{t_{n+1}(p)}{t_n(p)}. \tag{33}$$

Because $t_{n+1}(p) > t_n(p)$ and $t_{n+1}(p) < 2t_n(p)$, hence

$$1 < \Phi_p < 2. \tag{34}$$

Using Eq. (33), for integers u and v, it is easy to see that

$$\lim_{n\to\infty} \frac{t_{n+u}(p)}{t_n(p)} = \Phi_p^u, \quad \lim_{n\to\infty} \frac{t_{n+u}(p)}{t_{n+v}(p)} = \Phi_p^{u-v}. \tag{35}$$

5.1 Φ_p in the limit $p \to \infty$

Consider the *syllable* p-sequence $S_S(p \to \infty) = \{1,\ 2,\ 4,\ 8,\ 16, \cdots\}$, and the *extended syllable* p-sequence $S(p \to \infty) \equiv \{(0,\ 1),\ S_S(p \to \infty)\} = \{0,\ 1,\ 1,\ 2,\ 4,\ 8,\ 16, \cdots\}$, where each term is the sum of all the previous terms except the first two. For both these sequences, we have

$$\Phi_{p\to\infty} = \lim_{n\to\infty} \frac{t_{n+1}(p)}{t_n(p)} = 2. \tag{36}$$

Moreover, from Table 9, $\Phi_p = 2$ for $p \geq 18$.

5.2 Propositions

In Tables 5, 6, 7 and 8, we constructed p-sequences for $p = 2,\ 3,\ 4,\ 5$, and found their limiting ratio values. Similarly, one can construct tables of higher p-sequences and find their limiting ratio values. See Table 9 for the values of limiting ratios. In this regard, we propound the following two propositions.

(P1) The limiting ratio value of any p-sequence $S(p)$ is Φ_p. It is independent of the initial conditions (i.e., the seeds).

(P2) $\Phi_{p\geq 18} = 2$.

6 More additive sequences

In this section, motivated by Pingala's syllable problem, Fibonacci's rabbit pair problem, and Narayan Pandit's cow's progeny problem, we investigate a general problem: *A creature gives birth to α female young ones in one unit of time. Baby creature grows and gives birth when β units of time old. The creature ceases to give birth after γ terms, and dies when δ units of time old.. What is the total number of progeny at the end of n units of time? Initially, there is a single adult creature.* Following, we consider a few illustrations sans δ. See Tables 10, 11, 12 and 13. We invite the readers to investigate the problem taking into account δ also.

n	$S_1(2)$	$S_0(2)$	$S_C(2)$	$S_S(2)$	$S_G(2)$
0	0	1	1	1	2
1	1	0	1	2	21
2	1	1	2	3	23
3	2	1	3	5	44
4	3	2	5	8	67
5	5	3	8	13	111
6	8	5	13	21	178
7	13	8	21	34	289
8	21	13	34	55	467
9	34	21	55	89	756
10	55	34	89	144	1223
11	89	55	144	233	1979
12	144	89	233	377	3202
13	233	144	377	610	5181
14	377	233	610	987	8383
15	610	377	987	1597	13564
16	987	610	1597	2584	21947
17	1597	987	2584	4181	35511
18	2584	1597	4181	6765	57458
19	4181	2584	6765	10946	92969
20	6765	4181	10946	17711	150427
21	10946	6765	17711	28657	243396
22	17711	10946	28657	46368	393823
23	28657	17711	46368	75025	637219
24	46368	28657	75025	121393	1031042
25	75025	46368	121393	196418	1668261

Table 5: 2-sequences. (i) $S_C \equiv S_1 + S_0$. (ii) $S_X = S_1$. (iii) $S_1 \sim S_0 \sim S_C \sim S_S$. (iv) S_G is a general 2-sequence with seeds $s_1 = 2,\ s_2 = 21$. (v) For each of these 2-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.61803$.

n	$S_2(3)$	$S_1(3)$	$S_0(3)$	$S_C(3)$	$S_X(3)$	$S_S(3)$
0	0	0	0	1	0	1
1	0	1	0	1	1	2
2	1	0	1	1	2	4
3	1	1	1	3	3	7
4	2	2	2	5	6	13
5	4	3	4	9	11	24
6	7	6	7	17	20	44
7	13	11	13	31	37	81
8	24	20	24	57	68	149
9	44	37	44	105	125	274
10	81	68	81	193	230	504
11	149	125	149	355	423	927
12	274	230	274	653	778	1705
13	504	423	504	1201	1431	3136
14	927	778	927	2209	2632	5768
15	1705	1431	1705	4063	4841	10609
16	3136	2632	3136	7473	8904	19513
17	5768	4841	5768	13745	16377	35890
18	10609	8904	10609	25281	30122	66012
19	19513	16377	19513	46499	55403	121415
20	35890	30122	35890	85525	101902	223317
21	66012	55403	66012	157305	187427	410744
22	121415	101902	121415	289329	344732	755476
23	223317	187427	223317	532159	634061	1389537
24	410744	344732	410744	978793	1166220	2555757
25	755476	634061	755476	1800281	2145013	4700770

Table 6: 3-sequences. (i) $S_C \equiv S_2 + S_1 + S_0$. (ii) $S_2 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 3-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.83929$.

n	$S_3(4)$	$S_2(4)$	$S_1(4)$	$S_0(4)$	$S_C(4)$	$S_X(4)$	$S_S(4)$
0	0	0	0	1	1	0	1
1	0	0	1	0	1	1	2
2	0	1	0	0	1	2	4
3	1	0	0	0	1	3	8
4	1	1	1	1	4	6	15
5	2	2	2	1	7	12	29
6	4	4	3	2	13	23	56
7	8	7	6	4	25	44	108
8	15	14	12	8	49	85	208
9	29	27	23	15	94	164	401
10	56	52	44	29	181	316	773
11	108	100	85	56	349	609	1490
12	208	193	164	108	673	1174	2872
13	401	372	316	208	1297	2263	5536
14	773	717	609	401	2500	4362	10671
15	1490	1382	1174	773	4819	8408	20569
16	2872	2664	2263	1490	9289	16207	39648
17	5536	5135	4362	2872	17905	31240	76424
18	10671	9898	8408	5536	34513	60217	147312
19	20569	19079	16207	10671	66526	116072	283953
20	39648	36776	31240	20569	128233	223736	547337
21	76424	70888	60217	39648	247177	431265	1055026
22	147312	136641	116072	76424	476449	831290	2033628
23	283953	263384	223736	147312	918385	1592363	3919944
24	547337	507689	431265	283953	1770244	3068654	7555935
25	1055026	978602	831290	547337	3412255	5623572	14564533

Table 7: 4-sequences. (i) $S_C \equiv S_3 + S_2 + S_1 + S_0$. (ii) $S_3 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 4-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.92756$.

n	$S_4(5)$	$S_3(5)$	$S_2(5)$	$S_1(5)$	$S_0(5)$	$S_C(5)$	$S_X(5)$	$S_S(5)$
0	0	0	0	0	1	1	0	1
1	0	0	0	1	0	1	1	2
2	0	0	1	0	0	1	2	4
3	0	1	0	0	0	1	3	8
4	1	0	0	0	0	1	4	16
5	1	1	1	1	1	5	10	31
6	2	2	2	2	1	9	20	61
7	4	4	4	3	2	17	39	120
8	8	8	7	6	4	33	76	236
9	16	15	14	12	8	65	149	464
10	31	30	28	24	16	129	294	912
11	61	59	55	47	31	253	578	1793
12	120	116	108	92	61	497	1136	3525
13	236	228	212	181	120	977	2233	6930
14	464	448	417	356	236	1921	4390	13624
15	912	881	820	700	464	3777	8631	26784
16	1793	1732	1612	1376	912	7425	16968	52656
17	3525	3405	3169	2705	1793	14597	33358	103519
18	6930	6694	6230	5318	3525	28697	65580	203513
19	13624	13160	12248	10455	6930	56417	128927	400096
20	26784	25872	24079	20554	13624	110913	253464	786568
21	52656	50863	47338	40408	26784	218049	498297	1546352
22	103519	99994	93064	79440	52656	428673	979626	3040048
23	203513	196583	182959	156175	103519	842749	1925894	5976577
24	400096	386472	359688	307032	203513	1656801	3786208	11749641
25	786568	754784	707128	603609	400096	3257185	7443489	23099186

Table 8: 5-sequences. (i) $S_C \equiv S_4 + S_3 + S_2 + S_1 + S_0$. (ii) $S_4 \sim S_0 \sim S_S$. (iii) For each of these 5-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.96595$.

p	2	3	4	5
Φ_p	1.61803	1.83929	1.92756	1.96595
p	6	7	8	9
Φ_p	1.98358	1.99196	1.99603	1.99803
p	10	11	12	13
Φ_p	1.99902	1.99951	1.99976	1.99988
p	14	15	16	17
Φ_p	1.99994	1.99997	1.99998	1.99999
p	18	19	20	21
Φ_p	2.0	2.0	2.0	2.0

Table 9: The limitining ratio value $\Phi_p := \lim_{n\to\infty} \frac{t_{n+1}}{t_n}$ for p-sequences, $2 \le p \le 21$. We have limited ourselves here to five decimal places (for no sacred reasons). Evidently, $\Phi_2 < \Phi_3 < \cdots < \Phi_{p\ge 18} = 2$.

n	creature	baby (b_n) at start	total (t_n) at end
0	1	0	0
1	1	1	1
2	1	1	2
3	1+1	2	3
4	2+1	3	5
5	3+2	5	8
6	5+3	8	13
7	8+5	13	21
8	13+8	21	34
9	21+13	34	55
10	34+21	55	89
11	55+34	89	144
12	89+55	144	233
13	144+89	233	377
14	233+144	377	610
15	377+233	610	987

Table 10: $\alpha = 1$, $\beta = 2$ and $\gamma =$ NA. This yields Pingala (Fibonacci) sequence. Here $t_{n\ge 1} = b_n + b_{n-1}$, $t_{n\ge 3} = b_n + b_{n-2} + b_{n-3}$ (sum of $\beta + 1$ terms), $t_{n\ge 3} = t_{n-1} + t_{n-2}$ (recurrence relation), $x^2 = x+1$ (characteristic equation) and $x = \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.618$ (limiting ratio).

n	creature	baby (b_n)	total (t_n)
0	1	0	0
1	1	1	1
2	1	1	2
3	1	1	3
4	1	1	4
5	1+1	2	5
6	2+1	3	7
7	3+1	4	10
8	4+1	5	14
9	5+2	7	19
10	7+3	10	26
11	10+4	14	36
12	14+5	19	50

Table 11: $\alpha = 1$, $\beta = 3$ and $\gamma =$ NA. This sequence corresponds to Narayana Pandit's cow's progeny problem posed in *Ganit Kaumudi*. Here $t_{n\geq 3} = b_n + b_{n-1} + b_{n-2} + b_{n-3}$ (sum of $\beta{+}1$ terms), $t_{n\geq 5} = t_{n-1}{+}t_{n-4}$ (recurrence relation), $x^4 = x^3{+}1$ (characteristic equation) and $x = \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.38$ (limiting ratio). See also Wilson's *Meru 4*.

n	creature	baby (b_n)	total (t_n)	
0	1	0	0	0
1	1	α	α	2
2	1	α	2α	4
3	1	α	3α	6
4	$(1+\alpha)-1$	α^2	$\alpha^2+2\alpha$	8
5	$(\alpha+\alpha)-\alpha$	α^2	$2\alpha^2+\alpha$	10
6	$(\alpha+\alpha^2)-\alpha$	α^2	$3\alpha^2$	12
7	$(\alpha^2+\alpha^2)-\alpha^2$	α^3	$\alpha^3+2\alpha^2$	16
8	$(\alpha^2+\alpha^2)-\alpha^2$	α^3	$2\alpha^3+\alpha^2$	20
9	$(\alpha^2+\alpha^3)-\alpha^2$	α^3	$3\alpha^3$	24
10	$(\alpha^3+\alpha^3)-\alpha^3$	α^4	$\alpha^4+2\alpha^3$	32
11	$(\alpha^3+\alpha^3)-\alpha^3$	α^4	$2\alpha^4+\alpha^3$	40
12	$(\alpha^3+\alpha^3)-\alpha^3$	α^4	$3\alpha^4$	48

Table 12: $\alpha = 2$, $\beta = 2$ and $\gamma = 3$. Here $t_{n\geq 2} = b_n + b_{n-1} + b_{n-2}$ (sum of $\beta + 1$ terms). If $n = a\gamma + b$ then $t_n = b\alpha$ (when $a = 0$) and $t_n = [b(\alpha - 1) + \gamma]\alpha^a$ (when $a \geq 1$). Also note that $t_{n\leq 2}(\alpha = 1) = b$ and $t_{n\geq 3}(\alpha = 1) = \gamma = 3$.

n	creature	baby (b_n)	total (t_n)	
0	1	0	0	0
1	1	α	α	2
2	1	α	2α	4
3	0	0	2α	4
4	0	0	2α	4
5	α	α^2	$\alpha^2 + \alpha$	6
6	$\alpha + \alpha$	$2\alpha^2$	$3\alpha^2$	12
7	$(2\alpha + 0) - \alpha$	α^2	$4\alpha^2$	16
8	$(\alpha + 0) - \alpha$	0	$4\alpha^2$	16
9	$(0 + \alpha^2) - 0$	α^3	$\alpha^3 + 3\alpha^2$	20
10	$(\alpha^2 + 2\alpha^2) - 0$	$3\alpha^3$	$4\alpha^3 + \alpha^2$	36
11	$(3\alpha^2 + \alpha^2) - \alpha^2$	$3\alpha^3$	$7\alpha^3$	56
12	$(3\alpha^2 + 0) - 2\alpha^2$	α^3	$8\alpha^3$	64
13	$(\alpha^2 + \alpha^3) - \alpha^2$	α^4	$\alpha^4 + 7\alpha^3$	72
14	$(\alpha^3 + 3\alpha^3) - 0$	$4\alpha^4$	$5\alpha^4 + 4\alpha^3$	112
15	$(4\alpha^3 + 3\alpha^3) - \alpha^3$	$6\alpha^4$	$11\alpha^4 + \alpha^3$	184

Table 13: $\alpha = 2$, $\beta = 3$ and $\gamma = 2$. Here $t_{n\geq 3} = b_n + b_{n-1} + b_{n-2} + b_{n-3}$ (sum of $\beta + 1$ terms).

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