

Is The Riemann Hypothesis True? Yes, It Is. (V3)

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Abstract. In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$.

We give the proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

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**To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To the memory of my friend Abdelkader Sellal (1950 - 2017)**

1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [1]:

Conjecture 1. *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \frac{1}{2}$.

1.1. The function ζ . We denote $s = \sigma + it$ the complex variable of \mathbb{C} . For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.1. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 2. *(The Riemann Hypothesis,[2]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

In addition to the properties cited by the theorem 1.1 above, the function $\zeta(s)$ satisfies the functional relation [2] called also the reflection functional equation for $s \in \mathbb{C} \setminus \{0, 1\}$:

$$(1.1) \quad \zeta(1-s) = 2^{1-s} \pi^{-s} \cos \frac{s\pi}{2} \Gamma(s) \zeta(s)$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (1.1), we will use the one presented by G.H. Hardy [3] namely Dirichlet's eta function [2]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [2].

We have also the theorem (see page 16, [3]):

Theorem 1.2. For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2. A Equivalent statement to the Riemann Hypothesis. Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [2]:

Equivalence 1.3. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$(1.2) \quad \eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (1.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([3], pages 20-21). We can rewrite:

$$(1.3) \quad \eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0$$

$\eta(s)$ is a complex number, it can be written as :

$$(1.4) \quad \eta(s) = \rho \cdot e^{i\alpha} \implies \rho^2 = \eta(s) \cdot \overline{\eta(s)}$$

and $\eta(s) = 0 \iff \rho = 0$.

2. Preliminaries of the proof that the zeros of the function $\eta(s)$ are on the critical line $\Re(s) = \frac{1}{2}$.

Proof. . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \iff (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(2.1)

s , is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \implies \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(2.2)

s , is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η :

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-s \text{Log} n} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it) \text{Log} n} = \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} \cdot e^{-it \text{Log} n} \\ &= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma \text{Log} n} (\cos(t \text{Log} n) - i \sin(t \text{Log} n)) \end{aligned}$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \left| \sum_{n=1}^N \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^n \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \text{Log} k)}{k^\sigma} - i \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \text{Log} k)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $\lim_{n \rightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\cos(t \operatorname{Log} k)}{k^\sigma} = 0$$

$$\lim_{n \rightarrow +\infty} \sum_{k=1}^n (-1)^{k-1} \frac{\sin(t \operatorname{Log} k)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

$$(2.3) \quad \forall \epsilon_1 > 0 \exists n_r, \forall N > n_r, |\Re(\eta(s)_N)| < \epsilon_1 \implies \Re(\eta(s)_N)^2 < \epsilon_1^2$$

$$(2.4) \quad \forall \epsilon_2 > 0 \exists n_i, \forall N > n_i, |\Im(\eta(s)_N)| < \epsilon_2 \implies \Im(\eta(s)_N)^2 < \epsilon_2^2$$

Then:

$$0 < \sum_{k=1}^N \frac{\cos^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k, k'=1; k < k'}^N \frac{(-1)^{k+k'} \cos(t \operatorname{Log} k) \cdot \cos(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^N \frac{\sin^2(t \operatorname{Log} k)}{k^{2\sigma}} + 2 \sum_{k, k'=1; k < k'}^N \frac{(-1)^{k+k'} \sin(t \operatorname{Log} k) \cdot \sin(t \operatorname{Log} k')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > \max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$(2.5) \quad 0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k, k'=1; k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

We can write the above equation as :

$$(2.6) \quad 0 < \rho_N^2 < 2\epsilon^2$$

or $\rho(s) = 0$.

3. Case $\sigma = \frac{1}{2} \implies 2\sigma = 1$.

We suppose that $\sigma = \frac{1}{2} \implies 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([2], page 24):

Theorem 3.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 1. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j :

$$0 < \sum_{k=1}^N \frac{1}{k} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \rightarrow +\infty$, the series $\sum_{k=1}^N \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$(3.1) \quad \boxed{\lim_{N \rightarrow +\infty} \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t_j \text{Log}(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty}$$

if not, we will have a contradiction with the fact that :

$$\lim_{N \rightarrow +\infty} \sum_{k=1}^N (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \iff \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4. Case $0 < \Re(s) < \frac{1}{2}$.

4.1. Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$, it follows from the proposition (2.1) that $\zeta(s)$ has also no zeros with $0 < \sigma < \frac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. We deduce from the proposition (2.2) that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \frac{1}{2}$ and from the equivalent statement 1.3, we conclude that **the Riemann Hypothesis is true.**

4.2. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \implies \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \implies s$ lies inside the critical band. We write the equation (2.5):

$$0 < \sum_{k=1}^N \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^N \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $\lim_{N \rightarrow +\infty} \sum_{k=1}^N \frac{1}{k^{2\sigma}} \rightarrow +\infty$ and then, we obtain :

$$(4.1) \quad \boxed{\sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t \operatorname{Log}(k/k'))}{k^\sigma k'^\sigma} = -\infty}$$

5. Case $\frac{1}{2} < \Re(s) < 1$.

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.1, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \implies \rho(s') = 0$. By applying (2.5), we get:

$$(5.1) \quad 0 < \sum_{k=1}^N \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k < k'}^N (-1)^{k+k'} \frac{\cos(t' \operatorname{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2$$

As $0 < \sigma < \frac{1}{2} \implies 2 > 2\sigma' = 2(1 - \sigma) > 1$, then the series $\sum_{k=1}^N \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$(5.2) \quad \sum_{k,k'=1;k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \operatorname{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

Firstly, we suppose that $t = 0 \implies t' = 0$. The equation (5.2) becomes:

$$(5.3) \quad \sum_{k, k'=1; k < k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$(5.4) \quad \eta(s') = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'}} = 0$$

Let us define the sequence S_m as:

$$(5.5) \quad S_m(s') = \sum_{n=1}^m \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^m \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma')$$

From the definition of S_m , we obtain :

$$(5.6) \quad \lim_{m \rightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma')$$

We have also:

$$(5.7) \quad S_1(\sigma') = 1 > 0$$

$$(5.8) \quad S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad \text{because } 2^{\sigma'} > 1$$

$$(5.9) \quad S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

$$1. m = 2q \implies S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}, \text{ it gives:}$$

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \implies S_{m+1}(\sigma') > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$(5.10) \quad T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $\lim_{m \rightarrow +\infty} S_m(\sigma') = \eta(\sigma') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$(5.11) \quad \boxed{\text{The equation (5.3) is false for the case } t' = t = 0.}$$

Secondly, we suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$, we have:

$$(5.12) \quad \sum_{k, k'=1; k < k'}^{+\infty} (-1)^{k+k'} \frac{\cos(t' \text{Log}(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty$$

the left member of the equation (5.12) above is finite and depends of σ' and t' , but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all σ'' so that $2\sigma'' > 1$, we have $\zeta(2\sigma'')$:

$$\zeta(2\sigma'') = \zeta_1(2\sigma'') = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma''}} < +\infty$$

It depends only of σ'' , then in particular for all σ'' with $2 > 2\sigma'' > 1$, $\zeta(2\sigma'')$ depends only of σ'' , then the result giving by the equation (5.12) is false:

$$(5.13) \quad \boxed{\text{It follows that the equation (5.12) is false for the cases } t' \neq 0.}$$

From (5.11-5.13), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the second case of the paragraph (4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (1.3), it follows that **the Riemann hypothesis is verified.** ■

From the calculations above, we can verify easily the following known proposition:

Proposition 2. For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

6. Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (1.3), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 6.1. *The Riemann Hypothesis is true:*

All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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