Title:

An Inconsistency Between the Gravitational Time Dilation Equation and the Twin Paradox

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Abstract:

It is shown in this monograph that the well-known Equivalence Principle relating gravitation and acceleration, together with the Gravitational Time Dilation Equation, produces results that contradict the required outcome at the reunion of the twins in the famous twin 'paradox'. The Equivalence Principle Version of the Gravitational Time Dilation Equation (the "EPVGTD" equation) produces results that say that, when the traveling twin (he) instantaneously changes his velocity, in the direction TOWARD the distant home twin (her), that he will conclude that her age instantaneously becomes INFINITE. It is well known that, according to her, at their reunion, she will be older than him, but both of their ages will be finite. They clearly must be in agreement about the correspondence between their ages, because they are co-located there.

Section 1. The Gravitational Time Dilation Equation

The Gravitational Time Dilation Equation is described in Wikipedia:

https://en.wikipedia.org/wiki/Gravitational_time_dilation#References .

It says, in particular, that for two clocks in a constant and uniform gravitational field of force per unit mass "g", separated by the constant distance "d" in the direction of the field, the clock that is closer to the source of the field will run slower than the other clock, by the factor exp(g d).

The equivalence principle says that for two clocks that are accelerating with the same acceleration "A", separated by the constant distance "d" in the direction of the acceleration, the trailing clock will run slower than the other clock, by the factor exp(A d). The two values "g" and "A" are numerically the same.

Section 2. A Possible Proof that Negative Aging Doesn't Occur in Special Relativity

Consider the following scenario:

At some instant, the perpetually-inertial "home twin" (she) is 20 years old, and is holding a display that always shows her current age. Facing her is the "helper friend" (the "HF") of an observer (he) who is "d" ly away to her right. Both the HF and he are also 20 years old, and are stationary wrt her at that instant. Like her, he and the HF are each holding a display that always shows their current ages.

Now, suppose that he and his helper then both start accelerating at a constant "A" ly/y/y toward the right. He knows that his helper friend (the HF) is then ageing at a constant rate that is slower than his own rate of ageing, by the factor exp(Ad).

An instant later, his display shows the time 20 + epsilon_1, where epsilon_1 is a very small positive number. He knows that HF's display shows the time 20 + epsilon_2, where epsilon_2 = epsilon_1 / exp(Ad).

She can still see HF's display (because HF has only moved an infinitesimal distance away from her, to her right). She will see that HF's display reads 20 + epsilon_1 / exp(Ad). And likewise, HF can still see her display. What does HF see on her display? Does HF see that she is now slightly younger than 20? No! It would clearly be absurd for someone essentially co-located with her to see her get younger. HF would see her display reporting that she was some very small amount epsilon_3 OLDER that she was at the instant before the acceleration. HF then sends a message to him, telling him that she was 20 + epsilon_3 right then. When he receives that message, he then knows that her current age, when he was 20 + epsilon_1, was 20 + epsilon_3. So he KNOWS that she didn't get younger when he accelerated away from her. That contradicts what CMIF simultaneity says.

In the above, I asked

"What does HF see on her display?".

And I answered

"HF would see her display reporting that she was some very small amount epsilon_3 OLDER that she was at the instant before the acceleration."

Since the above argument makes use of very small (unspecified) quantities, it could be argued that time delays due to the speed of light might also need to be taken into account when describing what the HF sees on her display.
But I think any such concerns can be addressed by pointing out that the separation "d" between him and her can be made arbitrarily large, and CMIF simultaneity says that the amount of negative ageing that occurs is proportional to their separation. Since the errors involved due to the finite speed of light between her and the HF are independent of the distance "d", those errors become negligible for sufficiently large "d".

There is another argument that shows that the HF ("Helper Friend") can't conclude that the home twin (she) is less than 20 years old when the HF is 20 + epsilon_2. We can require that she transmits NO light messages to him when she is 20 years old or younger. Suppose the HF receives a light message from her when he is 20 + epsilon_2 years old. By the requirement, she must have been older than 20 years old when she sent that message. When the HF receives that message, he knows that she must be older than when she sent the message, so she must definitely be older than 20 years old when the HF is 20+epsilon_2. Therefore, she did NOT get younger, according to him, when he accelerated away from her.

A still simpler argument is that, if the HF ever concluded that she got younger when he accelerated away from her, he would be concluding that she was less than 20 years old at that instant of his acceleration. But the HF was co-located with her when she was less than 20, and he couldn't be two places a that same instant.

It seems to me that, once the distant accelerating observer has a way to set up an array of clocks (with attending observers) that he can use to define his concept of "NOW" (analogous to how Einstein did it for perpetually-inertial observers), it becomes impossible for the home twin to age negatively, according to the distant accelerating observer. It's true that those clocks aren't synchronized as they are in the perpetually-inertial case, but they don't have to be, since the distant accelerating observer knows exactly how the rates of those clocks compare to his own clock.

The way the accelerating observer (the "AO") defines his "NOW" instant at distant locations comes directly from the gravitational time dilation equation, via the equivalence principle. It says that a "helper friend" (HF) who always is accelerating exactly as the AO is accelerating, with acceleration A, will age at a rate that is a fixed known ratio of the AO's rate. The given HF and the AO are always a constant distance "d" apart. If the chosen HF is BEHIND the AO (compared to the direction of the acceleration), that HF will age SLOWER than the AO by the factor exp(A d). To keep things as simple as possible, we can always let all of the HF's and the AO's ages be the same, immediately before they all start accelerating. Then the ratio of the age of the "behind" HF's age to the AO's age is just 1/exp(A d). And if, instead, another HF is AHEAD of the AO (compared to the direction of the acceleration), then the ratio of that "ahead" HF's age to the AO's age is just exp(A d). (Of course, different "behind" HF's will have different distances "d" to the AO, and likewise for the "ahead" HF's.) So, at some instant T in the AO's life, he computes that the original "behind" HF's current age is T/exp(A d). Or, alternatively, he computes that the "ahead" HF's current age is T exp(A d) The way he SELECTS the HF from among all possible HF's (both ahead and behind him) is such that the chosen HF is momentarily co-located with the home twin (her) at the instant the AO wants to know her current age.

So, if all of the above is correct, that allows the AO to construct an array of (effectively) synchronized clocks and helper observers attached to him, similar to what a perpetually-inertial observer can do, that can put an observer momentarily co-located with the distant twin (her) at the instant in the AO's life when he wants to know her current age. And in both the perpetually-inertial and the accelerated cases, it would be ABSURD for that momentarily co-located observer to observe a large and abrupt change in her age at that instant.

Section 3. Instantaneous Velocity Changes in the Equivalence Principle Version of the Gravitational Time Dilation Equation

When using the CMIF simultaneity method, the analysis is GREATLY simplified by using instantaneous velocity changes, rather than finite accelerations that last for a finite amount of time. So I decided to try using instantaneous velocity changes in the Equivalence-Principle Version of the Gravitational Time Dilation equation (the "EPVGTD" equation). The result (assuming I haven't made a mistake somewhere) is unexpected and disturbing. My analysis found that the age change of the HF, produced by an instantaneous velocity change by the AO and the HF, from zero to 0.866 lightseconds/second (ls/s), directed toward the home twin (her), is INFINITE!

I'll describe my analysis, and perhaps someone can find an error somewhere.

Before the instantaneous velocity change, the AO (he), HF, and the home twin (she) are all mutually stationary. She and the HF are initially co-located, and the AO (he) is "d" lightseconds away from her and the HF.

I start by considering a constant acceleration "A" ls/s/s that lasts for a very short but finite time of "tau" seconds. That acceleration over tau seconds causes the rapidity, theta, (which starts at zero) to increase to

theta = A tau ls/s,

and so we get the following relationship:

A = theta / tau.

We will need the above relationship shortly.

(Rapidity has a one-to-one relationship to velocity. Velocity of any object that has mass can never be equal to or greater than the velocity of light in magnitude, but rapidity can vary from -infinity to +infinity.)

We want the velocity, beta, to be 0.866 ls/s after the acceleration. Rapidity, theta, is related to velocity, beta, by the equation

theta = arctanh (beta) = (1/2) ln [(1 + beta) / (1 - beta)].

("arctanh" just means the inverse of the hyperbolic tangent function.)

So velocity = 0.866 corresponds to a rapidity of about 1.317 ls/s.

The "EPVGTD" equation says that the acceleration A will cause the HF to age faster than the AO by the factor exp(A d), where d is the constant separation between the AO and the HF.

Note that the argument in the exponential exp(A d) can be separated like this:

exp(A d) = [exp(d)] sup A,

where "sup A" means "raise the quantity exp(d) to the power "A" ". The rationale for doing that is because the quantity exp(d) won't change as we make the acceleration greater and greater, and the duration of the acceleration shorter and shorter. That will make the production of the table below easier.

The change in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau [exp(d)] sup A,

because [exp(d)] sup A is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

But we earlier found that A = theta / tau, so we get

tau [exp(d)] sup {theta / tau}

for the change in the age of the HF due to the short acceleration. So we have an expression for the change in the age of the HF that is a function of only the single variable tau ... all other quantities in the equation (d and theta) are fixed. We can now use that equation to create a table that shows the change in the age of the HF, as a function of the duration of the acceleration (while keeping the area under the acceleration curve constant).

In order to make the table as easy to produce as possible, I chose the arbitrary value of the distance "d" to be such that

exp(d theta) = 20000.

Therefore we need

ln[exp (d theta)] = d theta = ln (20000) = 9.903,

and since theta = 1.317, d = 7.52 lightseconds.

If we were creating this table for the CMIF simultaneity method, we would find that as the duration of the acceleration decreases (with a corresponding increase in the magnitude of the acceleration, so that the product remains the same), the amount of ageing by the HF approaches a finite limit. I.e., in CMIF, eventually it makes essentially no difference in the age of the HF when we halve the duration of the acceleration, and make the acceleration twice as great.

But here is what I got for the EPVGTD simultaneity method:

(in the table, "10sup4" means "10 raised to the 4th power".)

tau	(tau) (2000)sup(1/tau)
1.0	2x10sup4 = 20000
0.5	2x10sup8
0.4	2.26x10sup10
0.3	6.3x10sup13
0.2	0.64x10sup21
0.1	1.02x10sup42
0.01	1.27x10sup428
0.001	? (My calculator overflowed at 10sup500)

Clearly, for the EPVGTD simultaneity method, the HF's age goes to infinity as the acceleration interval goes to zero. That seems like an absurd answer to me. And it is radically different from what happens with CMIF simultaneity, where the HF's age quickly approaches a finite limit as tau goes to zero.

Section 4. Instantaneous Velocity Changes in the Equivalence Principle Version of the Gravitational Time Dilation Equation - Revised Model (the LGTD Model)

I repeated my previous analysis of the instantaneous increase in the home person's (her) age (according to the accelerating person, AO, him), according to the Equivalence Principle Version of the Gravitational Time Dilation Equation, (the "EPVGTD" equation), and replaced it with the new equation, which I'll call the "Linearized Gravitational Time Dilation Equation", (the "LGTD" equation). I simply replace the exponential exp(A d) with the quantity (1 + A d). (This is the same approximation that Einstein used in his 1907 paper). In what follows below, I'll repeat each affected calculation that I made in my last post, and show the revised calculation.

[Previous]:

The "EPVGTD" equation says that the acceleration A will cause the HF to age faster than the AO by the factor exp(A d), where d is the constant separation between the AO and the HF.

[Revised]:

The "LGTD" equation says that the acceleration A will cause the HF to age faster than the AO by the factor (1 + A d), where d is the constant separation between the AO and the HF.

(Both of the above are for the case where the AO accelerates TOWARD the unaccelerated person (her).)

[...]

[Previous]:

The change in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau [exp(d)] sup A,

because [exp(d)] sup A is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[Revised]:

The change in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 + A d),

because (1 + A d) is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[Previous]:

But we earlier found that A = theta / tau, so we get

tau [exp(d)] sup {theta / tau}

[Revised]:

But we earlier found that A = theta / tau, so we get

tau (1 + [(theta d) / tau] = tau + (theta d)

[...]

It is still true that d = 7.52 lightseconds and theta = 1.317.

Therefore the revised result is that the change in HF's age during the acceleration is equal to

tau + (theta d) = tau + (1.317)(7.52) = tau + 9.904.

So, in the revised model, as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches 9.904 seconds from above. So the HF's age increased by a finite amount, unlike the infinite increase that the EPVGTD equation gave.

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds older than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds older than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously increases by 6.51 seconds, so the LGTD and CMIF don't agree.

Section 5. LGTD, When the Direction of the Velocity Change is AWAY From Her

I just repeated my previous analysis of instantaneous velocity changes in the "linearized" (LGTD) version of the equivalence principle version of the gravitational time dilation equation, but for the case where the instantaneous velocity change is AWAY FROM the home twin (her). The result is exactly like the previous result, except that she instantaneously gets YOUNGER, not older. (This contradicts my previous possible proof that negative ageing doesn't occur.)

Below, I'll repeat the previous calculations, and show the changes.

[Previous]:

I simply replace the exponential exp(A d) with the quantity (1 + A d).

[New]:

I simply replace the exponential exp(-A d) with the quantity (1 - A d).

[Previous]:

The "LGTD" equation says that the acceleration A will cause the HF to age FASTER than the AO by the factor (1 + A d), where d is the constant separation between the AO and the HF.

(The above is for the case where the AO accelerates TOWARD the unaccelerated person (her).)

[New]:

The "LGTD" equation says that the acceleration A will cause the HF to age SLOWER than the AO by the factor (1 - A d), where d is the constant separation between the AO and the HF.

(The above is for the case where the AO accelerates AWAY FROM the unaccelerated person (her).)

[...]

[Previous]:

The change in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 + A d),

because (1 + A d) is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[New]:

The change in the age of the HF, caused by an acceleration "A" that lasts "tau" seconds is just

tau (1 - A d),

because (1 - A d) is the constant rate at which the HF is ageing, during the acceleration, and tau is how long that rate lasts.

[Previous]:

But we earlier found that A = theta / tau, so we get

tau (1 + [(theta d) / tau]) = tau + (theta d)

[New]:

But we earlier found that A = theta / tau, so we get

tau (1 - [(theta d) / tau]) = tau - (theta d) [both Previous and New]:

It is still true that d = 7.52 lightseconds and theta = 1.317.

[Previous]:

Therefore the revised result is that the change in HF's age during the acceleration is equal to

tau + (theta d) = tau + (1.317)(7.52) = tau + 9.904.

[New]:

Therefore the revised result is that the change in HF's age during the acceleration is equal to

tau - (theta d) = tau - (1.317)(7.52) = tau - 9.904.

[Previous]

So, in the revised model, as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches 9.904 seconds from above. So with an instantaneous velocity change, the HF's age INCREASED instantaneously by a finite amount.

[New]

So, in the revised model, as tau approaches zero (to give an instantaneous velocity change), the change in the HF's age during the speed change approaches -9.904 seconds from above. So with an instantaneous velocity change, the HF's age DECREASED instantaneously by a finite amount.

[Previous]:

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds OLDER than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds OLDER than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously increases by 6.51 seconds, so the LGTD and CMIF don't agree.

[New]:

Before the instantaneous velocity change, the AO, the HF, and the home twin (she) were all the same age. She and the HF were co-located. So after the instantaneous speed change, the AO hasn't aged at all, but the HF is 9.904 seconds YOUNGER than he was before the speed change, according to the AO. And since she and the HF have been colocated during the instantaneous speed change, they couldn't have ever differed in age during the speed change ... it would be absurd for either of them to see the other have an age different from their own age at any instant. So after the instantaneous speed change, the AO must conclude that she and the HF both instantaneously got 9.904 seconds YOUNGER than they were immediately before the speed change.

By comparison, the CMIF simultaneity method says that the AO will conclude that her age instantaneously decreases by 6.51 seconds, so the LGTD and CMIF don't agree.

Section 6. What to Make of All These Different and Contradictory Results?

The "EPVGTD Equation" (the one with the exponential), says that, if the AO (he) instantaneously changes his velocity in the direction TOWARD the home time (her), she instantaneously gets INFINITELY older, according to him. That's nonsense, because it gives incorrect ages for the twins when they are reunited.

On the other hand, when he instantaneously changes his velocity in the direction AWAY from her, the EPVGTD equation says that her age doesn't change instantaneously. While it's not certain that that result itself is incorrect, it seems to result in an inconsistency at the reunion. The EPVGTD equation says that, with zero acceleration, he and all the HF's age at the same rate. That seems to require that on the outbound and inbound legs, his conclusion about the correspondence between his and her ages must be the same. And clearly, on the OUTBOUND leg, he MUST say she is ageing SLOWER than he is, by the factor gamma. So he must say that, on the INBOUND leg, she is ageing slower by the factor gamma. But in that case, their conclusions about the correspondence between their ages at the reunion won't be consistent: she says she is the OLDER, but he says she is the YOUNGER. So his conclusions won't match her conclusions at the reunion, which is impossible since they are colocated then and they MUST agree about the correspondence between their ages then.

So much for the EPVGTD equation. What about the LGTD equation? The linearized equation (the LGTD equation) gives results that are qualitatively similar to the CMIF simultaneity method: her age instantaneously changes, according to him, during his instantaneous velocity change (instantaneously increasing when his momentarily infinite acceleration is TOWARD her, and instantaneously decreasing when his momentarily infinite acceleration is AWAY FROM her). But the AMOUNT of the instantaneous change is greater than CMIF says it should be. It is interesting that the amount of the instantaneous age changes would be exactly the same for CMIF and LGTD if the linearized equation multiplied the distance "d" by the velocity "v", rather than by the rapidity "theta". But, in determining the velocity effect obtained by integrating the acceleration "A", it IS necessary to use the rapidity "theta", not the velocity "v", as the variable of integration. (Taylor and Wheeler go over this in detail).

WHY does the EPVGTD equation fail so miserably in this example? Isn't the GTD equation a well-established result in general relativity? And the equivalence principle is certainly well-established. Is the GTD equation WRONG?

And WHY goes the LGTD work better than the EPVGTD, at least qualitatively? The LGTD should be a justified approximation of the EPVGTD only when the argument (A d) is small, and an infinite "A" (even though it lasts only an infinitesimal time) certainly isn't small! The LGTD equation shouldn't give results that are even qualitatively correct, but it does. Why?