Signal transmission in the Schwarzschild metric: An analogy with Special Relativity

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From the perspective of a distant observer, a free-falling body in the Schwarzschild metric would require an infinite time to reach the Schwarzschild radius, whereas a comoving observer would measure just a finite interval of proper time along that path. This paradoxical situation is commonly interpreted considering the perspective of the distant observer as a simple "artifact" due to the enormous delay of the light signals emitted by the free-falling body during its fall, "already" completed. This interpretation of relativistic mechanics is intrinsically inconsistent, as shown in this article. We propose an alternative elucidation based on the analogy between the asymptotic trajectory of a free-falling body approaching the horizon event of a Schwarzschild black hole and an accelerated body exponentially asymptotically tending to the speed of light in Special Relativity.

1 Introduction

The scientific consensus holds that the event horizon at $R_S = \frac{2GM}{c^2}$ in the Schwarzschild metric represents a surface that can be crossed *inwards* but not *outwards*, a "one-way valve" [2, see p. 335]. This assumption gives rise to several theoretical problems, being the paradox of information probably the most remarkable one [5, 6]. However, when the geodesics of a body freely falling into a black hole are studied in Schwarzschild coordinates, an asymptotic approach to R_S is obtained. If that result were taken seriously, as an actual fact, it would imply that a black hole's event horizon consists in a surface that cannot be crossed *neither* inwards nor outwards, and thus a genuine frontier between two disconnected areas of space, the "inner" and the "outer" space (inside and outside the event horizon, respectively). Consequently, the paradox of information would automatically vanish: as no piece of information (nor matter) could ever cross the horizon, it would never be lost behind it. On the other hand, there is a fact that seems to point out in the opposite direction to that conclusion: although in Schwarzschild coordinates the trajectory of a free-falling body into a black hole asymptotically approaches R_S without ever crossing it, a comoving observer would measure a finite time along its way to reach it. In other words, in contrast with the "Schwarzschild time", the proper time required to arrive at the event horizon corresponds to a finite value. The consistency of both points of view is the essential issue that this article pretends to clarify.

Physics textbooks -and most publications- consider that an object freely falling into a black hole actually attains the event horizon in a finite time, and state that the "distant observers" *see it* as an asymptotic process only because the light signals emitted during the approach of the body to the event horizon reach us with a large delay [2, see p. 334-335]. We regard that this interpretation of mathematical results is intrinsically inconsistent, as it can be proven through the following reasoning: (1) Geodesic equations describe how a body actually *moves* according to a determinate system of reference, not just how it is *seen* by observers in it. (2) In order to determine "apparent trajectories" (that is, how a concrete observer would actually *see* the motion of the body), geodesics of light signals from the body to the observer must be also taken into account. (3) In a time-independent metric, the time that a body or a light signal spends in moving from a point A to a point B along a certain path is the same time that it would spend in moving from B to A along the reverse path. (4) Therefore, as the speed of light cannot be superseded by any massive body, in a time-independent metric if a body has moved from A to B in a determined interval of time, a light signal emitted by the body from B will arrive at A in a *lesser* interval of time. In other words, if an object departs from A at time t_1, emits a light signal when it arrives at B at time t_2, and the light signal reaches A at time t_3, then it must be accomplished that $t_2 - t_1 > t_3 - t_2$. Thus, $t_2 - t_1 > \frac{t_3 - t_1}{2}$, that is, the time spent by the body to move from A to B must be greater than half the time interval from the departure of the body to the return of the light signal to A. (5) Consequently, if a distant observer sees that a body falling into a Schwarzschild black hole spends an infinite time to reach its event horizon, then it must actually spend an infinite time to attain it.

How to conciliate then a genuine everlasting asymptotic approach to the event horizon with the fact that a comoving observer would spend only a finite time to attain it? Our proposition is the following one: In truth, the comoving observer would need just a finite interval of proper time to reach the event horizon, but the process never completes; otherwise stated, after an infinite period of time -as experienced by an external observer- the comoving observer will have lived only a finite amount of proper time. In other to illustrate this apparently bizarre postulate, we propose an analogy with an "equivalent" situation in Special Relativity, that of a body exponentially asymptotically approaching the speed of light.

In section 3, we review radial geodesics for a free-falling body in the Schwarzschild metric (subsection 3.1), as well as those for light (subsection 3.2). In subsection 3.3 the geodesics for both a free-falling body and a light signal are used in order to obtain an expression for the "apparent" motion of the body as *seen* by a distant observer. In subsection 3.4 the propagation of light signals from the distant observer to the free-falling body are studied (instead of those traveling from the free-falling body to the distant observer, already studied in the previous subsection). In subsection 3.5 the proper time measured by a comoving observer is computed, and we check that it truly correspondents to a finite value. In subsection 3.6 the orbital period of a body in circular motion around the central mass in the Schwarzschild metric is calculated, in order to provide a physical interpretation for the time measured by a distant observer. In section 4 a first analogy with Special Relativity is considered, that of a uniformly accelerated particle, which is determined to have some common aspects but also some important differences with the motion of a free-falling body in the Schwarzschild metric. In section 5 a second analogy is studied, that of an accelerated particle exponentially asymptotically tending to the speed of light, and the correspondence is shown to be much closer. Finally, in section 6 the Discussion of all these results is exposed.

2 Conventions

Natural units are used for simplicity ($c = 1, G = 1$). The chosen metric signature is $(+,-,-,-)$.

3 Motion of a free-falling body in the radial direction in the Schwarzschild metric

3.1 Geodesics of a massive body in the Schwarzschild metric

As it is well known, the Schwarzschild metric in polar units is given by the expression [1, see p. 263]

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta\, d\phi^2\right), \tag{1}$$

where M is the total mass of the black hole.

The general expression for geodesic equations may be written in either of the following ways [2, see p. 263 and 264]:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\,, \tag{2}$$

$$\frac{du_\mu}{ds} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}u^\alpha u^\beta = 0\,, \tag{3}$$

where $g_{\mu\nu}$ are the components of the metric, u^μ those of the four-velocity, and $\Gamma^\mu_{\alpha\beta}$ the Christoffel symbols.

By introducing the metric components of eq. (1) in eq. (3), we obtain the geodesic equations for the Schwarzschild metric:

$$\frac{du_0}{d\tau} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial t}u^\alpha u^\beta = 0\,, \tag{4}$$

$$\begin{aligned}\frac{du_1}{d\tau} &= \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial r}u^\alpha u^\beta = \\ &= \frac{M}{r^2}\left(u^0\right)^2 + \left(1-\frac{2M}{r}\right)^{-2}\frac{M}{r^2}\left(u^1\right)^2 - \\ &-r\left(u^2\right)^2 - r\,\sin^2\theta\left(u^3\right)^2\,,\end{aligned} \tag{5}$$

$$\frac{du_2}{d\tau} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial \theta}u^\alpha u^\beta = -r^2\sin\theta\,\cos\theta\left(u^3\right)^2\,, \tag{6}$$

$$\frac{du_3}{d\tau} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial \phi}u^\alpha u^\beta = 0\,, \tag{7}$$

where we have had into account that, in natural units and with the chosen sign convention, $ds = d\tau$.

From eq. (4), it may be inferred that u_0 must be a constant,

$$u_0 = A\,, \tag{8}$$

where the value of A depends on the energy per unit of mass of the moving particle [1, see p. 179 and 180].

Rising the index with the metric in eq. (8), the relationship between the differentials of Schwarzschild coordinates' time t and proper time τ may be found:

$$\frac{dt}{d\tau} = u^0 = g^{0\mu}u_\mu = g^{00}u_0 = A\left(1-\frac{2M}{r}\right)^{-1}\,, \tag{9}$$

and, by inversion of the derivative,

$$\frac{d\tau}{dt} = \left(\frac{dt}{d\tau}\right)^{-1} = \frac{1}{A}\left(1-\frac{2M}{r}\right)\,. \tag{10}$$

On the other hand, from eq. (1), dividing every term by dt^2, we have

$$\begin{aligned}\left(\frac{d\tau}{dt}\right)^2 &= \left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 - \\ &-r^2\left(\frac{d\theta}{dt}\right)^2 - r^2 sin^2\theta\left(\frac{d\phi}{dt}\right)^2\,.\end{aligned} \tag{11}$$

We will focus our attention in purely radial motions, so that

$$\frac{d\theta}{dt} = \frac{d\phi}{dt} = 0\,, \tag{12}$$

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1-\frac{2M}{r}\right) - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2\,. \tag{13}$$

If we insert eq. (10) into eq. (13), we obtain

$$\frac{1}{A^2}\left(1-\frac{2M}{r}\right)^2=\left(1-\frac{2M}{r}\right)-\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 . \quad (14)$$

From eq. (14), $\frac{dr}{dt}$ can be isolated:

$$\frac{dr}{dt}=\pm\left(1-\frac{2M}{r}\right)\sqrt{1-\frac{1}{A^2}\left(1-\frac{2M}{r}\right)}, \quad (15)$$

where the plus sign (+) implies a centrifugal movement and the minus sign (−) a centripetal one.

As we had already pointed out, the value of A is related to the energy per unit of mass of the moving particle. Its minimal value corresponds to the case of a particle initially at rest at $r = r_0$, and it can be deduced from eq. (15):

$$\sqrt{1-\frac{1}{A^2_{min}}\left(1-\frac{2M}{r_0}\right)}=0 \Rightarrow A_{min}=\sqrt{1-\frac{2M}{r_0}}<1 . \quad (16)$$

On the other hand, in the ultrarelativistic limit, $A \to \infty$.

In order to solve the differential equation (15), the following change of variables will be useful:

$$1-\frac{2M}{r}=z , \quad r=\frac{2M}{1-z} , \quad \frac{dr}{dz}=\frac{2M}{(1-z)^2} . \quad (17)$$

Consequently,

$$\frac{dr}{dt}=\frac{2M}{(1-z)^2}\frac{dz}{dt} , \quad \frac{2M}{(1-z)^2}\frac{dz}{dt}=\pm z\sqrt{1-\frac{z}{A^2}} , \quad (18)$$

$$\frac{dz}{dt}=\frac{\pm 1}{2M}z(1-z)^2\sqrt{1-\frac{z}{A^2}} , \quad (19)$$

and the following differential equation is obtained:

$$\frac{\pm dt}{2M}=\frac{dz}{z(1-z)^2\sqrt{1-\frac{z}{A^2}}} . \quad (20)$$

To be able to integrate the right side of eq. (20), it will be advantageous to divide it in several terms. As can be easily verified,

$$\frac{1}{z(1-z)^2}=\frac{1}{z}+\frac{1}{1-z}+\frac{1}{(1-z)^2} \quad (21)$$

and, therefore,

$$\frac{1}{z(1-z)^2\sqrt{1-\frac{z}{A^2}}}=F_1+F_2+F_3 , \quad (22)$$

where

$$F_1\equiv\frac{1}{z\sqrt{1-\frac{z}{A^2}}} , \quad F_2\equiv\frac{1}{(1-z)\sqrt{1-\frac{z}{A^2}}} , \quad F_3\equiv\frac{1}{(1-z)^2\sqrt{1-\frac{z}{A^2}}} . \quad (23)$$

Hence, integration of eq. (20) yields

$$\int_{t_0}^{t}\frac{\pm dt'}{2M}=\int_{z(r_0)}^{z(r)}\frac{dz}{z(1-z)^2\sqrt{1-\frac{z}{A^2}}} , \quad (24)$$

$$\frac{\pm(t-t_0)}{2M}=I_1+I_2+I_3 , \quad (25)$$

where

$$I_1\equiv\int_{z(r_0)}^{z(r)}F_1dz , \quad I_2\equiv\int_{z(r_0)}^{z(r)}F_2dz , \quad I_3\equiv\int_{z(r_0)}^{z(r)}F_3dz . \quad (26)$$

In the resolution of I_1, I_2 and I_3, a second change of variable will be helpful:

$$\sqrt{1-\frac{z}{A^2}}=w , \quad z=A^2(1-w^2) , \quad dz=-2A^2w\,dw , \quad (27)$$

so that

$$w(r)=w\left(z(r)\right)=\sqrt{1-\frac{1}{A^2}\left(1-\frac{2M}{r}\right)}=\sqrt{1-\frac{r-2M}{A^2r}} . \quad (28)$$

With this new change, the integration of I_1, I_2 and I_3 can be accomplished in the following way:

(i) I_1

$$I_1 = \int_{w(r_0)}^{w(r)} \frac{-2dw}{(1-w^2)} = -\int_{w(r_0)}^{w(r)} \left(\frac{1}{1-w} + \frac{1}{1+w}\right) dw =$$

$$= [\ln(1-w) - \ln(1+w)]_{w(r_0)}^{w(r)} = \left[\ln\left(\frac{1-w}{1+w}\right)\right]_{w(r_0)}^{w(r)} =$$

$$= \left[\ln\left(\frac{1-w^2}{(1+w)^2}\right)\right]_{w(r_0)}^{w(r)} = \left[\ln(1-w^2) - 2\ln(1+w)\right]_{w(r_0)}^{w(r)} =$$

$$= \ln\left(\frac{1}{A^2}\left(1-\frac{2M}{r}\right)\right) - \ln\left(\frac{1}{A^2}\left(1-\frac{2M}{r_0}\right)\right) -$$

$$-2\ln\left(1+\sqrt{1-\frac{r-2M}{A^2 r}}\right) + 2\ln\left(1+\sqrt{1-\frac{r_0-2M}{A^2 r_0}}\right) =$$

$$= \ln\left(\frac{r-2M}{r_0-2M}\right) + \ln\left(\frac{r_0}{r}\right) - 2\ln\left(\frac{1+\sqrt{1-\frac{r-2M}{A^2 r}}}{1+\sqrt{1-\frac{r-2M}{A^2 r_0}}}\right). \tag{29}$$

(ii) I_2

$$I_2 = \int_{w(r_0)}^{w(r)} \frac{-2A^2 dw}{(1-A^2+A^2w^2)} = \int_{w(r_0)}^{w(r)} \frac{-2dw}{\left(w^2+\frac{1}{A^2}-1\right)}. \tag{30}$$

Equation (30) has three different solutions, depending on the value of A:

If (a) $A < 1$,

$$I_{2,a} = \left[\frac{-2}{\sqrt{\frac{1}{A^2}-1}} \arctan\left(\frac{w}{\sqrt{\frac{1}{A^2}-1}}\right)\right]_{w(r_0)}^{w(r)} =$$
$$= \frac{2}{\sqrt{\frac{1}{A^2}-1}} \arctan\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}{\sqrt{\frac{1}{A^2}-1}}\right) - \tag{31}$$
$$-\frac{2}{\sqrt{\frac{1}{A^2}-1}} \arctan\left(\frac{\sqrt{1-\frac{r-2M}{A^2 r}}}{\sqrt{\frac{1}{A^2}-1}}\right).$$

If (b) $A = 1$,

$$I_{2,b} = \left[\frac{2}{w}\right]_{w(r_0)}^{w(r)} = \frac{2}{\sqrt{1-\frac{r-2M}{r}}} - \frac{2}{\sqrt{1-\frac{r_0-2M}{r_0}}} =$$
$$= 2\sqrt{\frac{r}{2M}} - 2\sqrt{\frac{r_0}{2M}}. \tag{32}$$

If (c) $A > 1$,

$$I_{2,c} = \left[\frac{1}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\frac{w+\sqrt{1-\frac{1}{A^2}}}{w-\sqrt{1-\frac{1}{A^2}}}\right)\right]_{w(r_0)}^{w(r)} =$$
$$= \left[\frac{1}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\frac{\left(w+\sqrt{1-\frac{1}{A^2}}\right)^2}{w^2-\left(1-\frac{1}{A^2}\right)}\right)\right]_{w(r_0)}^{w(r)} =$$
$$= \left[\frac{2}{\sqrt{1-\frac{1}{A^2}}} \ln\left(w+\sqrt{1-\frac{1}{A^2}}\right) - \right.$$
$$\left. - \frac{1}{\sqrt{1-\frac{1}{A^2}}} \ln\left(w^2+\frac{1}{A^2}-1\right)\right]_{w(r_0)}^{w(r)} =$$
$$= \frac{2}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\sqrt{1-\frac{r-2M}{A^2 r}} + \sqrt{1-\frac{1}{A^2}}\right) -$$
$$-\frac{2}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\sqrt{1-\frac{r_0-2M}{A^2 r_0}} + \sqrt{1-\frac{1}{A^2}}\right) +$$
$$+\frac{1}{\sqrt{1-\frac{1}{A^2}}} \left(\ln\left(\frac{1}{A^2}\frac{2M}{r_0}\right) - \ln\left(\frac{1}{A^2}\frac{2M}{r}\right)\right) = \tag{33}$$
$$= \frac{2}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\sqrt{1-\frac{r-2M}{A^2 r}} + \sqrt{1-\frac{1}{A^2}}\right) -$$
$$-\frac{2}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\sqrt{1-\frac{r_0-2M}{A^2 r_0}} + \sqrt{1-\frac{1}{A^2}}\right) +$$
$$+\frac{1}{\sqrt{1-\frac{1}{A^2}}} \ln\left(\frac{r}{r_0}\right).$$

(iii) I_3

$$I_3 = \int_{w(r_0)}^{w(r)} \frac{-2A^2 dw}{(1-A^2+A^2w^2)^2} = \int_{w(r_0)}^{w(r)} \frac{\frac{-2}{A^2}dw}{\left(w^2+\frac{1}{A^2}-1\right)^2} \ . \quad (34)$$

Likewise eq. (30), eq. (34) has three different solutions depending on the value of A:

If (a) $A<1$,

$$I_{3,a} = \frac{-1}{1-A^2}\left(\left[\frac{w}{w^2+\frac{1}{A^2}-1}\right]_{w(r_0)}^{w(r)} + \int_{w(r_0)}^{w(r)} \frac{dw}{w^2+\frac{1}{A^2}-1}\right) =$$

$$= \frac{-1}{1-A^2}\left[\frac{w}{w^2+\frac{1}{A^2}-1}\right]_{w(r_0)}^{w(r)} -$$

$$-\frac{1}{1-A^2}\left[\frac{1}{\sqrt{\frac{1}{A^2}-1}}\arctan\left(\frac{w}{\sqrt{\frac{1}{A^2}-1}}\right)\right]_{w(r_0)}^{w(r)} =$$

$$= \frac{1}{1-A^2}\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2r_0}}}{\frac{2M}{A^2r_0}} - \frac{\sqrt{1-\frac{r-2M}{A^2r}}}{\frac{2M}{A^2r}}\right)+$$

$$+\frac{1}{1-A^2}\frac{1}{\sqrt{\frac{1}{A^2}-1}}\arctan\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2r_0}}}{\sqrt{\frac{1}{A^2}-1}}\right)-$$

$$-\frac{1}{1-A^2}\frac{1}{\sqrt{\frac{1}{A^2}-1}}\arctan\left(\frac{\sqrt{1-\frac{r-2M}{A^2r}}}{\sqrt{\frac{1}{A^2}-1}}\right)=$$

$$= \frac{A^2}{1-A^2}\left(\frac{r_0}{2M}\sqrt{1-\frac{r_0-2M}{A^2r_0}} - \frac{r}{2M}\sqrt{1-\frac{r-2M}{A^2r}}\right)+$$

$$+\frac{A}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2r_0}}}{\sqrt{\frac{1}{A^2}-1}}\right)-$$

$$-\frac{A}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{\sqrt{1-\frac{r-2M}{A^2r}}}{\sqrt{\frac{1}{A^2}-1}}\right) . \quad (35)$$

If (b) $A=1$,

$$I_{3,b} = -2\int_{w(r_0)}^{w(r)} \frac{dw}{w^4} = -2\left[\frac{-1}{3}\frac{1}{w^3}\right]_{w(r_0)}^{w(r)} = \frac{2}{3}\left[\frac{1}{w^3}\right]_{w(r_0)}^{w(r)} =$$

$$= \frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - \frac{2}{3}\left(\frac{r_0}{2M}\right)^{\frac{3}{2}} . \quad (36)$$

If (c) $A>1$,

$$I_{3,c} = \frac{1}{A^2-1}\left(\left[\frac{w}{w^2+\frac{1}{A^2}-1}\right]_{w(r_0)}^{w(r)} + \int_{w(r_0)}^{w(r)} \frac{dw}{w^2+\frac{1}{A^2}-1}\right) =$$

$$= \frac{1}{A^2-1}\left[\frac{w}{w^2+\frac{1}{A^2}-1}\right]_{w(r_0)}^{w(r)} +$$

$$+\frac{1}{A^2-1}\left[\frac{1}{2\sqrt{1-\frac{1}{A^2}}}\ln\left(\frac{w-\sqrt{1-\frac{1}{A^2}}}{w+\sqrt{1-\frac{1}{A^2}}}\right)\right]_{w(r_0)}^{w(r)} =$$

$$= \frac{A^2}{A^2-1}\left(\frac{r}{2M}\sqrt{1-\frac{r-2M}{A^2r}} - \frac{r_0}{2M}\sqrt{1-\frac{r_0-2M}{A^2r_0}}\right)+$$

$$+\frac{1}{2(A^2-1)\sqrt{1-\frac{1}{A^2}}}\left[\ln\left(\frac{w^2-1+\frac{1}{A^2}}{\left(w+\sqrt{1-\frac{1}{A^2}}\right)^2}\right)\right]_{w(r_0)}^{w(r)} =$$

$$= \frac{A^2}{A^2-1}\left(\frac{r}{2M}\sqrt{1-\frac{r-2M}{A^2r}} - \frac{r_0}{2M}\sqrt{1-\frac{r_0-2M}{A^2r_0}}\right)+$$

$$+\frac{A\ln\left(\frac{r_0}{r}\right)}{2(A^2-1)^{\frac{3}{2}}} - \frac{A}{(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{\sqrt{1-\frac{r-2M}{A^2r}}+\sqrt{1-\frac{1}{A^2}}}{\sqrt{1-\frac{r_0-2M}{A^2r_0}}+\sqrt{1-\frac{1}{A^2}}}\right) . \quad (37)$$

Therefore, inserting the results of integration of I_1, I_2 and I_3 into eq. (25), we obtain the following solutions:

If (a) $A<1$,

$$\begin{aligned}\frac{\pm(t-t_0)}{2M} &= C_a + \ln\left(\frac{r-2M}{r_0-2M}\right) + \ln\left(\frac{r_0}{r}\right) - \\ &-2\ln\left(\frac{1+\sqrt{1-\frac{r-2M}{A^2 r}}}{1+\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}\right) - \\ &-\frac{3A-2A^3}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{\sqrt{1-\frac{r-2M}{A^2 r}}}{\sqrt{\frac{1}{A^2}-1}}\right) - \\ &-\frac{A^2}{1-A^2}\frac{r}{2M}\sqrt{1-\frac{r-2M}{A^2 r}}\,,\end{aligned} \tag{38}$$

where

$$\begin{aligned}C_a \equiv \frac{3A-2A^3}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}{\sqrt{\frac{1}{A^2}-1}}\right) + \\ +\frac{A^2}{1-A^2}\frac{r_0}{2M}\sqrt{1-\frac{r_0-2M}{A^2 r_0}}\,.\end{aligned} \tag{39}$$

If (b) $A = 1$,

$$\begin{aligned}\frac{\pm(t-t_0)}{2M} &= C_b + \ln\left(\frac{r-2M}{r_0-2M}\right) + \ln\left(\frac{r_0}{r}\right) - \\ &-2\ln\left(\frac{1+\sqrt{\frac{2M}{r}}}{1+\sqrt{\frac{2M}{r_0}}}\right) + 2\sqrt{\frac{r}{2M}} + \frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}}\,,\end{aligned} \tag{40}$$

where

$$C_b \equiv -2\sqrt{\frac{r_0}{2M}} - \frac{2}{3}\left(\frac{r_0}{2M}\right)^{\frac{3}{2}}\,. \tag{41}$$

If (c) $A > 1$,

$$\begin{aligned}\frac{\pm(t-t_0)}{2M} &= C_c + \ln\left(\frac{r-2M}{r_0-2M}\right) + \ln\left(\frac{r_0}{r}\right) - \\ &-2\ln\left(\frac{1+\sqrt{1-\frac{r-2M}{A^2 r}}}{1+\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}\right) - \frac{2A^3-3A}{2(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{r_0}{r}\right) + \\ &+\frac{2A^3-3A}{(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{\sqrt{1-\frac{r-2M}{A^2 r}}+\sqrt{1-\frac{1}{A^2}}}{\sqrt{1-\frac{r_0-2M}{A^2 r_0}}+\sqrt{1-\frac{1}{A^2}}}\right) + \\ &+\frac{A^2}{A^2-1}\frac{r}{2M}\sqrt{1-\frac{r-2M}{A^2 r}}\,,\end{aligned} \tag{42}$$

where

$$C_c \equiv -\frac{A^2}{A^2-1}\frac{r_0}{2M}\sqrt{1-\frac{r_0-2M}{A^2 r_0}}\,. \tag{43}$$

In a centripetal movement, after a long enough time, r will be close to $2M$. It is easy to notice that in all three equations (38), (40) and (42) the dominant term when approaching $2M$ will be $\ln\left(\frac{r-2M}{r_0-2M}\right)$, as $\lim_{r\to 2M}\ln\left(\frac{r-2M}{r_0-2M}\right) = -\infty$ while all the other terms tend to finite values. Let us obtain the first-order Taylor series approximation of all the terms which tend to finite values around $r = 2M$. By introducing these approximations in eqs. (38), (40) and (42), in all cases the result will be an expression of the style

$$\frac{-(t-t_0)}{2M} \approx \hat{C} + \ln\left(\frac{r-2M}{r_0-2M}\right) + B(r-2M)\,, \tag{44}$$

where $\hat{C}$ corresponds to the sum of the C constant of each equation and the values in $r = 2M$ of all terms other than $\ln\left(\frac{r-2M}{r_0-2M}\right)$, while B corresponds to the sum of the derivatives in $r = 2M$ of all terms other than $\ln\left(\frac{r-2M}{r_0-2M}\right)$.

If (a) $A < 1$,

$$\begin{aligned}\hat{C}_a = C_a + \ln\left(\frac{r_0}{2M}\right) - 2\ln\left(\frac{2}{1+\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}\right) - \\ -\frac{3A-2A^2}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{1}{\sqrt{\frac{1}{A^2}-1}}\right) - \frac{A^2}{1-A^2}\,.\end{aligned} \tag{45}$$

If (b) $A = 1$,

$$\hat{C}_b = C_b + \ln\left(\frac{r_0}{2M}\right) - 2\ln\left(\frac{2}{1+\sqrt{\frac{2M}{r_0}}}\right) + \frac{8}{3}\,. \qquad (46)$$

If (c) $A > 1$,

$$\hat{C}_c = C_c + \left(1 - \frac{2A^3-3A}{2(A^2-1)^{\frac{3}{2}}}\right)\ln\left(\frac{r_0}{2M}\right) + \frac{2A^3-3A}{(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{1+\sqrt{1-\frac{1}{A^2}}}{\sqrt{1-\frac{r_0-2M}{A^2r_0}}+\sqrt{1-\frac{1}{A^2}}}\right) - 2\ln\left(\frac{2}{1+\sqrt{1-\frac{r_0-2M}{A^2r_0}}}\right) + \frac{A^2}{A^2-1}\,. \qquad (47)$$

Concerning the calculation of coefficients B, it could be effectively accomplished by the derivation of all terms other than $\ln\left(\frac{r-2M}{r_0-2M}\right)$ in equations (38), (40) and (42). However, it is possible to calculate them in a more direct way, as all these equations are solutions of eq. (25), whose differential form is eq. (20). As we are explicitely excluding the term $\ln\left(\frac{r-2M}{r_0-2M}\right)$ in the definition of B,

$$B = \lim_{r\to 2M}\frac{d}{dr}\left\{I_1 + I_2 + I_3 - \ln\left(\frac{r-2M}{r_0-2M}\right)\right\} =$$

$$= \lim_{z\to 0}\left\{\frac{dz}{dr}\frac{d}{dz}(I_1+I_2+I_3) - \frac{1}{r-2M}\right\} =$$

$$= \lim_{z\to 0}\left\{\frac{(1-z)^2}{2M}(F_1+F_2+F_3) - \frac{1}{2M}\left(\frac{1}{z}-1\right)\right\} =$$

$$= \frac{1}{2M}\lim_{z\to 0}\left\{\frac{(1-z)^2}{\sqrt{1-\frac{z}{A^2}}}\left(\frac{1}{z}+\frac{1}{1-z}+\frac{1}{(1-z)^2}\right) - \frac{1}{z} + 1\right\} =$$

$$= \frac{1}{2M}\lim_{z\to 0}\left\{\frac{1}{z}\left[\frac{(1-z)^2}{\sqrt{1-\frac{z}{A^2}}} - 1\right]\right\} + \frac{3}{2M}\,, \qquad (48)$$

where we have used the relationship between z and r given by eq. (17) and the definition of functions F_1, F_2 and F_3 in eqs. (23). In order to solve the remaining uncertainty, we will use L'Hôpital's rule:

$$\lim_{z\to 0}\left\{\frac{1}{z}\left[\frac{(1-z)^2}{\sqrt{1-\frac{z}{A^2}}} - 1\right]\right\} = \lim_{z\to 0}\left\{\frac{\frac{(1-z)^2}{\sqrt{1-\frac{z}{A^2}}} - 1}{z}\right\} =$$

$$= \lim_{z\to 0}\left\{\frac{\frac{2(1-z)(-1)}{\sqrt{1-\frac{z}{A^2}}} + (1-z)^2\left(\frac{-1}{2}\right)\left(1-\frac{z}{A^2}\right)^{\frac{-3}{2}}\left(\frac{-1}{A^2}\right)}{1}\right\} = -2 + \frac{1}{2A^2}\,. \qquad (49)$$

Therefore, eq. (48) becomes

$$B = \tfrac{1}{2M}\left(1 + \tfrac{1}{2A^2}\right)\,. \qquad (50)$$

Thus, the expression of B results to be independent of the fact that $A < 1$, $A = 1$ or $A > 1$:

$$B_a = B_b = B_c \equiv B\,. \qquad (51)$$

Hence, eq. (44) may be written in a completely general way as:

$$\frac{-(t-t_0)}{2M} \approx \hat{C} + \ln\left(\frac{r-2M}{r_0-2M}\right) + \frac{r-2M}{2M}\left(1+\frac{1}{2A^2}\right)\,, \qquad (52)$$

which is the expression that, for the sake of simplicity, we will use in our further calculations. As we shall soon see, it allows direct comparison with the expression for the trajectory of a light signal.

On the other hand, it can be pointed out that at large times even a more simplified equation could be suitable to describe the centripetal movement of a free-falling body in the radial direction in the Schwarzschild metric. As a matter of fact, as $r - 2M \to 0$,

$$\frac{-(t-t_0)}{2M} \sim \hat{C} + \ln\left(\frac{r-2M}{r_0-2M}\right)\,. \qquad (53)$$

Under this approximation, an expression for $r(t)$ may be easily isolated:

$$r(t) \sim 2M + (r_0 - 2M)\, e^{-\hat{C} - \frac{1}{2M}(t-t_0)}\,. \qquad (54)$$

3.2 Geodesics of light in the Schwarzschild metric

For light $d\tau = 0$, so that $A \equiv \frac{dt}{d\tau} = \infty$. Thus, eq. (15) becomes

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right) = \pm\frac{r-2M}{r}\,, \qquad (55)$$

where again the plus sign (+) implies a centrifugal movement and the minus sign (−) a centripetal one.

Let us isolate the variables:

$$\pm dt = \frac{r}{r-2M}dr\,, \qquad (56)$$

In order to solve the right side of eq. (56), the following change of variables will be useful:

$$r - 2M = \Delta r, \quad dr = d\Delta r \,, \tag{57}$$

so that

$$\pm dt = \frac{\Delta r + 2M}{\Delta r} d\Delta r \,, \quad \pm dt = \left(1 + \frac{2M}{\Delta r}\right) d\Delta r \,. \tag{58}$$

By integration,

$$\pm \int_{t_0}^{t} dt' = \int_{r_0-2M}^{r-2M} \left(1 + \frac{2M}{\Delta r}\right) d\Delta r \,, \tag{59}$$

$$\pm(t - t_0) = [\Delta r + 2M \ln \Delta r]_{r_0-2M}^{r-2M} \,, \tag{60}$$

$$\pm(t - t_0) = (r - 2M) - (r_0 - 2M) + 2M \ln\left(\frac{r - 2M}{r_0 - 2M}\right) \,, \tag{61}$$

$$\frac{\pm (t - t_0)}{2M} = \frac{-(r_0 - 2M)}{2M} + \ln\left(\frac{r - 2M}{r_0 - 2M}\right) + \frac{r - 2M}{2M} \,. \tag{62}$$

If we rewrite eq. (62) as

$$\frac{\pm (t - t_0)}{2M} = \hat{C}_{Light} + \ln\left(\frac{r - 2M}{r_0 - 2M}\right) + \frac{r - 2M}{2M} \,, \tag{63}$$

with

$$\hat{C}_{Light} \equiv \frac{-(r_0 - 2M)}{2M} \,, \tag{64}$$

the analogy with eq. (52) results evident. As a matter of fact, in eq. (52) there is an additional factor $\left(1 + \frac{1}{2A^2}\right)$ in the $\frac{r-2M}{2M}$ term, which has into account that the speed of a massive free-falling body will always be lesser than the speed of light. Nevertheless, in the ultrarelativistic limit, when $A \to \infty$, $\left(1 + \frac{1}{2A^2}\right) \to 1$, so that we recover eq. (63). On the other hand, we can ask ourselves what is the relationship between the constants $\hat{C}$ and $\hat{C}_{Light}$. While $\hat{C}_{Light}$ presents a rather simple expression given by eq. (64), $\hat{C}$ depends on complicated expressions provided by eqs. (45)-(47), which in turn depend on the values of C provided by eqs. (39), (41) and (43). However, the relation between them may be established through a quite plain reasoning.

Let us imagine that *simultaneously*, from a point situated at radius r_0 around a Schwarzschild black hole, a body is thrown towards the black hole and a light signal emitted in the same sense and direction. At very large times, the motion of the free-falling body can be approximated by the expression given by eq. (54). Similarly, the motion of the light signal, described by eq. (63), when $r - 2M \to 0$, can be approximated by the expression

$$\frac{-(t - t_0)}{2M} \sim \hat{C}_{Light} + \ln\left(\frac{r - 2M}{r_0 - 2M}\right) \,, \tag{65}$$

From eq. (65), $r_{Light}(t)$ may be straightforwardly isolated:

$$r_{Light}(t) \sim 2M + (r_0 - 2M)\, e^{-\hat{C}_{Light} - \frac{1}{2M}(t-t_0)} \,. \tag{66}$$

As light moves always faster than any massive body and both the light signal and the free-falling body have departed from the same point at the same time, the light signal must always *precede* the falling body. As their motion is centripetal, this means that the radius reached by the light signal must always be *lesser* than the one reached by the body:

$$r_{Light}(t) < r_{Body}(t) \,. \tag{67}$$

Consequently,

$$2M + (r_0 - 2M)e^{-\hat{C}_{Light} - \frac{1}{2M}(t-t_0)} < 2M + (r_0 - 2M)e^{-\hat{C} - \frac{1}{2M}(t-t_0)}, \tag{68}$$

$$\begin{aligned} &(r_0 - 2M)\, e^{-\hat{C}_{Light} - \frac{1}{2M}(t-t_0)} < (r_0 - 2M)\, e^{-\hat{C} - \frac{1}{2M}(t-t_0)} \,, \\ &e^{-\hat{C}_{Light} - \frac{1}{2M}(t-t_0)} < e^{-\hat{C} - \frac{1}{2M}(t-t_0)} \,, \\ &e^{-\hat{C}_{Light}} < e^{-\hat{C}} \,, \\ &-\hat{C}_{Light} < -\hat{C} \,, \end{aligned} \tag{69}$$

$$\hat{C} < \hat{C}_{Light} = \frac{-(r_0 - 2M)}{2M} \,. \tag{70}$$

Therefore, we can express $\hat{C}$ as

$$\hat{C} = \hat{C}_{Light} - \Delta\hat{C} \,, \tag{71}$$

with $\Delta\hat{C} > 0$.

Thus, we can rewrite eq. (52) as

$$\frac{-(t - t_0)}{2M} \approx \hat{C}_{Light} - \Delta\hat{C} + \ln\left(\frac{r - 2M}{r_0 - 2M}\right) + \frac{r - 2M}{2M}\left(1 + \frac{1}{2A^2}\right) \,, \tag{72}$$

or as

$$\begin{aligned} \frac{-(t - t_0)}{2M} \approx \frac{-(r_0 - 2M)}{2M} - \Delta\hat{C} + \\ + \ln\left(\frac{r - 2M}{r_0 - 2M}\right) + \frac{r - 2M}{2M}\left(1 + \frac{1}{2A^2}\right) \,. \end{aligned} \tag{73}$$

or as

$$\frac{-(t-t_0)}{2M} \approx \frac{r-r_0}{2M} - \Delta\hat{C} + \ln\left(\frac{r-2M}{r_0-2M}\right) + \frac{1}{2A^2}\frac{r-2M}{2M} \,. \tag{74}$$

Equation (62) may be also written in a more synthetic way and with similarity to eq. (74) as

$$\frac{\pm(t-t_0)}{2M} = \frac{r-r_0}{2M} + \ln\left(\frac{r-2M}{r_0-2M}\right) \,. \tag{75}$$

Additionally, we can get even more information about $\Delta\hat{C}$ by comparing the trajectories of a light signal and a massive body simultaneously emitted (or thrown) from a point at radius $r = r_0$. We have already compared the radii of both at the same time. We can also compare the different times in which they reach a specific radius r. We know that the signal of light must reach it *earlier*, that is, in a *lesser* period of time. From eqs. (74) and (75):

$$\frac{-\left(t_{Body}-t_0\right)}{2M} \approx \frac{r-r_0}{2M} - \Delta\hat{C} + \ln\left(\frac{r-2M}{r_0-2M}\right) + \frac{1}{2A^2}\frac{r-2M}{2M} \,, \tag{76}$$

$$\frac{-\left(t_{Light}-t_0\right)}{2M} = \frac{r-r_0}{2M} + \ln\left(\frac{r-2M}{r_0-2M}\right) \,. \tag{77}$$

By the subtraction of both expressions, we have

$$\frac{t_{Body}-t_{Light}}{2M} \approx \Delta\hat{C} - \frac{1}{2A^2}\frac{r-2M}{2M} > 0 \quad \forall r < r_0 \,. \tag{78}$$

Consequently,

$$\Delta\hat{C} > \frac{1}{2A^2}\frac{r-2M}{2M} \quad \forall r < r_0 \,. \tag{79}$$

As both the light signal and the massive body were *simultaneously* at $r = r_0$, so that $t_{Light} = t_{Body}$, we may be tempted to use this equality to identify $\Delta\hat{C} = \frac{1}{2A^2}\frac{r_0-2M}{2M}$. However, we must remember that eq. (74) proceeds ultimately from eq. (44), which by construction is only valid when r approaches $2M$.

3.3 "Apparent" trajectory of the free-falling body

By "apparent" trajectory of a body, we mean the trajectory of the body as *seen* by a specific observer located in a "determined spatial point" at each moment in time (it could be observer in motion, but at each moment it must occupy a unique and concrete position). In order to be *seen* by the observer, the moving body must emit light signals (either by emission or by reflection), so that the body will be seen by the observer *only* when these light signals reach the observer in its concrete position. In this section, we are specifically interested in the apparent trajectory of a free-falling body in the Schwarzschild metric according to a fixed observer at $r = R$ (which is not an inertial observer).

Let us assume that the free-falling body departs precisely from $r_0 = R$ at time $t_0 = 0$ and reaches $r = r_1$ at time $t = t_1$. From eq. (74),

$$\frac{-t_1}{2M} \approx \frac{r_1-R}{2M} - \Delta\hat{C} + \ln\left(\frac{r_1-2M}{R-2M}\right) + \frac{1}{2A^2}\frac{r_1-2M}{2M} \,. \tag{80}$$

The light emitted by the body from $r_0 = r_1$ at time $t_0 = t_1$ will arrive at $r = R$ at time $t = t_2$. According to eq. (75),

$$\frac{+(t_2-t_1)}{2M} = \frac{R-r_1}{2M} + \ln\left(\frac{R-2M}{r_1-2M}\right) \,, \tag{81}$$

or

$$\frac{+(t_2-t_1)}{2M} = \frac{R-r_1}{2M} - \ln\left(\frac{r_1-2M}{R-2M}\right) \,. \tag{82}$$

By the subtraction of equations (80) and (82), we obtain

$$\frac{-t_2}{2M} \approx \frac{-2(R-r_1)}{2M} - \Delta\hat{C} + 2\ln\left(\frac{r_1-2M}{R-2M}\right) + \frac{1}{2A^2}\frac{r_1-2M}{2M} \,. \tag{83}$$

As the light signal proceeding from the body at r_1 has reached the observer at time t_2, the apparent position of the body at time $t = t_2$ is $r_{app} = r_1$:

$$\frac{-t}{2M} \approx \frac{-2\left(R-r_{app}\right)}{2M} - \Delta\hat{C} + 2\ln\left(\frac{r_{app}-2M}{R-2M}\right) + \frac{1}{2A^2}\frac{r_{app}-2M}{2M} \,. \tag{84}$$

At large times, when $r_{app} \sim 2M$, the following approximation is possible:

$$\frac{-t}{2M} \sim \frac{-2(R-2M)}{2M} - \Delta\hat{C} + 2\ln\left(\frac{r_{app}-2M}{R-2M}\right) \,, \tag{85}$$

from where the expression for $r_{app}(t)$ may be isolated:

$$r_{app}(t) \sim 2M + (R-2M)\, e^{\frac{R-2M}{2M} + \frac{\Delta\hat{C}}{2} - \frac{1}{4M}t} \,. \tag{86}$$

It may be noticed that both $r(t)$ and $r_{app}(t)$ are asymptotic expressions, with different time constants, $\frac{-1}{2M}$ vs $\frac{-1}{4M}$. On the other hand, it is also important to emphasize the relationship between the time interval $\Delta t_1 = t_1 - 0$ that the body spends in moving from $r = R$ to $r = r_1$ and the time interval $\Delta t_2 = t_2 - t_1$ that it takes light to describe the reverse path. From equations (80) and (81),

$$\Delta t_1 = \Delta t_2 + 2M\Delta\hat{C} - \frac{1}{2A^2}(r_1 - 2M) \,. \tag{87}$$

Considering the results of eqs. (79) and (87) together, it can be inferred that $\Delta t_1 > \Delta t_2$. Logically, the free-falling body inverts more time in arriving at $r = r_1$ from $r_0 = R$ than the light emitted by it in traveling the reverse way.

3.4 Signal transmission

Let us consider again the trajectory of a free-falling body in the radial direction which departs from $r = R$ at $t_0 = 0$. According to eq. (74):

$$\frac{-t}{2M} \approx \frac{r-R}{2M} - \Delta\hat{C} + \ln\left(\frac{r-2M}{R-2M}\right) + \frac{1}{2A^2}\frac{r-2M}{2M} \,, \tag{88}$$

Let us suppose that at $t = \Delta t$ a fixed observer at $r = R$ sends a light signal towards the body:

$$\frac{-(t-\Delta t)}{2M} = \frac{r-R}{2M} + \ln\left(\frac{r-2M}{R-2M}\right) \,, \tag{89}$$

according to eq. (75).

We will assume that the signal reaches the body at $t = t_a$ at $r = r_a$. By subtracting equations (88) and (89), we obtain

$$\frac{\Delta t}{2M} \approx \Delta\hat{C} - \frac{1}{2A^2}\frac{r_a - 2M}{2M} \,, \tag{90}$$

so that

$$r_a \approx 2M + 2A^2(2M\Delta\hat{C} - \Delta t) \,. \tag{91}$$

If $\Delta t < 2M\Delta\hat{C}$, then $r_a > 2M$ and there will be an actual time t_a when $r = r_a$. In order to obtain the value of t_a, we only need to replace r by r_a in eq. (88):

$$t_a \approx (R-r_a) + 2M\Delta\hat{C} - 2M\ln\left(\frac{r-2M}{R-2M}\right) - \frac{1}{2A^2}(r_a - 2M) \,, \tag{92}$$

or, replacing r_a by its expression in eq. (91),

$$t_a \approx (R-2M) - 2A^2(2M\Delta\hat{C} - \Delta t) + \Delta t - \\ - \ln\left(\frac{2A^2(2M\Delta\hat{C} - \Delta t)}{R-2M}\right) \,. \tag{93}$$

On the contrary, if $\Delta t > 2M\Delta\hat{C}$, then $r_a < 2M$ and there is no time at which this radius will be ever reached. In other words, as $r > 2M \;\forall t$, signals after

$$\Delta t_{max} \approx 2M\Delta\hat{C} \tag{94}$$

will never reach the free-falling body.

3.5 Proper time

We can obtain the proper velocity of the free-falling body from eqs. (9) and (15):

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = \\ = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \frac{1}{A^2}\left(1 - \frac{2M}{r}\right)}\,A\left(1 - \frac{2M}{r}\right)^{-1} = \\ = \pm A\sqrt{1 - \frac{1}{A^2}\left(1 - \frac{2M}{r}\right)} \,. \tag{95}$$

Thus, by separating the variables,

$$\pm d\tau = \frac{dr}{A\sqrt{1 - \frac{1}{A^2}\left(1 - \frac{2M}{r}\right)}} \,. \tag{96}$$

Integration yields

$$\pm\Delta\tau = \frac{1}{A}\int_{r_0}^{r}\frac{dr'}{\sqrt{1 - \frac{1}{A^2}\left(1 - \frac{2M}{r'}\right)}} \,. \tag{97}$$

In order to solve the right side of eq. (97), the change of variables specified in eq. (17) will be again useful:

$$\frac{1}{A}\int_{r_0}^{r}\frac{dr'}{\sqrt{1 - \frac{1}{A^2}\left(1 - \frac{2M}{r'}\right)}} = \frac{1}{A}\int_{z(r_0)}^{z(r)}\frac{\frac{2M}{(1-z)^2}}{\sqrt{1 - \frac{z}{A^2}}}dz = \\ = \frac{2M}{A}\int_{z(r_0)}^{z(r)}\frac{dz}{(1-z)^2\sqrt{1 - \frac{z}{A^2}}} = \\ = \frac{2M}{A}\int_{z(r_0)}^{z(r)} F_3\,dz = \frac{2M}{A}I_3 \,, \tag{98}$$

where we have used the identities in eqs. (23) and (26). The expression of I_3, as we have seen in section 3, depends on A being lesser, equal or greater than 1.

In a centripetal movement from $r_0 = R$ to $r = 2M$, where the minus sign ($-$) must be chosen, the interval of proper time $\Delta\tau$ inverted by a free-falling body in its trajectory will be given by te expressions:

(i) If $A < 1$,

$$\Delta\tau_a = \frac{A}{1-A^2}\left(2M - r_0\sqrt{1-\frac{r_0-2M}{A^2 r_0}}\right)+$$
$$+\frac{2M}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{1}{\sqrt{\frac{1}{A^2}-1}}\right)-$$
$$-\frac{2M}{(1-A^2)^{\frac{3}{2}}}\arctan\left(\frac{\sqrt{1-\frac{r_0-2M}{A^2 r_0}}}{\sqrt{\frac{1}{A^2}-1}}\right). \tag{99}$$

(ii) If $A = 1$,

$$\Delta\tau_b = \frac{4M}{3A}\left(\left(\frac{r_0}{2M}\right)^{\frac{3}{2}} - 1\right). \tag{100}$$

(iii) If $A > 1$,

$$\Delta\tau_c = \frac{A}{A^2-1}\left(r_0\sqrt{1-\frac{r_0-2M}{A^2 r_0}} - 2M\right)+$$
$$+\frac{2M}{2(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{2M}{r_0}\right)+$$
$$+\frac{2M}{(A^2-1)^{\frac{3}{2}}}\ln\left(\frac{1+\sqrt{1-\frac{1}{A^2}}}{\sqrt{1-\frac{r_0-2M}{A^2 r_0}}+\sqrt{1-\frac{1}{A^2}}}\right). \tag{101}$$

In all three cases, $\Delta\tau$ presents a finite value, in contrast to $lim_{r\to 2M}\ t = \infty$.

3.6 Circular orbits in the Schwarzschild metric

Let us consider a planet describing a circular orbit around the black hole in the plane $\theta = \frac{\pi}{2}$, at $r = R$. As both coordinates θ and r are constant,

$$u^1 = \frac{dr}{d\tau} = 0, \quad u^2 = \frac{d\theta}{d\tau} = 0, \tag{102}$$

and also, as the metric is diagonal,

$$u_1 = 0, \quad u_2 = 0. \tag{103}$$

With these considerations into account, eq. (5) may be rewritten in the following manner:

$$0 = \frac{M}{R^2}\left(\frac{dt}{d\tau}\right)^2 - R\ \sin^2\left(\frac{\pi}{2}\right)\left(\frac{d\phi}{d\tau}\right)^2. \tag{104}$$

Therefore, as $\sin\frac{\pi}{2} = 1$,

$$\frac{M}{R^2}\left(\frac{dt}{d\tau}\right)^2 = R\left(\frac{d\phi}{d\tau}\right)^2, \quad \frac{M}{R^2}dt^2 = Rd\phi^2, \tag{105}$$

$$\frac{d\phi}{dt} = \frac{1}{R}\sqrt{\frac{M}{R}}, \quad \phi = \phi_0 + \frac{1}{R}\sqrt{\frac{M}{R}}t. \tag{106}$$

As ϕ is an angular variable, and consequently periodic modulo 2π, the period of the orbit will be

$$T = \frac{2\pi}{\frac{d\phi}{dt}} = 2\pi R\sqrt{\frac{R}{M}}. \tag{107}$$

A similar development may be found in Schutz [1, see p. 284-286].

4 A first analogy with Special Relativity: Uniformly accelerated motion

In this section we will review the motion of a particle exposed to a constant force in the context of Special Relativity. A constant force F implies a constant variation of the lineal moment p over time:

$$F = \frac{dp}{dt}, \quad p = p_0 + Ft. \tag{108}$$

For simplicity, we shall assume that $p_0 = 0$, so that

$$p = Ft. \tag{109}$$

As it is well known, the relationship between the lineal moment p and the velocity v in Special Relativity is given by the expression

$$p = \frac{mv}{\sqrt{1+v^2}}, \tag{110}$$

from where we can isolate v:

$$v = \frac{p}{\sqrt{m^2+p^2}} = \frac{Ft}{\sqrt{m^2+(Ft)^2}}. \tag{111}$$

By integration, we obtain the expression for the position of the particle over time:

$$s(t) = \int_0^t v(t')dt' = \int_0^t \frac{Ft'}{\sqrt{m^2+(Ft')^2}}dt',$$
$$= \int_0^t \frac{Ft'}{m\sqrt{1+\left(\frac{F}{m}\right)^2 t'^2}}dt' = \left[\frac{m}{F}\sqrt{1+\left(\frac{F}{m}\right)^2 t'^2}\right]_0^t = \tag{112}$$
$$= \sqrt{t^2+\left(\frac{m}{F}\right)^2} - \frac{m}{F}.$$

where we have assumed that the motion begins at $s = 0$ at $t = 0$.

Let us study the signal transmission between a fixed observer at $s = 0$ and the accelerating body. A light signal emitted at time $t = \Delta t$ from $s = 0$ will present the following trajectory:

$$s = t - \Delta t \,, \tag{113}$$

where we have had again into account that in natural units $c = 1$.

At what time t_1 will the light signal reach the body? Of course, it will happen when $s_{light}(t_1) = s_{body}(t_1)$:

$$t_1 - \Delta t = \sqrt{t_1^2 + \left(\frac{m}{F}\right)^2} - \frac{m}{F} \,, \tag{114}$$

so that

$$t_1 - \Delta t + \frac{m}{F} = \sqrt{t_1^2 + \left(\frac{m}{F}\right)^2} \,, \tag{115}$$

$$\begin{aligned} t_1^2 + \left(\tfrac{m}{F}\right)^2 + (\Delta t)^2 - 2\tfrac{m}{F}\Delta t + 2\left(\tfrac{m}{F} - \Delta t\right) t_1 = \\ = t_1^2 + \left(\frac{m}{F}\right)^2 , \end{aligned} \tag{116}$$

$$t_1 = \frac{2\frac{m}{F}\Delta t - (\Delta t)^2}{\frac{m}{F} - \Delta t} \tag{117}$$

Light signals sent after

$$\Delta t_{max} = \frac{m}{F} \tag{118}$$

will never reach the accelerating particle, in a completely analogous way eq. to (94), corresponding to the case of a free-falling body in the Schwarzschild metric.

On the other hand, there is an important difference between both situations. If we calculate the interval of proper time of the particle along its trajectory, we find

$$\begin{aligned} \Delta\tau = \int_0^t \sqrt{1 - v^2(t')}\, dt' = \int_0^t \frac{1}{\sqrt{1 - \left(\frac{F}{m}t'\right)^2}}\, dt' = \\ = \frac{m}{F} \operatorname{arcsinh}\left(\frac{F}{m}\, t\right). \end{aligned} \tag{119}$$

In the case of the uniformly accelerated particle, $\lim\limits_{t\to+\infty} \tau = +\infty$, while in the case of a free-falling body in the Schwarzschild metric $\Delta\tau$ tends to a finite value when $t \to +\infty$, as detailed in eqs. (99)-(101).

5 A closer analogy: Accelerated motion exponentially asymptotically tending to the speed of light

In the previous section we have compared the motion of a uniformly accelerated particle in Special Relativity with that of a free-falling body in the Schwarzschild metric. In this one, we will consider again an accelerated particle in the context of Special Relativity, but instead of a *uniformly* accelerated one it will be a particle tending exponentially asymptotically to the speed of light:

$$v = 1 - e^{-\alpha t} \,, \tag{120}$$

with α being a constant.

The relativistic moment of the particle will be given by the expression

$$p = \frac{mv}{\sqrt{1 - v^2}} = \frac{m(1 - e^{-\alpha t})}{\sqrt{2e^{-\alpha t} - e^{-2\alpha t}}} = \frac{m(e^{\alpha t} - 1)}{\sqrt{2e^{\alpha t} - 1}} \,. \tag{121}$$

Consequently, the force necessary for generating the temporal evolution of speed in eq. (120) should be

$$F = \frac{dp}{dt} = \frac{m\alpha e^{2\alpha t}}{\left(\sqrt{2e^{\alpha t} - 1}\right)^3} \,. \tag{122}$$

The above expression does not correspond to any force straightforwardly identifiable with any physical situation, but as F is finite $\forall t$ it is at least theoretically possible, as long as a sufficient power source is provided.

Let us study also signal transmission in this new analogy. The position of the particle over time may be obtained by integration of the speed function:

$$\begin{aligned} s(t) = \int_0^t v(t')dt' = \int_0^t (1 - e^{-\alpha t'})dt' = \\ = \left[t' + \frac{e^{-\alpha t'}}{\alpha}\right]_0^t = t - \frac{1}{\alpha}(1 - e^{-\alpha t}). \end{aligned} \tag{123}$$

Let us assume again that we send a light signal at time $t = \Delta t$ from $s = 0$, as in eq. (113). If it reaches the particle at time $t = t_1$, we have

$$t_1 - \Delta t = t_1 - \frac{1}{\alpha}(1 - e^{-\alpha t_1}) \,, \tag{124}$$

so that

$$t_1 = \frac{-1}{\alpha} \ln(1 - \alpha\Delta t) \,. \tag{125}$$

Again, we find the same kind of behavior as in eqs. (93) and (117): $t_1 \to \infty$ as $\Delta t \to \Delta t_{max}$. Signals after

$$\Delta t_{max} = \frac{1}{\alpha} \tag{126}$$

will never reach the particle.

Concerning proper time, its relation with Schwarzschild time t in this case will be

$$\begin{aligned} \Delta\tau = \int_0^t \sqrt{1 - v^2(t')}\, dt' = \int_0^t \sqrt{2e^{-\alpha t'} - e^{-2\alpha t'}}\, dt' = \\ = \int_0^t \sqrt{2} e^{\frac{-\alpha t'}{2}} \sqrt{1 - \frac{e^{-\alpha t'}}{2}}\, dt' \,. \end{aligned} \tag{127}$$

In order to solve the integral, we perform the following change of variables:

$$e^{\frac{-\alpha t'}{2}} = \sqrt{2} \sin\xi \,, \tag{128}$$

$$\frac{-\alpha}{2} e^{\frac{-\alpha t'}{2}} \; dt' = \sqrt{2} \cos\xi \; d\xi \,. \tag{129}$$

Then,

$$\begin{aligned}
\Delta\tau &= \tfrac{-4}{\alpha} \int_{\xi(0)}^{\xi(t)} \sqrt{1 - \sin^2\xi} \; \cos\xi \; d\xi = \\
&= \frac{-4}{\alpha} \int_{\xi(0)}^{\xi(t)} \cos^2\xi \; d\xi = \frac{-4}{\alpha} \int_{\xi(0)}^{\xi(t)} \frac{1}{2} \left(1 + \cos(2\xi)\right) \; d\xi = \\
&= \frac{-2}{\alpha} \left[\xi + \frac{\sin(2\xi)}{2} \right]_{\xi(0)}^{\xi(t)} = \frac{-2}{\alpha} \left[\xi + \sin\xi \; \cos\xi \right]_{\xi(0)}^{\xi(t)} = \\
&= \frac{-2}{\alpha} \left[\arcsin\left(\frac{e^{\frac{-\alpha t'}{2}}}{\sqrt{2}} \right) + \frac{e^{\frac{-\alpha t'}{2}}}{\sqrt{2}} \sqrt{1 - \frac{e^{-\alpha t'}}{2}} \right]_0^t = \\
&= \frac{2}{\alpha} \left(\frac{\pi}{4} + \frac{1}{2} - \arcsin\left(\frac{e^{\frac{-\alpha t}{2}}}{\sqrt{2}} \right) - \frac{e^{\frac{-\alpha t}{2}} \sqrt{2 - e^{-\alpha t}}}{2} \right) = \\
&= \frac{1}{\alpha} \left(\frac{\pi}{2} + 1 - 2 \arcsin\left(\frac{e^{\frac{-\alpha t}{2}}}{\sqrt{2}} \right) - e^{\frac{-\alpha t}{2}} \sqrt{2 - e^{-\alpha t}} \right) .
\end{aligned} \tag{130}$$

In this case, the interval of proper time measured by the particle needed to reach the speed of light would be

$$\lim_{t\to\infty} \Delta\tau = \frac{1}{\alpha} \left(\frac{\pi}{2} + 1 \right) . \tag{131}$$

Therefore, a finite value, as the proper time interval that a free-falling body would invert in arriving at the Schwarzschild radius.

6 Discussion

It is often emphasized that the mathematical singularity in the Schwarzschild radius is not a "physical" singularity, since it can be removed by a suitable change of coordinates, as the works by Kruskal and Szekeres confirm. However, the fact that there is not an intrinsic singularity at the Schwarzschild radius does not automatically prove that it can be actually crossed. Why should it not be crossed -could someone object- if there is not a physical barrier in it? Simply because, according to calculations correctly performed, it takes an infinite time to attain it. However, it could be still objected that an infinite time is only required from the perspective of a distant observer, while a comoving observer would measure just a finite interval.

In the Introduction we have argued against the common explanation of the fact that the infinite time of asymptotic approach which a distant observer would perceive is basically an "optical illusion" due to the *delay* in the arrival of light signals proceeding from the body. As we have pointed out, in a time-independent metric, it is completely nonsensical to hold that the time inverted by a light signal along a distance can be greater than the time spent by a massive body to travel the same distance. Calculations in subsection 3.3 reinforce our reasoning and show that the "apparent" asymptotic movement is only possible since the actual motion of the body is also genuinely asymptotic.

In opposition to the thesis which we defend in this paper, i.e., that the asymptotic approach to the event horizon of a Schwarzschild black hole by a free-falling body is *real*, it could be objected as well that if the body is *still* falling and has not yet crossed the event horizon we should be able to recover any light signal that we send towards the body in any moment of time. If the body is still "in front of" the horizon, the light signals could be "reflected" by it and return to the transmitter. Instead, in section 3.4 we have seen that there is a time limit, Δt_{max}, after which no light signal that we send towards the body will ever return. Does it mean that after that time the body has become situated "behind" the horizon? Certainly not. By comparing equations (15) and (55), describing respectively the speed of a massive body and that of a light signal, it is straightforward to notice that both tend to the same value when approaching the Schwarzschild radius, as $\lim_{r\to 2M} \sqrt{1 - \frac{1}{A^2}\left(1 - \frac{1}{2M}\right)} = 1$. Therefore, the free-falling body is asymptotically tending to the speed of light as it approaches the event horizon. In sections 4 and 5 two accelerated motions with speeds asymptotically approaching the speed of light have been studied in the context of Special Relativity: the first one a uniformly accelerated motion, the second one an accelerated movement with an exponential asymptotic tendency to the speed of light. In both cases, there is also a limit time after which no light signal sent towards the body will ever reach it. Hence, the fact of not being able to recover the light signals sent towards a free-falling body in the Schwarzschild metric after a certain limit time may be perfectly explained by its asymptotic approximation to light speed. Thus, it does not provide at all an argument in favor of the crossing of the event horizon.

On the other hand, if we accept that the trajectory of a free-falling body towards the Schwarzschild radius is truly asymptotic, how can be this fact conciliated with the finite time that a comoving observer would measure? According to our judgment, that fact is not as paradoxical as it could seem if we have into account the two conditions that affect the passage of time in General Relativity: (i) the speed of the moving body, and (ii) the gravitational field itself. In Special Relativity, as gravitational fields are excluded, only the speed of the moving body affects the course of time in it. When a free-falling body is approaching the event horizon, it is both moving towards more intense gravitational fields and attaining speeds closer to light. Consequently, it should not be surprising that for the free-falling body the march of time becomes “frozen”.

In order to illustrate this phenomenon, as the understanding of several aspects in General Relativity seems to be obscure, we have proposed an analogy with Special Relativity, in which we have at least the effect of time dilation due to speed. In the case of a uniformly accelerated movement, proper time tends to infinity as Minkowski time does, and therefore this situation is not comparable to the free-falling body in the Schwarzschild metric. Instead, the case of an accelerated particle exponentially asymptotically tending to the speed of light results to be completely analogous to the motion of the free-falling body asymptotically approaching the Schwarzschild radius. As a matter of fact, if the movement of the exponentially asymptotically accelerated particle were indefinitely maintained over time, it would reach the speed of light in a finite proper time. In truth, it could be argued that such a kind of movement cannot be indefinitely held over time, as an infinite amount of energy would be required. However, no fundamental law avoids keeping it during an arbitrarily long period of time, as long as a sufficient power supply be available. If we *analytically* extend that behavior to infinite, a finite proper time would be obtained, as we have already pointed out. No sensible physicist would dare to state that ”the exponentially accelerated particle has already reached the speed of light, but we do not perceive it as the signals that it sends to us arrive with an increasingly big delay”. Stating it for the free-falling body in the Schwarzschild metric is equally absurd, but due to a historical concatenation of mistakes is up today the dominating paradigm. Certainly, it could be demurred that, contrarily to the exponentially accelerated particle, the free-falling movement in the Schwarzschild metric can be indefinitely maintained. Well, according to General Relativity *alone* perhaps it could be, but thermodynamically it is accepted that black holes must emit radiation [3, 4] and that the collapsing matter around them should radiate as well [6, 7, 9, 15]. Therefore, as sooner or later the fee-falling body shall completely “vanish” due to thermal radiation, its process of asymptotic approach to the Schwarzschild radius can be neither indefinitely held over time.

If the event horizon of a black hole can never be crossed by a free-falling body outside it, and therefore black holes cannot grow but only accumulate matter around them, the process itself of black hole formation from gravitational collapse should be impossible [6, 10, 11]. Furthermore, gravitational collapse should be itself an asymptotic process [12–14].

On the other hand, it could still be argued that all the calculations presented in this paper have been performed in the Schwarzschild metric, and it does not “represent the geometry properly” in the vicinity of the Schwarzschild radius (nor inside it), according to most authors, for instance [1, see p. 301]. Probably we could agree that the Schwarzschild metric is problematic *in* the Schwarzschild radius, but not *around* it. As a matter of fact, the metric itself prevents any body from arriving at the problematic point.

Concerning “analytical extensions” of Schwarzschild solution, as Kruskal-Szekeres coordinates, those imply the use of obscure variables with a dubious physical meaning [16, 17]. Certainly, we could say that all coordinate systems are equally legitimate in General Relativity, as the theory was precisely constructed under that assumption, but not they all allow us to glimpse the physical implications of calculations with identical clarity. To begin with, by definition the Schwarzschild metric is a *time-independent* metric, but this fact becomes completely concealed in Kruskal-Szekeres, Lemaître or Eddington-Finkelstein coordinates. Schwarzschild coordinates constitute a privileged coordinate system, as they directly reveal what they signify.

In section 3.6, circular orbits of a planet around a black hole have been calculated in order to emphasize the fact that the movement of an astronomical object around a black hole (or around any star) is *cyclical* in the Schwarzschild coordinate time t. Values of t are not mere mathematical constructions, but correspond to concrete physical phenomena, for instance, the number of turns of a planet around the central mass. Before a free-falling body reached the Schwarzschild radius, a planet around the black hole should travel infinite times in turn of it.

7 Conclusion

The asymptotic approach of a free-falling body towards the event horizon of a Schwarzschild black hole is mathematically analogous to that of an accelerated particle asymptotically tending to the speed of light in Special Relativity. Consequently, if in the case of the accelerated particle in Special Relativity it is clear that it never reaches the speed of light, it should not be disputed that the free-falling body never attains the event horizon of the Schwarzschild black hole, even if in both cases only a finite interval of proper time would be required.

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