In Situ Experiment on Fractal Corresponds with Cosmological Observations and Conjectures

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Abstract

Fractal geometry is an accepted mathematical description of nature. Standard examples include trees, clouds and waves. From the earliest galaxy surveys, it was questioned whether the universe is also fractal. For this to be true, it would mean the cosmological principle, the foundation assumption of general relativity, would be challenged. The 2012 WiggleZ Dark Energy Survey found in agreement with fractal-cosmology proponents that the small-scale observable universe is fractal; however, beyond this, the large-scale—and thus the universe—is not fractal. Current fractals models assume a forward—progressive—looking trunk, bough, branch and twig 'Romanesco broccoli' structural of fractal development. In this paper, an alternative backward—regressive—looking, twig, branch, bough, then trunk model—observed from within (in situ) the growing fractal— was examined. Can a growing fractal model from this alternative perspective correspond with and explain all cosmological observations and conjectures? An experiment was conducted on a 'simple' (Koch snowflake) fractal. New triangle sizes, of arbitrary size, were held constant and earlier triangles were allowed to expand as the fractal set iterated (grew). Classical kinematic equations—velocity and acceleration—were calculated for the total area total and the distance between arbitrary points. Hubble-Lemaitre's Law, accelerated expansion, and changing size distribution, all corresponding to cosmological observations and conjectures were tested for. Results showed: the area expanded exponentially from an arbitrary starting size; and as a consequence, the distances between measured points—from any location within the set—receded away from the 'observer' at increasing velocities and accelerations. It was concluded, at the expense of the cosmological principle, that the fractal is a geometrical match to the cosmological observations and conjectures, including the inflation epoch, Hubble-Lemaitre and accelerated expansion. Large-scale smoothness is a property of a fractal and is expected. From this model: from planet Earth, we are observing within the branches looking out and back in time to the boughs (the large quasar structures) and the trunk (the CMB—which was once seedling size the Planck area) of a fractal structured universe. The universe is not only fractal, it is a fractal. Other problems—including the cosmological catastrophe are addressed with the fractality. It was concluded the fractal may offer a direct mechanism to the cosmological problem and can further explain the quantum problem—unifying the two realities as being two aspects of the same geometry.

Keywords: Cosmological Principle, Dark Energy, Inflation, Hubble-Lemaitre's Law, Quantum Mechanics

1 INTRODUCTION

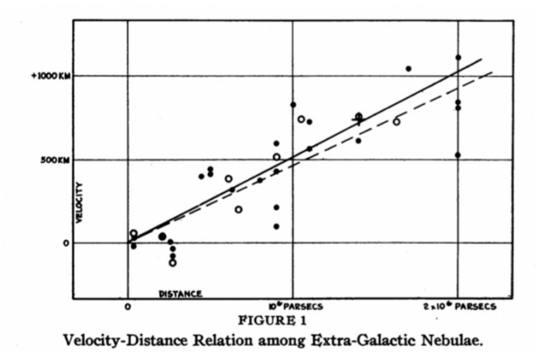
This paper should be read in conjunction with the author's complement paper: *The* Fractal *Corresponds to Light and Quantum Foundation Problems*[1].

From observations, the universe is not only expanding from a conjectured 'big bang' singularity point beginning, but this expansion is accelerating. These claims are derived, respectively, by the late 1920s Hubble galaxy observations[2] (Figure 1a) and Lemaitre[3] mathematics; the 1964 Penzias and Wilson discovery of the cosmic microwave background (CMB) [4] (Figure 1c); and the 1998 supernova observations of accelerated expansion by Riess et al. [5] and Perlmutter et al.[6]. These discoveries, along with the so-termed 'dark matter'—that is pulling and holding the universe together—form The Standard 'lambda, cold dark matter' (ΛCDM) Model of Cosmology. As successful and great a human achievement as this model is; it is, however, by all accounts, in a self-titled crisis. Nobody—it is claimed—has any idea how to 'make sense of it', let alone be able to marry it in any 'simple' way with its equal enigma, our quantum reality.

Notwithstanding these problems, the standard model is also based on Albert Einstein's 1915—very successful—General Relativity which in turn assumes—fundamentally—that the cosmological principle holds on all cosmological scales. The cosmological principle assumes that no matter one's observation position in the universe—be it, for instance, at the outer edge of the observable universe—one should observe similar 'smooth and uniform' (homogenous) distribution patterns throughout and in all (isotropic) directions. By the cosmological principle, the universe should be homogenous and isotropic on all cosmic scales; however, by observations, it is found not to be, and this is the root problem.

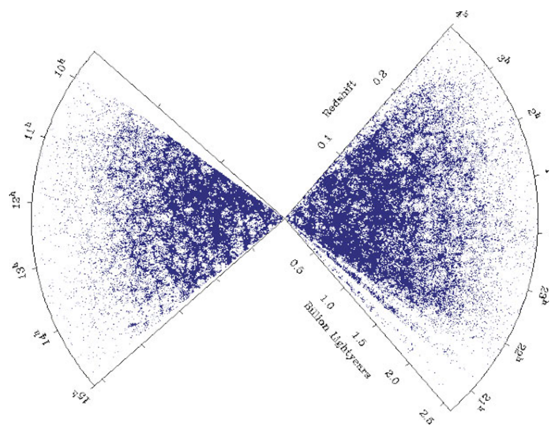
The first large-scale astronomical surveys of galaxies, beginning in the 1980s, quickly revealed this counter of the cosmological principle. They found, (Figure 1 b and c) as we look further out, galaxies are distributed in 'clusters' followed by 'super-clusters'—at least on 'small-scales'

a



(a) The original Hubble Diagram show the relationship between the recession velocity of galaxy by their distance[2];

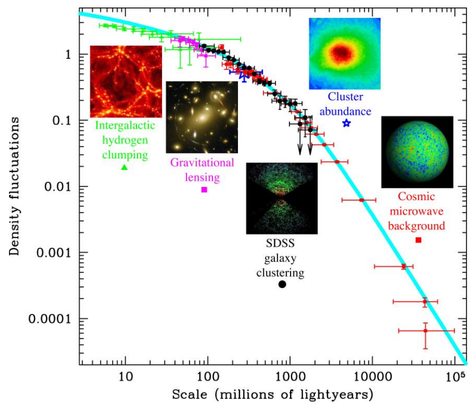
b



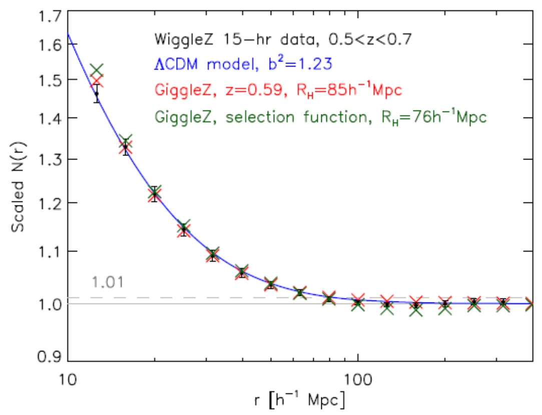
(b) 2003 2df Redshift Survey among the first surveys to show small scale 'fractal' clustering[7];

c

d



(c) The Log. mass-density curve of the universe by measurement method showing rough and dense galaxy distribution near Earth out to the smooth CMB 'edge';



(d) The WiggleZ Dark Energy Survey figure 13, page 16 of the Wigglez survey, corresponding to 'a' and revealing changing galaxy distributions from small-scale to large-scale [8].

Figure 1. Redshift Survey Evolution.

1.1 Fractal-cosmology

One geometric candidate— posited soon after its conception and long after Einstein's work—to explain this 'small-scale galactic structure anomaly', was the fractal [9]. Fractals are known as the geometry of chaos[10] and are defined by the 'epic' imagery of the Mandelbrot Set—named after the discoverer and fractal name giver, Benoit Mandelbrot. They are a scale-free mathematical object that is based on the repetition or iteration of a 'simple' pattern (for the Mandelbrot set, $z_n + 1 = z_n{}^2 + c$ [9]) and are a well-accepted mathematical description of our reality, with examples including trees, clouds, and market prices. From fractals a new field of cosmology—fractal-cosmology—was formed [11]. Fractals do not appear homogenous nor isotropic and thus would challenge the standard model.

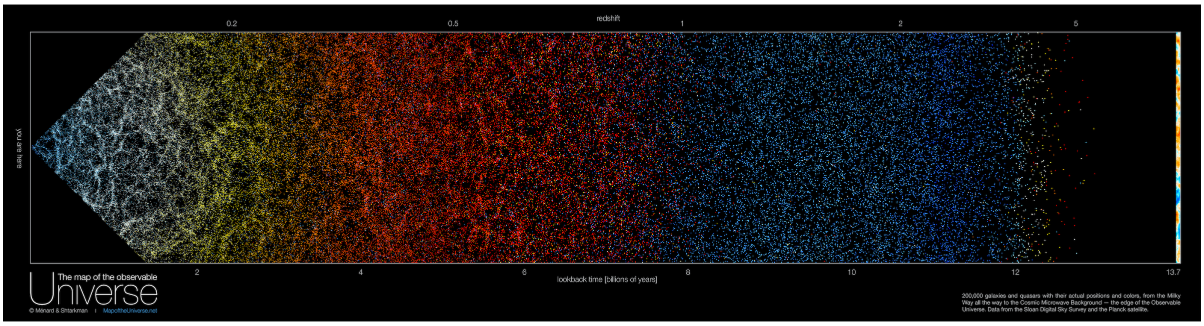
In the late 1990s, a debate ensued over whether the observed clustering and super clustering galaxy distribution is fractal [12]. Proponents—the fractal-cosmologists—argued that, with better technology, even larger galactic structures would be discovered beyond the then observed [12],[13],[14],[15],[16]. The opponents, Hogg and others, argued: yes, the universe appears rough and fractal on the small-scale, but on the large-scale universe it is smooth and therefore, the cosmological principle holds[17].

The debate came to a head with the findings of the 2012 WiggleZ Dark Energy Survey[8] (Figure 1d), and others [18]. It was concluded—but granted—that the universe does indeed show direct evidence of small-scale fractal galaxy distribution for distances less than 70 to 100 Mega parsecs away (3 billion light-years); however, the universe is assumed overall 'smooth', homogenous and isotropic, beyond this on large-scales. As it stands today—for cosmology—'fractals are out'[19].

Meanwhile, not long after the WiggleZ survey paper was released, improved large-scale surveys were revealing 'very large', 'thin', and 'old' galactic structures in the assumed 'smooth universe'. They are the 4 billion light-years sized Huge 'Large Quasar Group'[20] and the 10 billion light-years sized Hercules Corona Borealis Great Wall [21]). These structures are beyond the small—granted fractal—

scale and contradict all of the above fractal rebuttals. It seems the debate over a fractal universe had passed and nothing has been made of the LQGs.

Figure 2 is latest 2020 DESI survey map of the universe. The DESI survey is an exquisite visualisation of the observable universe showing the evolution and distribution of galaxies and the absolute differences between small-scale and large-scale structure. This map—'the map of the observable universe'[22]—is arguably not in accord with the cosmological principle. Viewing from the outreaches is not the same as viewing from planet Earth. To the left, from planet Earth, we see the clustering and super-cluster that were observed in the early surveys. This is followed by (firstly) the red and the blue 'old' (LCQ—quasar) smoothing out to the 'origins' CMB limit.



Showing the changing distribution and demographics of galaxy structure observed from Earth (left) out to the cosmic microwave background (right).

Figure 2. Map of the Observable Universe[22].

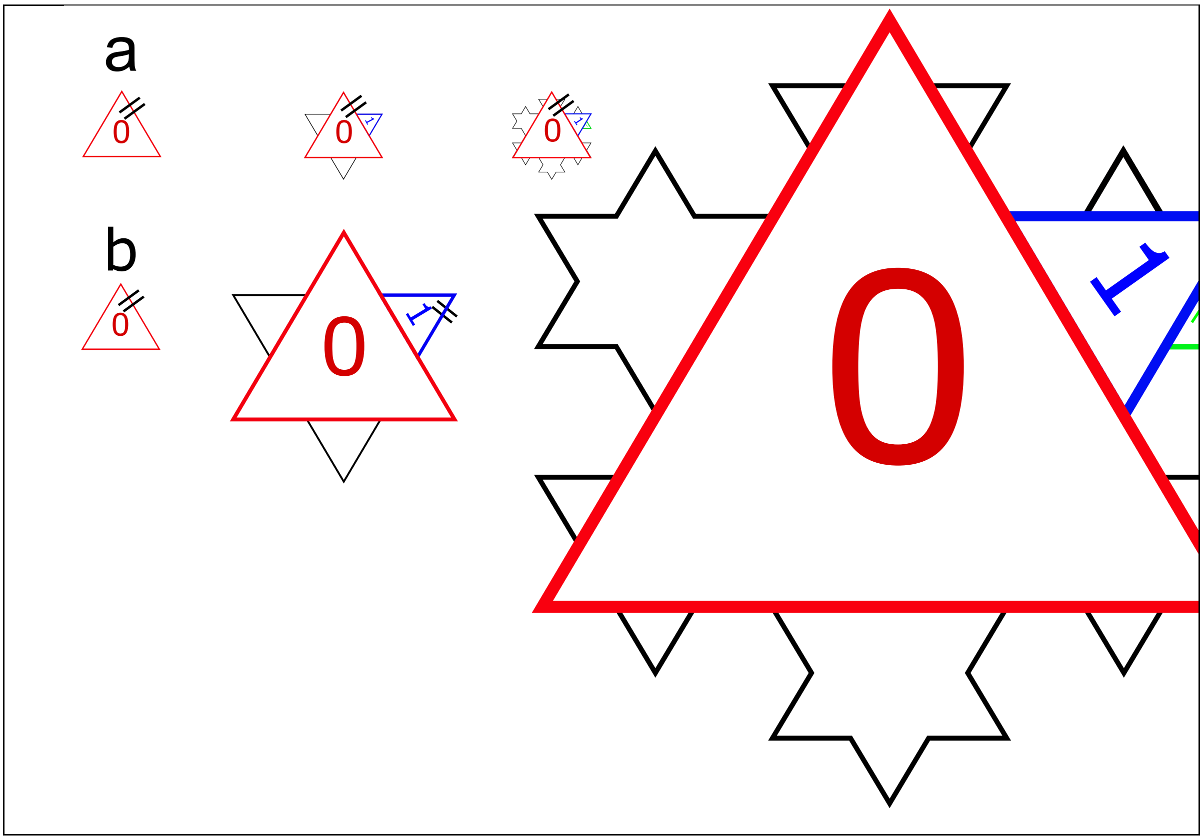
Can a different perspective or model of the fractal, that assumes general relativity a given, help explain all of the cosmic observations and conjectures? Is the 'rough' to 'smooth' pattern—as shown in Figure 2—expanding from a point beginning exactly what one would expect to observe if observing within (in situ) a fractal? Is it that the universe—as the fractal cosmologists currently claim—is not only fractal but that it is a fractal in its totality? Do we have the fractal model wrong? To test these questions, we need to study the growth of a fractal from the perspective of a fractal.

1.2 An inverted fractal model

Current fractal cosmology studies assume a traditional or classical perspective of fractal growth, popularised by the view of the Romanesco broccoli. This is a 'forward-looking' *progressive* perspective into or at the fractal set is best demonstrated by the Koch snowflake fractal.

Figure 3 a shows this convergent 'snowflake' fractal structure emerges in around 7 ± 2 iterations by the addition of new but diminishing sized bits (blue 1 and green 2) to the initial constant in size (thatched red '0') triangle bit. This progressive perspective of the fractal is explained and analysed by the author in the paper 'The Fractal Corresponds to Light and Quantum Foundation Problems'[1]. However, the fractal can be—simultaneously—modelled from an alternative *retrospective* or backwards-looking perspective

Figure 3 b.



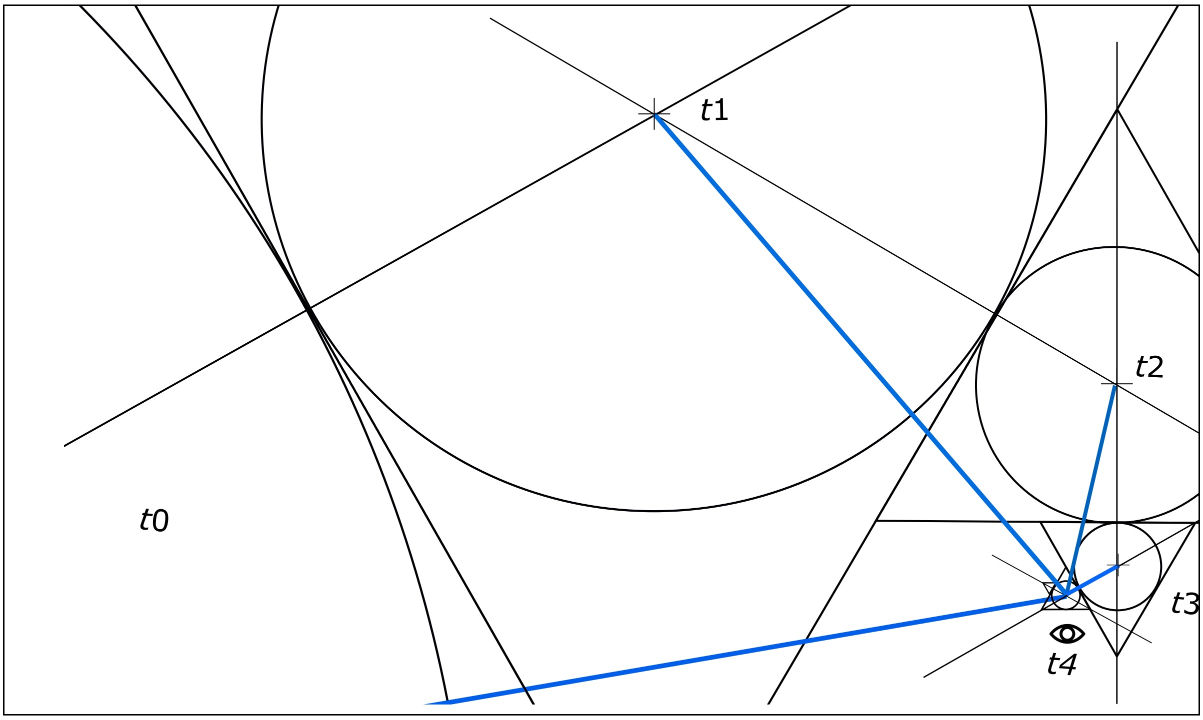
The schematics above demonstrate fractal development by (a) the (classical) forward or evolving Snowflake perspective, where the standard-sized thatched (iteration '0') is the focus, and the following triangles diminish in size from colour red iteration 0 to colour green iteration 2. (b) The inverted retrospective perspective assumes a fixed location within the fractal. The new (thatched) triangle is the focus and held at standard size while the original red iteration 0 triangle expands in area—as the fractal iterates.

Figure 3. Dual Perspectives of (Koch Snowflake) Fractal Growth.

To model this retrospective perspective the fractal was 'inverted' where bits grow rather than diminish. The new 'thatched' bit sizes remain constant in size (the same size as the original bit size '0') while the older generations of bit sizes grow with iteration-time as demonstrated with colours red (the original size) blue (the 1st iteration), green (the 2nd). With iteration-time, the size of the initial red iteration 0 triangle expands relative to the size of the new blue triangle.

The closest analogy or practical example of this fractal model may be to think of the growth and structure of a tree—an example of a 'natural' fractal. Is it that we are observing the universe as if in a tree structure—from a position of the outer branches? Surrounded by other similar-sized branches, we see, looking back and down, larger branches then boughs and finally the—once seedling-sized trig—trunk. All new branches on average start at a similar size to the seedling size and expand with time. On a mature tree, there will be many young seedling-sized branches on its outer. To complement this tree analogy, in a recent paper, it was found all trees accelerate in size with age [23]. This is not to say the universe is like a tree, but that a tree is a fractal, and fractals do come in many forms. Maybe the universe is one form.

Figure 4 shows an observer, represented by an 'eye image', at iteration-time 4 (*t*4) within the growing Koch snowflake fractal. The blue lines show the displacement between the observer and the centre points of the equilibrium triangles of earlier ages (*t*0,*t*1,*t*2,*t*3).



Showing the displacement between a fixed observation position (eye t4) with the iterating (Koch snowflake) fractal

Figure 4. Displacement Measurements within an Iterating Fractal

In this investigation, the following questions were asked about the retrospective iterating fractal from the perspective of an in-situ observer within this fractal.

1. Does the fractal demonstrate accelerated expansion?
2. Can a Hubble-Lemaitre diagram be produced from the observing perspective of an arbitrary location within the set?
3. Does the fractal expand from a single-point Planck area to an arbitrary size of 1 in a time comparable to the conjectured inflation epoch?
4. Concerning arbitrary centre points of component fractal bits (triangles in the Koch snowflake), is the distribution of these bits change dependent on the location of the observer?

It was hypothesised that current observations of the universe are all what one would expect to see if inside a growing fractal that began at a single point.

Specifically, the retrospective fractal will demonstrate:

1. a 'singularity' (Big Bang) beginning; the presence, and dominance of a 'uniform' 'edge' corresponding to a Cosmic Microwave Background limit;
2. inflation epoch expansion;
3. accelerating (exponential) 'dark energy' expansion;

4. a cosmological constant;
5. a Hubble-Lemaitre Law of expansion[24],[3];
6. the retrospective fractal model will—with respect to galaxy distribution and demographics—contravene the two assumptions of the cosmological principle.

To test these questions a model of the fractal was developed measuring the change in the area, and displacement with each iteration from a fixed position of observation.

2 METHODS

To analyse and address the said questions a spreadsheet model [25] was developed to trace the area expansion of the retrospective fractal by iteration-time(t). The classical Koch Snowflake area equations were adapted to account for this perspective. A quantitative data series was made ready for analysis. The scope of this investigation was limited to two-dimensional—as a demonstration; three-dimensional space or volume can be inferred from this initial assumption. Changes in the areas of triangles and distances between points in the fractal set were measured and analysed to determine whether the fractal area and distance between points expand.

2.1 Area Expansion of the Total Inverted Fractal with Iteration-time

To answer question 1, the following tables were produced. A data table was produced (Table 1) to calculate the area growth at each iteration-time for a single triangle. The area (A) of a single triangle was calculated from the following equation (1) measured in standard (arbitrary) centimetres (cm)

$$A = \frac{l^2\sqrt{3}}{4} \tag{1}$$

where l is the triangle's base length. l was placed in Table 1 and was set to 1.51967128766173 cm so that the area of the first triangle (t_0) approximated an arbitrary area of 1 cm^2. To expand the triangle with iteration-time, the base length was multiplied by a factor of 3. The iteration-time number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Calculations were made to the 10th iteration, and the results were graphed.

With iteration, new triangles are (in discrete quantities) introduced into the set. While the areas of new triangles remain constant, the earlier triangles expand, and by this, the total fractal set expands. To calculate the area change of a total inverted fractal (as it iterated), the area of the single triangle (at each iteration-time) was multiplied by its corresponding quantity of triangles (at each iteration-time).

Two data tables (Tables 3 and 4 in the spreadsheet file) were developed. Table 3 columns were filled with the calculated triangle areas at each of the corresponding iteration-time — beginning with the birth of the triangle and continuing to iteration ten. Table 4 triangle areas of Table 3 were multiplied by the number of triangles in the series corresponding with their iteration-time. Values calculated in Tables 3 and 4 were totalled and analysed in a new table (Table 5). Analysed were: total area expansion per iteration, expansion ratio, expansion velocity, expansion acceleration, and expansion acceleration ratio. Calculations in the columns used kinematic equations developed below.

To answer question 3 the total area expansion per iteration equation was set to calculate the iteration-time taken to expand from one size to another. The time taken was calculated by setting the initial triangle area (the Planck area) using the Planck length constant ($1.61619926 \times 10^{-35}$) and the final area was set to an arbitrary area size of 1 (cm).

2.2 Acceleration

Acceleration (a) was calculated by the following equation

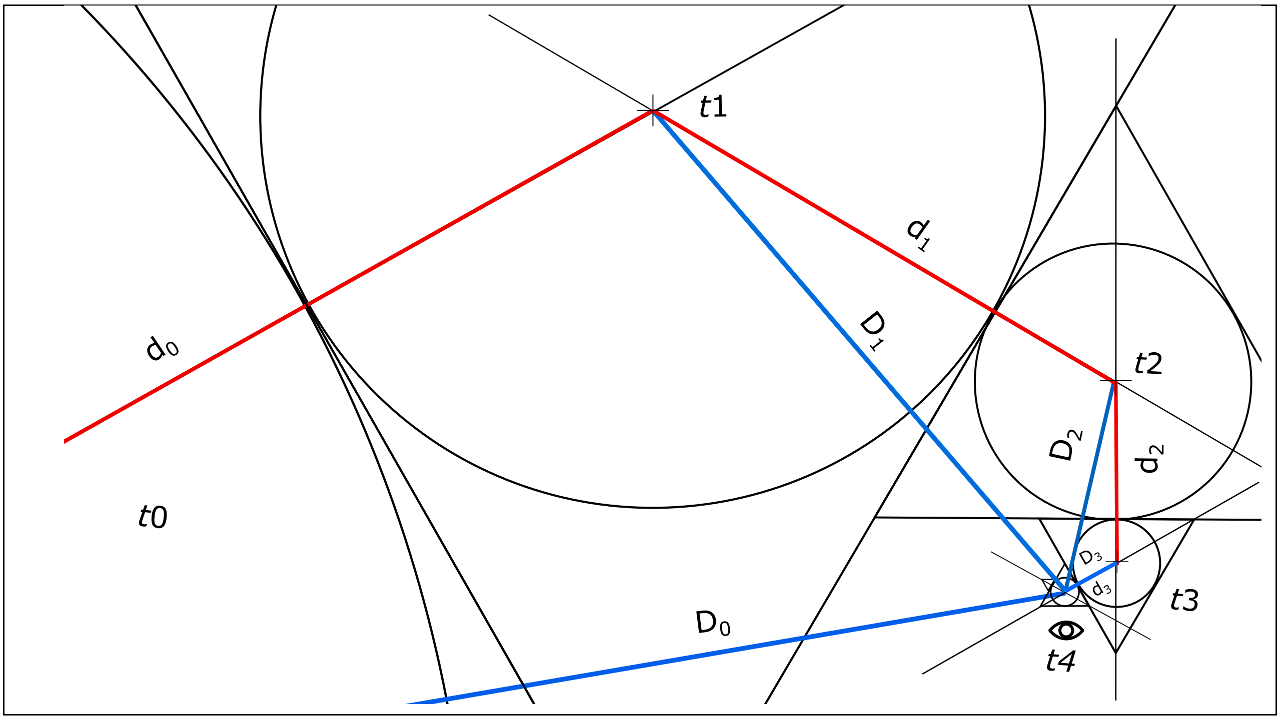
$$a = \frac{\Delta \boldsymbol{v}}{\Delta \mathbf{t}} \tag{2}$$

Acceleration is measured in standard units per iteration cm^{-1t-2} and cm^{-2t-2}. Using the same methods as used to develop the Hubble diagram (as described above in 2.3) an 'acceleration vs. distance' diagram was created, regressed, and an expansion constant derived. Ratios of displacement expansion and acceleration were calculated by dividing the outcome of t_1 by the outcome of t_0.

The same method of ratio calculation was used to determine the change or expansion of the area.

2.3 Distance and Displacement, Hubble-Lemaitre Diagram

To answer question 2, distances between an arbitrary observation position and points on retrospective triangle bits were measured and analysed. Calculations were made on a second data table (table 2) on the spreadsheet.



Measuring the blue line displacement (D) and the red line distance (d) from an observer (t4) to triangle centre points inside an iterating Koch Snowflake fractal. t = iteration-time.

Figure 5. Measurement of Displacement and Distance Inside Fractal.

Triangle geometric centre points were chosen as the points of measurement as shown in Figure 5. The blue and red lines trace the displacement (D) and distance (d) respectively between an arbitrary observation point at *t*4 and triangle centre points (*t*3 *t*2, and *t*1). Calculating the displacement was out of the scope of this study; distances were calculated instead. Distances were measured by calculating the inscribed radius for each equilateral triangle by equation (3) below. The total distance between points was calculated by adding the inscribed radius of the first triangle, for example, from t_0 to the inscribed radius of the next expanded triangle t_1.

$$r = \frac{\sqrt{3}}{6} l \tag{3}$$

From the radius distance measurements, total distance, distance expansion ratio, velocity, acceleration, and expansion acceleration ratio for every iteration-time were calculated using classical mechanics equations. Velocity (v) was calculated by the following equation

$$\boldsymbol{v} = \frac{\Delta \boldsymbol{d}}{\Delta \boldsymbol{t}} \tag{4}$$

where distance(d). Velocity is measured in standard units per iteration cm^{-1t-1} for receding points and cm^{-2t-1} for the increasing area.

To test for Hubble's Law, a Hubble (like) scatter graph titled 'The Fractal/Hubble diagram' was constructed from the results of the recession velocity and distance calculations (in Table 2 of the inverted fractal spreadsheet file). On the x-axis was the displacement (total distance) of triangle centre points at each iteration-time from t_0 and on the y-axis the expansion velocity at each iteration-time. A best-fitting linear regression line was calculated and a Hubble's Law equation (5) was derived

$$\boldsymbol{v} = \boldsymbol{H_0 d} \tag{5}$$

where H_0 is the (present) Hubble constant (the gradient).

2.3.1 Test Measuring Real Displacement

The propagation of triangles in the (inverted) Koch Snowflake fractal, is not linear but in the form of a logarithmic spiral or helical—as shown in Figure 5(above), and Appendix

Figure 17. The method thus far assumes and calculates the linear circumference of this spiral and not the true displacement (the radius). This method was justified by arguing the required radius (or displacement) of the logarithmic spiral calculation was too complex to calculate, (and beyond the scope of this investigation), and that expansion inferences from inverted fractal could be made from the linear circumference alone. A spiral model was created independently, and radii were measured to test whether spiral results were consistent with the linear results in the investigation. Measurements were made using geometric software (see Appendix I Figure 17). Displacements and the derived Hubble diagram from this radius model were expected to show significantly lower values than the above (calculated) circumference non-vector method but share the same (exponential) behaviour. Appendix Figure 1 shows the distance between the centre points, and blue, the displacement. See Appendix Figure 18, and Figure 19, and Table 1 for results.

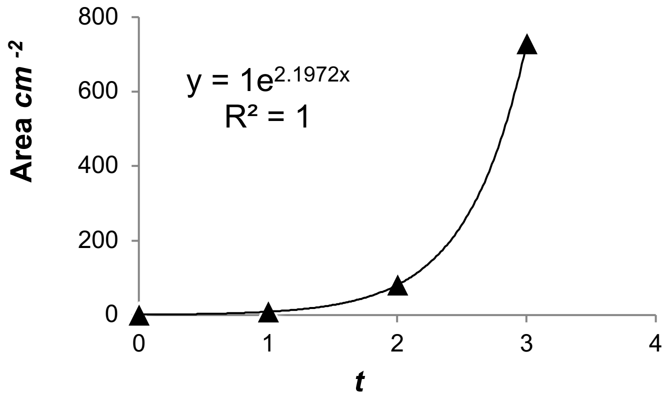
2.4 Small Scale Long Scale Point Distribution Analysis

To address question 4, the number of triangle sizes per total distance increment on the fractal-Hubble diagram was calculated by counting the number of triangle sizes (in the distance column in Table 2) and dividing this by the distance increments measured in the sample. See Table 2a of the spreadsheet model. The number of triangles at each increment was calculated by totalling the number of triangles (from Table 4) for each respective iteration-distance. An amended Fractal-Hubble diagram was created combining (recessional) velocity with the number of triangles at every distance. See Table 7 of the spreadsheet model[25].

3 Results

3.1 Accelerated Area Expansion

The area of the initial triangle of the inverted Koch Snowflake fractal increased exponentially, Figure 6.



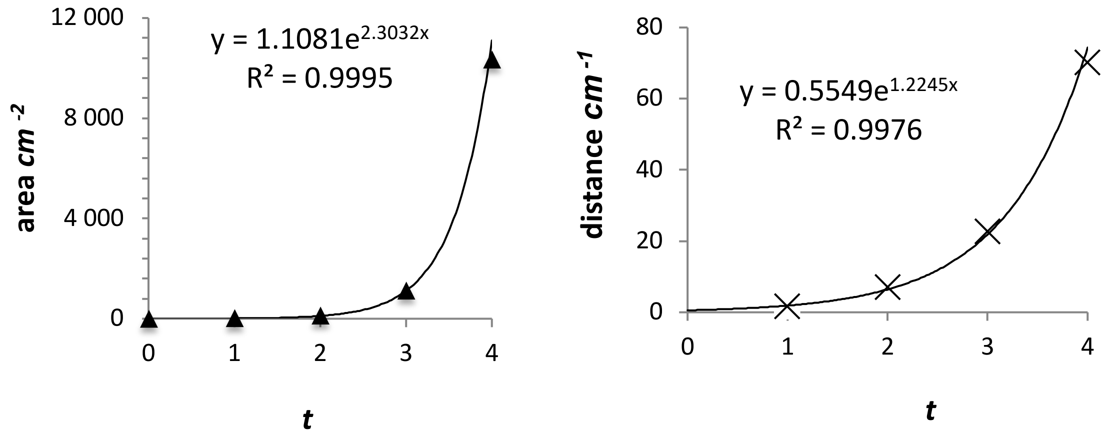
The area of the initial triangle bite on the inverted Koch Snowflake fractal increases exponentially with iteration-time. cm = centimetres.t= iteration-time.

Figure 6. Initial triangle exponential area expansion.

This expansion, with respect to iteration-time, is written as

$$\boldsymbol{A = 1e^{2.197t}} \tag{6}$$

The area of total area of the fractal (Figure 7 a) and the distance between centre points Figure 7 b) on the fractal increased exponentially.



(a) total area expansion

(b) distance between points. cm = centimetres. *t*= iteration-time.

Figure 7. Area/distance expansion per iteration-time on the Inverted Koch Snowflake fractal.

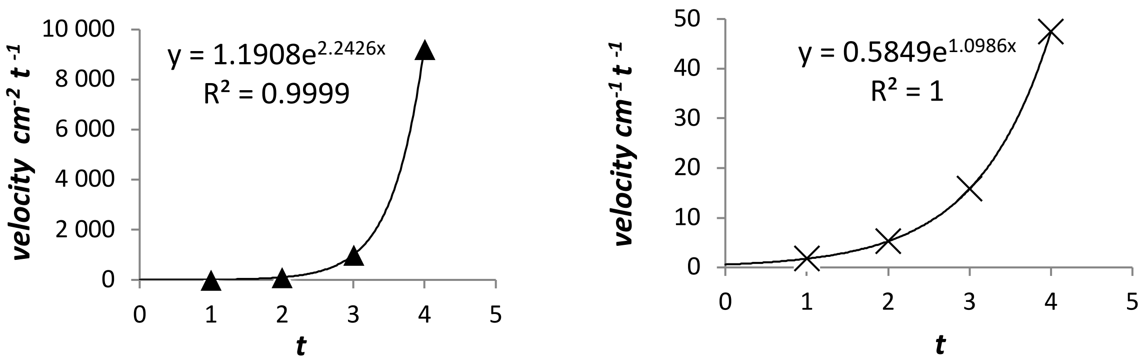
The expansion of the total area (A^T) is described as

$$A^T = 1.1081e^{2.3032t} \quad (7)$$

The expansion of distance between points (***d)*** is described by the equation

$$d = 0.5549e^{1.2245t} \quad (8)$$

The (recession) velocities for both total area and distance between points—Figure 8 a and b respectively—increased exponentially per iteration-time.



(a) expansion of the total area, at each corresponding iteration-time (t) cm=centimetres.

(b) distance between points.

Figure 8. (Expansion) velocity of Area and Points on the Inverted Koch Snowflake Fractal.

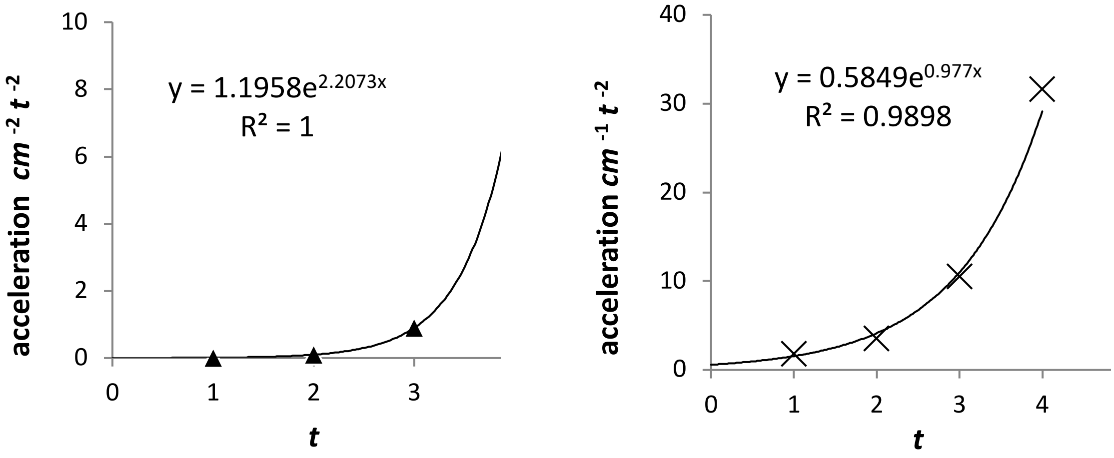
Velocity is described by the following equations respectively

$$v = 1.1908e^{2.2426t} \quad (9)$$

$$v^T = 0.5849e^{1.0986t} \quad (10)$$

where v^T is the (recession) velocity of the total area; and v the (recession) velocity of the distance between points.

The accelerations for both total area and (recession) distance between points—Figure 9 a and b respectively—increased exponentially per iteration-time.



(a) Expansion of the total area. cm = centimetres. t= iteration-time.

(b) Distance between points.

Figure 9. Expansion Acceleration of Area and Distance between Points on the Inverted Koch Snowflake Fractal.

Acceleration is described by the following equations respectively

$$\boldsymbol{a^T = 1.1958e^{2.2073t}} \tag{11}$$

$$\boldsymbol{a = 0.5849e^{0.977t}} \tag{12}$$

where a^T is the (recession) acceleration of the total area, and a, is the (recession) acceleration of distance between points.

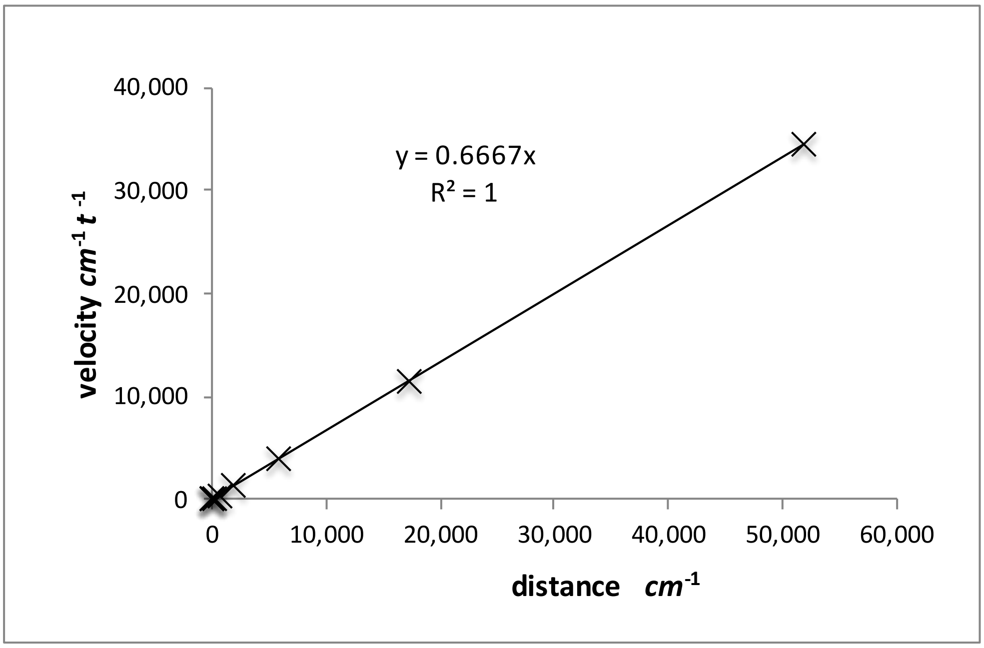
3.2 Inflation Epoch Expansion

From equation (11) the development of the fractal takes 72.59 (2s.f.) iteration-times to expand from this arbitrary small area to the arbitrary large area of 1 cm^{-2}.

$$\boldsymbol{t = \frac{1}{2.2073}\ln\left(2.61223 \times 10^{70}\right)} \tag{13}$$

3.3 The Fractal/ Hubble-Lemaitre Diagram

As the distance between centre points increased (with each corresponding iteration-time) the recession velocity of the points also increased—as shown in Figure 10 below.



As the (exponential) distance between triangle geometric centres increases with iteration-time, the recession velocity of the points increases. cm = centimetres.t= iteration-time.

Figure 10. The Fractal Hubble-Lemaitre diagram.

Recession velocity vs. distance of the fractal is described by the equation

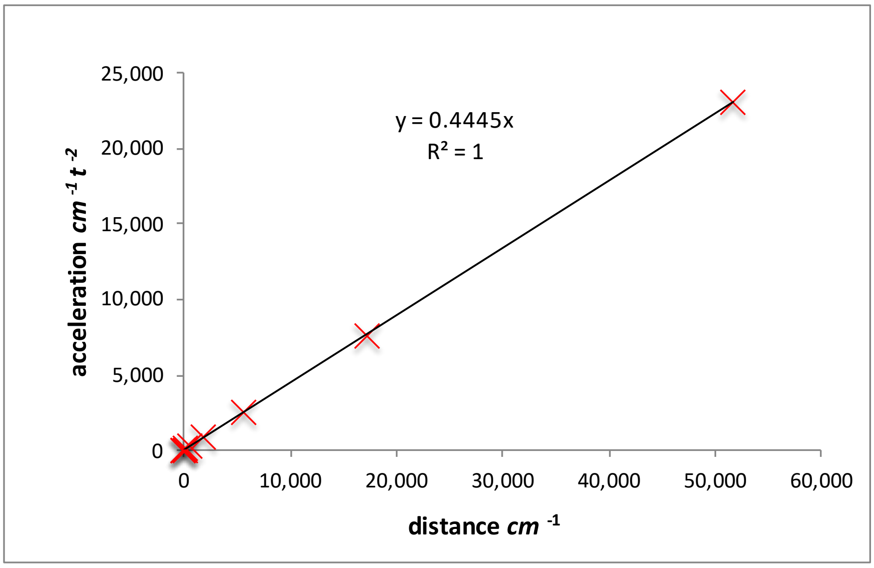
$$v = 0.6667d \tag{14}$$

where the constant factor is measured in units of $cm^{-1}t^{-1}$ cm^{-1}.

The spiral radius distance (d) velocity by experiment (see Appendix Figure 18 and Appendix Table 1 for details) resulted in a Fractal-Hubble equation of

$$v = 0.6581d \tag{15}$$

In terms of acceleration vs. distance from the observer; the recession of points accelerated away with increasing distance—as shown in Figure 11 below.



As the distance between triangle geometric centres increases with iteration, the recession acceleration of the points increases. cm = centimetres, t = iteration-time.

Figure 11. Recessional Acceleration vs. Distance on the Inverted Koch Snowflake Fractal.

The recession acceleration of points at each iteration-time at differing distances on the inverted fractal is described by the equation

$$\boldsymbol{a = 0.4445d} \tag{16}$$

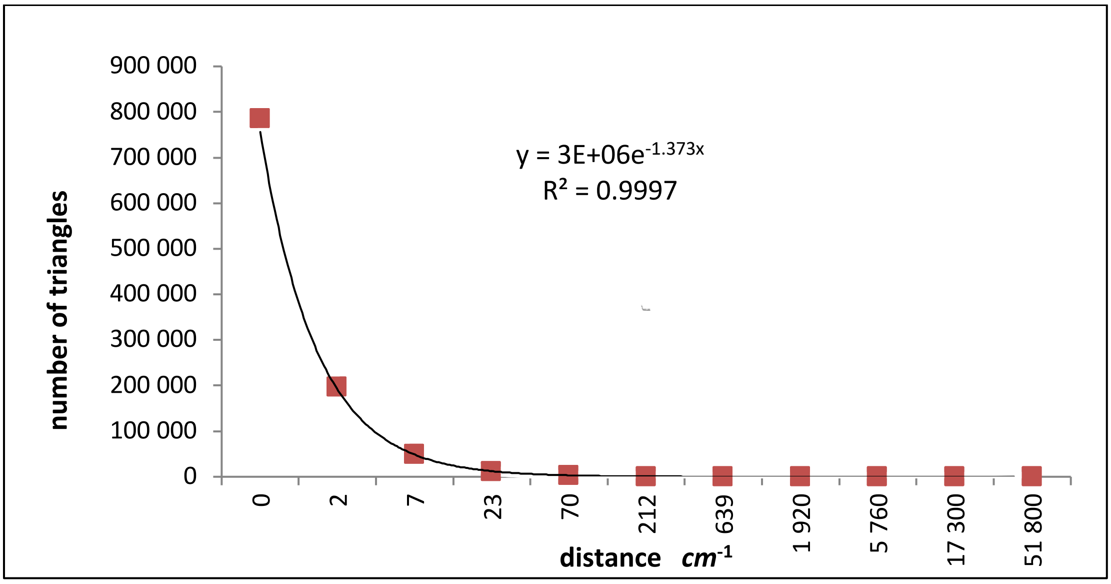
where the constant factor is measured in units of $cm^{-1}t^{-2}$. a = acceleration; d = distance.

The spiral radius displacement (D) acceleration by experiment (see Appendix Figure 19 and Appendix Table 1 for details), was described by the equation

$$\boldsymbol{a = 0.4295D}. \tag{17}$$

3.4 Distribution of Points and Triangles with Iteration-Time

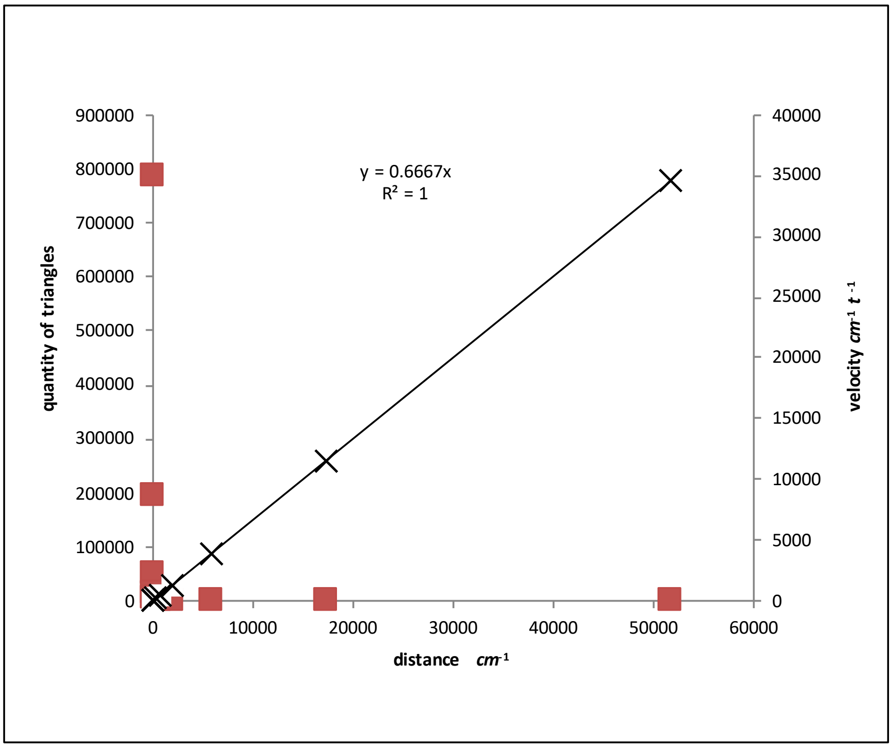
On the Hubble-fractal diagram (Figure 10), eight of the ten measurement points are located inside the first (1.20E+4cm^{-1}) increment distance. The remaining 2 measurement points are outside this range. Figure 12 shows the number of triangles at these 'measurement points' by the distance between the geometric centres (from the observer). The number of triangles decreased exponentially from 7.86E+05, at the observation point, iteration-distance 0, to a quantity of 1 at a distance of 51800cm^{-1} (iteration-10).



As the distance between triangles' geometric centres (exponentially) increases with iteration, and from the observer, the quantity of triangles per iteration decreases exponentially to 1—at iteration-time 0. cm = centimetre.

Figure 12. Count of Triangles at each Distance (Point) from the Observer on the Inverted Koch Snowflake Fractal first 10 iteration-times.

Figure 13 combines the fractal-Hubble diagram (Figure 10) with the number of triangles at each distance point (Figure 12) and shows the relationship between the clustering of measurement points close to the (low recessional velocity) origin and the smooth distribution (high recessional velocity) at large distances—towards the origin of the set.

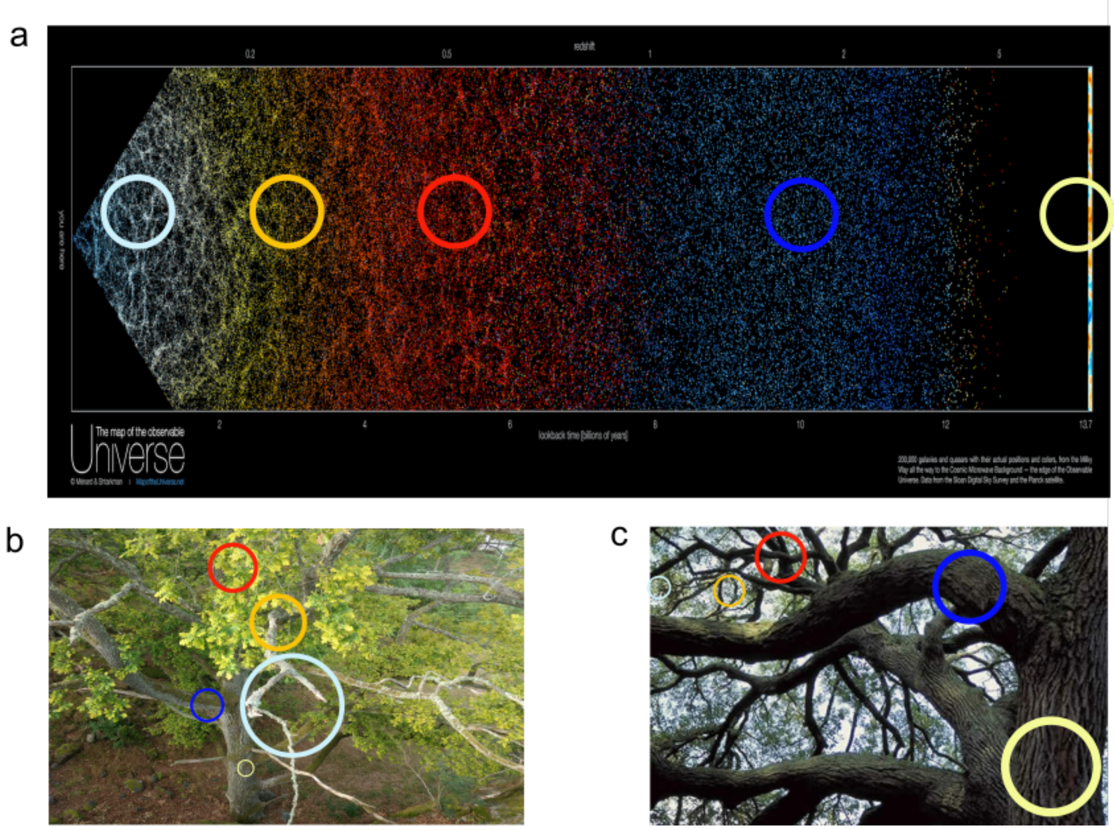


As the distance increases from the observer with respect to iteration-time: the recession velocity of the distance between geometric points increases; while the number of triangles at each distance decreases. cm = centimetre.

Figure 13. Fractal-Hubble Point Distribution Diagram

4 DISCUSSIONS

Figure 14 shows how the galaxy distribution and demographics from the current map of the universe correspond with a branch distribution on a living fractal, in this case, the common (oak) tree. In (a) the different regions/structures of the universe are highlighted by rings of differing colours, and in (b) and (c) these rings are shown to fit the corresponding parts—from rough to smooth—on the tree fractal, from trunk to twig branches.



(a) The Map of the Universe[22] with observation rings from clustered 'small scale' (light blue) to the smooth (large structured LQGs) outer universe (blue and light-yellow rings). (b)Observing from high in the branches of an oak tree (fractal) looking down and back to the trunk with corresponding universe observation rings; (c) observing from the trunk of an oak tree (fractal) looking up and out to the branches.

Figure 14. Non-isotropic and non-homogenous universe observations corresponding with a living tree fractal

4.1 Accelerating Growth and the Development of the Fractal Tree

The following refers to question 1 (section 1.2) about the acceleration of the fractal.

The growing tree is the perfect example of a fractal and stands as the perfect real-life metaphor of the inverted fractal model, they have similar properties. Using a tree as an analogy to model the fractal universe is not to say the universe is exactly a tree structure; it is only to say that just as a tree is a fractal structure, the universe too to be appears is a fractal structure.

If the retrospective observation perspective—from deep within a snowflake fractal—is substituted with an observation from high within a common branching tree, the clustering of points on the Fractal-Hubble diagram would equally correspond to the clustering of self-similar (sized) branches—in the tree—surrounding the observer. If the observer were to look down, inwards from the outer branches—towards the trunk of the tree — the branch (nodes) quantity would decrease, the volume of the single branches would increase, and the branch 'clustering' would smooth out.

4.1.1 Trees Growth Found to Accelerate

Trees and all plants are perfect examples of fractals. A tree's growth is generally described as being of 'natural' fractal geometry (or L systems). In a recent publication, it was found trees were found to be growing at an accelerating rate [23],[26]. The study measured up to 80 years of tree growth, on more than 600,000 trees, over 6 continents and found that the growth of 97 per cent of the trees was accelerating with age. This accelerated growth rate with time is a mystery to biologists.

This phenomenon of acceleration of plant growth may be explained by the plant's growth being fractal. If the productive leafy stem of the emergent tree Figure 15a becomes the focus of the tree's growth and is held constant in size—just as with the standard triangle size is to the inverted Koch snowflake (Figure 3b)—then the older branches and the load-bearing trunk of the tree will grow exponentially with iteration-time—again just as the snowflake did.

Figure 15a and b show this seedling versus outer branch assumption on a real tree. On a—left, the size of a leaf of a fully grown/developed tree and a—right shows the leaf size as a seedling. Figure 15b—right shows the seedling held in a hand alongside the outer branches leading down to the boughs and the truck behind.

a b

(a) Showing the one constant on an iterating tree fractal, the leaf size. The left branch is from the outer branches of the fully-grown tree, and the branch on the right is seedling-sized.

(b) The same seedling as in (a—right) is held alongside the outer branch size of a fully developed tree of (a—left). Behind are the trunk, boughs and branches of the same tree. The trunk of the tree, it can be deduced, was once the same size as the seedling in hand.

Figure 15. Fractal Tree Growth from a Constant Leaf Size.

4.2 Accelerating Expansion of the Fractal Explains 'Dark Energy' Conjecture

The acceleration property of the fractal (Figure 9) is consistent with the 1998 astronomical discovery (by observation) of the accelerating expanding universe and conjectures surrounding the term 'dark energy' and the cosmological constant (lambda). It can be inferred (from this inverted fractal model) that the accelerating expansion of the universe concerning distance (Figure 11) is a property of fractal geometry, and can be described by the equation

$$\boldsymbol{a = F_a d} \quad (18)$$

where F_a is the fractal (cosmological) recession acceleration constant measured in units of $cm^{-1}t^{-2}\ cm^{-1}$. The constant F_a in equation (18) may be interpreted as a fractal a 'cosmological constant'—lambda—concerning point acceleration and distance.

The acceleration between points with respect to time (equation (12) is described as

$$\boldsymbol{a = a_0 e^{F\lambda t}} \quad (19)$$

where the constant $\boldsymbol{F\lambda}$ may be interpreted as a fractal 'Cosmological Constant' Lambda with respect to point acceleration and iteration-time.

With the continual entry of new triangles into the fractal set, the total fractal area of the total universe grows exponentially (Figure 11 above). This total area expansion with respect to time is described by the function

$$\boldsymbol{A^T = A_0\, e^{F\Lambda t}} \quad (20)$$

where $\boldsymbol{F\Lambda}$ is a fractal constant with respect to total area expansion and time.

4.3 Fractal Growth Consistent with Inflation Epoch Expansion

The expansion rate of the isolated (unbounded) fractal demonstrates the conjected early universe "inflation epoch" expansion rate [27]. The opportunity is open to link this expansion with the photon properties of light. Papers on the 'universe ticking' of light' have conjectured the 'ticking' be at around 10^{-33} per second[28],[29].

If the production of triangle bits by iteration of the fractal is set to correspond to this oscillation frequency or 'ticking' of photons, this 72.59 iteration-times to expand from the Planck area size to a size of one may be found to be consistent with conjectured inflationary epoch speeds. This is a prediction and an opportunity for a better model or experiment to test this claim.

4.4 Hubble-Lemaitre Law

The following refers to question 2 (section 1.2) pertaining to (as it was known in its time) Hubble's Law and the fractal.

The shape of the fractal-Hubble curve (Figure 10) has direct significance in Georges Lemaitre's conjecture surrounding the expanding universe [3] and Edwin Hubble and Humason's 1929 concurring observations [2]. The fractal model (Figure 10) demonstrates the Hubble-Lemaitre Law where from any observation point within a fractal the recession speed of points increases with distance.

The fractal model needs no 'rising raisin bread' or 'rubber sheets' as used as demonstrations by experts today. When velocity (v) is plotted against the distance of points (D) (Figure 10, and Appendix Figure 18) the inverted fractal demonstrates Hubble's Law described by the equation

$$\boldsymbol{v} = \boldsymbol{F_v D} \tag{21}$$

where (F_V) is the slope of the line of best fit — where the fractal (Hubble) recession velocity is constant. The scale invariance of the Fractal-Hubble diagram concurs with the historical development of the Hubble diagram through the ages. From its 1929 original to the improved 1931 to its most recent, the shape of the diagram remains constant, just as with the fractal model.

4.5 Fractal Explains Small-Scale (and Large-Scale) Galaxy Distribution

The following refers to question 4 (section 1.2) pertaining to the distribution of bit size on the fractal.

Figure 10 (the Hubble-fractal diagram) also shows the distance between measurement points on the fractal Hubble curve is not linear but increases in what appears to be exponential. This increase is a consequence of the increasing size of the triangles with growth.

This changing concentration of measured centre-points on the diagram has significance in cosmology as the concentration corresponds with the similar changing distribution of observed galaxies. It pertains directly to the decreasing distribution—the smoothness—of galaxies looking back in time. The claimed 'small-cosmic scale fractality of galaxy distribution'.

In Figure 12, it is shown the quantity of triangle bits at the origin (iteration 0) on the Fractal-Hubble diagram is 786,432. All of these triangles are of the same size as the observer's triangle viewing position. This quantity of bits also corresponds to the clustering of the measurement points near the origin of the diagram and this is due to the location the observer is within the emergent (inverted) fractal and the relative size of these triangle bits near the observer. The observer is 'in the branches' so to speak. Again, it is as if the observer is on a branch of the tree (see section 4.9) surrounded by branches of similar age and size and is looking back—down—to the trunk of the tree, which was the origin of the tree and has now expanded.

4.5.1 LQGs and Large-scale Structure Observations Bough Branches

The 'very large', 'thin' and old large-scale survey structures in the assumed smooth universe concur totally with my model of the fractal. They are the 4 billion light-years in sized Huge 'Large Quasar Group'[20] and the 10 billion light-years sized Hercules—Corona Borealis Great Wall [21]. They represent the large 'bough branches', first out from the CMB 'trunk' of the fractal structure (see Figure 14c). To support this claim, the LQG structures are very large, and they are also old. They are composed of quasars and are thinly distributed, compared to the small-scale clustered region.

4.6 The Fractal Refutes the—Homogeneity and Isotropy—Cosmological Principle

Cosmological observations concur with an in-situ fractal perspective. The fractal model (see Figure 14) explains why the universe is neither homogenous nor isotropic. The standard model of cosmology's key assumption—the cosmological principle—maybe, as it stands today, be a mere illusion, a false paradigm.

Before explaining how this fractal model does not conform to the status quo—something that has been continually explained throughout this paper—it should be made clear it is already claimed and granted by cosmologists in their explanations of the cosmological principle that based on modern observations it only holds on large scales—scales larger a redshift z factor of .25 (about 4 billion light-years)—and that on small cosmic scales it does not. The recent discovery of—thinner and older—large quasar groups (LQGs) and the Hercules—Corona Borealis Great Wall (4.5.1) that are beyond this z-factor distance (beyond a *z*-factor of 4) add strength to the large-scale rebuttal [20],[30].

The following deals with this cosmological principle rebuttal.

1. *Homogeneity:* From this fractal experiment, distributions are not the same in all directions but rather the galaxy distribution will diminish with distance and time as explained in section 4.2. Smoothness will be observed on large—older—scales (towards the trunk), and clustered fractal activity on small—newer—scales (the branches); just as observed looking out from the Earth's position towards the singularity CMB smoothness. More on this in section 4.2. Also, as we look back in time, the fractal model concurs with observations and claims made about the evolution of galaxies—evidenced by dark matter halo merger trees structures (section 4.10). This corresponds to the 'old' LQGs discovery also.
2. *Isotropy*: In a fractal, observations will not be the same in all directions; points will be very different from different locations. As with a fractal tree modelled here, there is an obvious trunk to the structure and there are obvious clusters of branches, and these will not be observed isotopically in all directions. The view will be different if viewed from the perspective of the trunk, and if viewed from within the branches. In this fractal model, it remains true everything is receding away from any observer, but the view will be different—depending on the position of the observer—and thus not necessarily the same in all directions. There is a 'strange' fractal edge that has grown since the fractal's origin, and this edge appears—by the model—to also be 'the centre', though this has expanded and is viewed today—in part and consistent with the standard model—as the CMB. All space between this 'edge' and Earth observation is newer, and this is again supported by the evolution of galaxies.

4.7 Singularity Beginning

The following refers to question 3 (section 1.2) and pertains to the origins of the fractal.

The expansion of the first single triangle bit in this model demonstrates a singularity 'Big Bang' beginning. Its area begins from arbitrarily small size and may be set to the size value of the Planck area. This simplest of demonstrations is consistent with the observed very cool cosmic microwave background (CMB). It is not an explosion: it is an infinite exponential expansion of the area of the fractal set. This is consistent with descriptions that 'space itself that is expanding'. The fractal in isolation is expanding into 'nothing'. The set has a frontier; however, any position beyond this is unattainable. To an observer anywhere in the set, this initial triangle (t_0) will dominate the extreme horizon, but it will not be seen by all observers. If an observer is more than 7 ± 2 iterations distant from (triangle) bit t^0 and observing without any form of technology—to 'zoom' back in iteration-time, the said bit will not be seen: this is the fractal distance. The 7 ± 2 is derived from the classical emergent development of the fractal as shown in Figure 3 may be termed the equilibrium iteration count or observable fractal distance.

4.7.1 Centre of the Universe

The fractal model implies the beginning was a specific place. This fractal claim is consistent with the standard model's claim 'the Big Bang is everywhere'. However, the space we on Earth inherit is emergent; it is created and is newer and different from the original place the fractal began. From this, a discussion over where the universe's centre—its beginning— may develop.

4.7.2 Emergent History and the Big Bang

A fractal universe would imply an emergent structure—the whole is made of many parts—just as the tree is made of many branches. It may force us to question the initial conditions of the Big Bang beginning. Namely, whether all mass (in the universe) was together in one place and at one time. It could now be argued—from the principles of fractal emergence—the universe developed/evolved mass from the bottom up with time. It started small, from a seedling and developed structure. However, this does not explain the extreme temperatures claimed. There is a begging question from the hot dense 'Big Bang'; how can there be dense and heat before the time of—at least—photons? Was it emergent all the way?

4.8 The Particle Horizon Problem

Following from 4.3 and 4.8.1, the modelled *retrospective* inverted fractal demonstrates—and is consistent with—space's ability to expand 'extremely fast'. If we think about the production of the fractal from the classical fractal perspective

Figure 3 (a) and that this production has a speed, a rate of production that is propagated akin to the propagation of a light photon, then if we compare this speed with the inverted expanding area behind the fractal the complete model makes sense and the claim 'space expands faster than the speed of light' as proposed by Albert Einstein in his General Theory of Relativity and as conjectured by inflation theory.

Arbitrary points on the surface of the original—iteration 0—triangle may be assumed to be close enough to assumed to have 'causal contact'; however, with the exponential expansion of the fractal object, this contact will not remain and the points will exponentially expand apart at a rate demonstrated from this experiment (4.3). Concerning the speed of light; the fractal has a constant propagation speed, this speed can be assumed, in principle, to be able to be surpassed by the (accelerating) area expansion 'speed' of the fractal itself. This fractal expansion speed claim is also consistent with and addresses issues surrounding the particle horizon problem and the cosmological principle (axiom).

4.8.1 The Fractal and the Speed of Light.

From the 'classical view' of the fractal Figure 3A (fully developed in the author's paper *The Fractal Corresponds to Light and Quantum Foundation Problems*[1]) from the fractal there may be a strong insight into the nature and behaviour of light. If this is so, this may help understand why the universe expands and behaves the way it does and also help unify the large-scale universe with the quantum nature of the universe.

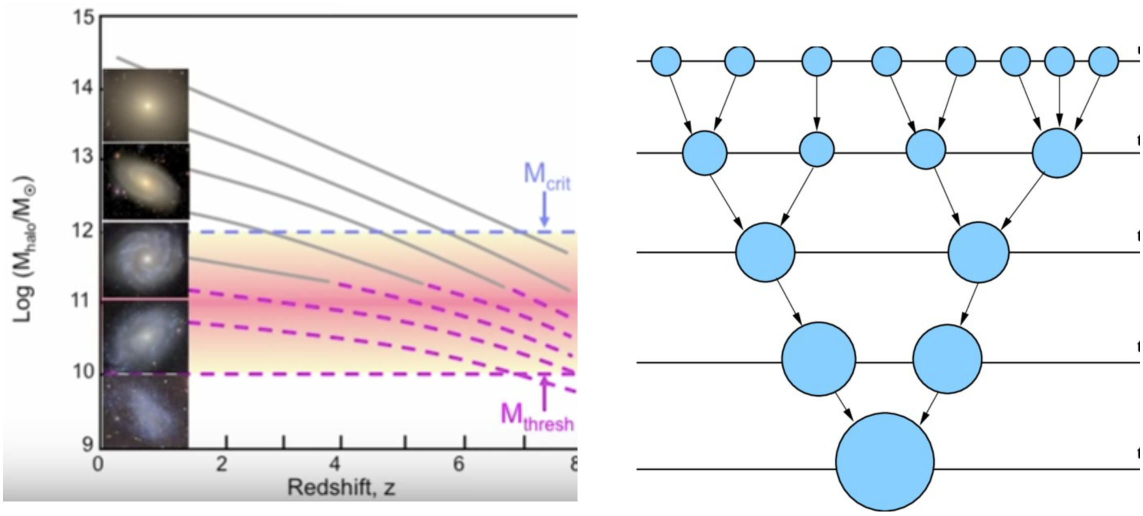
4.9 Vacuum Catastrophe

Continuing from the above (4.8) the 'vacuum catastrophe discrepancy' may also be resolved by understanding the universe as a fractal and that we, the observer, are in one. As described in the introduction, the fractal shares a duality of perspectives from an observer in one; the classical (forward) view and the expanding (back) view, together they are different aspects of the one geometry. This investigation focused on the expansion and has claimed this to be the dark energy cosmological constant. The classical aspect—outside this investigation—can be shown to behave as the quantum problem. The classical fractal demonstrates wave-like helical, smaller and smaller (wavelengths), and higher and higher frequencies; while the expansion (behind) is in terms of exponentials.

If a standard fixed area size, for instance, the area of iteration 0 triangle size, is used to calculate the total area of the set, the resulting total area will be very large. However, if the total area of the inverted fractal is divided by the real corresponding area sizes of the expanded triangles (allowing for their expansion at each iteration-time) the number will equate to a lower and more realistic number. The total area will equate to the total number of triangles propagated in the set. In principle, all triangles are as identical as the iteration 0 standard triangle, and only differ in scale due to the fractal expansion.

4.10 Dark Matter Halo Trees and the Evolution of Stars and Galaxies

Something that is rather beyond the scope of the investigation but important enough to mention as it is seen by the author to be inextricable to the fractal model is the evolution of galaxy demographics and distribution in the form of Dark Matter halo trees. From a presentation given by Sandra Faber on this subject, a fractal interpretation of the universe would give rise to this 'fractal' tree structure; again, from smooth and thin at far and early distances, to rough and clustered nearby.



(a) Diagram showing the age and size structure of the galaxies; that we (Earth) are surrounded by large and old galaxy clusters.

(b) the classic Dark Matter Halo tree—evolving from early t_1 (top) to large clusters t_5 (bottom)

Figure 16. Fractal Dark Matter Halo Trees and the Evolution of Atoms, Stars and Galaxies.

The significance of these merger halo structures is that they concur and correspond with an evolving emerging fractal model universe as revealed in this study. Halo trees are what one would expect to see if observing within a fractal space.

4.11 Raised Questions

4.11.1 Which Fractal Shape?

This investigation also does not in any way suggest the universe has the shape of a tree or a snowflake: fractal expansion could have equally been demonstrated using the Sierpinski triangle. The universe shares a feature special to fractals: fractals come in many forms, and that form is beyond the scope of this paper.

4.11.2 Fractal and the Age of the Universe.

The tree (fractal) grows by iteration-time, and not solar time. Trees can generally be as old—by counting the tree rings—as several hundreds of years; however, in terms of their fractal iteration age, they may only be some 4 to 7 iteration-times old. Could it be that the universe has a similar relationship or paradox in terms of its age and what we observe? Could it be the age of the universe we observe is not the true age of it— that it is older?

A fractal universe would suggest the 'big bang' evidence and conjecture is only part of a larger—potential infinite—'greater fractal'. Is the CMB a vail, hiding the—infinite— depth behind. If this is so, what we observe, in terms of galaxies, is all there are; before this is void.

4.11.3 General Relativity

The inverted fractal model does not affect General Relativity, but only its cosmological principle assumptions. The consequences of this are beyond the scope of this paper. It is conceivable general relativity may have to be adapted to take into account the geometry of the fractal. Work has already begun in this area: from noted theorist Laurent Nottale [31],[32] and others [33]. It should be made clear; this fractal model does not point to any new insight to do with gravity. This does not mean the fractal cannot explain the dark matter; this is beyond the scope of this paper.

4.11.4 Decreasing Fractal Dimension Looking Back

Recent studies have shown fractal dimension decreases with increased z-values [34]. This complements my model and claims as the complexity of the fractal system 'develops' with iteration-time.

4.11.5 Addressing Dark Flow, the Great Attractor

At the time of writing, there have been papers published [35]—based on the existence of so-called 'dark flow' and the Great Attractor which appears to be 'flowing' in the opposite direction as to 'dark energy accelerating observations—that challenge the observations pointing to an accelerating universe (and thus the existence of dark energy).

I believe the fractal model can address these rebuttals as being part of the fractal system. If an observer is assumed to be within the fractal set (the universe), which I am assuming we are in my model; then a flow in the opposite direction to the early and older parts of the fractal—as claimed in the paper—is to be expected, even predicted as part of the continued growth of the system.

4.11.6 Quantum Mechanics (Like) Properties of the Fractal

Viewed from an (arbitrary) position outside the set a fractal will grow at a decreasing rate to form the classical fractal shape—a snowflake as shown in Figure 3A. But from the perspective of an observer within the fractal set, the same expansion will appear to expand. This assumption of observation from within the set, looking forward from a fixed position has an uncanny resemblance to properties and problems shared with objects described only by quantum mechanics and the electromagnetic spectrum.

When isolated, the iterating (snowflake) fractal is an infinitely of discrete triangles (bits). The snowflake is a superposition of all triangles, in one place, at one time. The production of new triangles propagates in the geometry of a spiral: rotating in an arbitrary direction to form—when viewed from a side elevation—a logarithmic sinusoidal wave, comparable to the described electromagnetic spectrum. This spiralling wave-like propagation is illustrated below in Figure 3B and Appendix Figure 17. Location or position within this infinite set is only known when observed or measured; otherwise, all positions are possible—at the same time. These quantum-like features of the fractal are the essential background to this investigation and are covered in my complementary paper *The Fractal Corresponds to Light and Quantum Foundation Problems*[1]. Together, the dual perspectives will make sense of the universe.

4.11.7 Inverted Fractal and Shape of the Atom

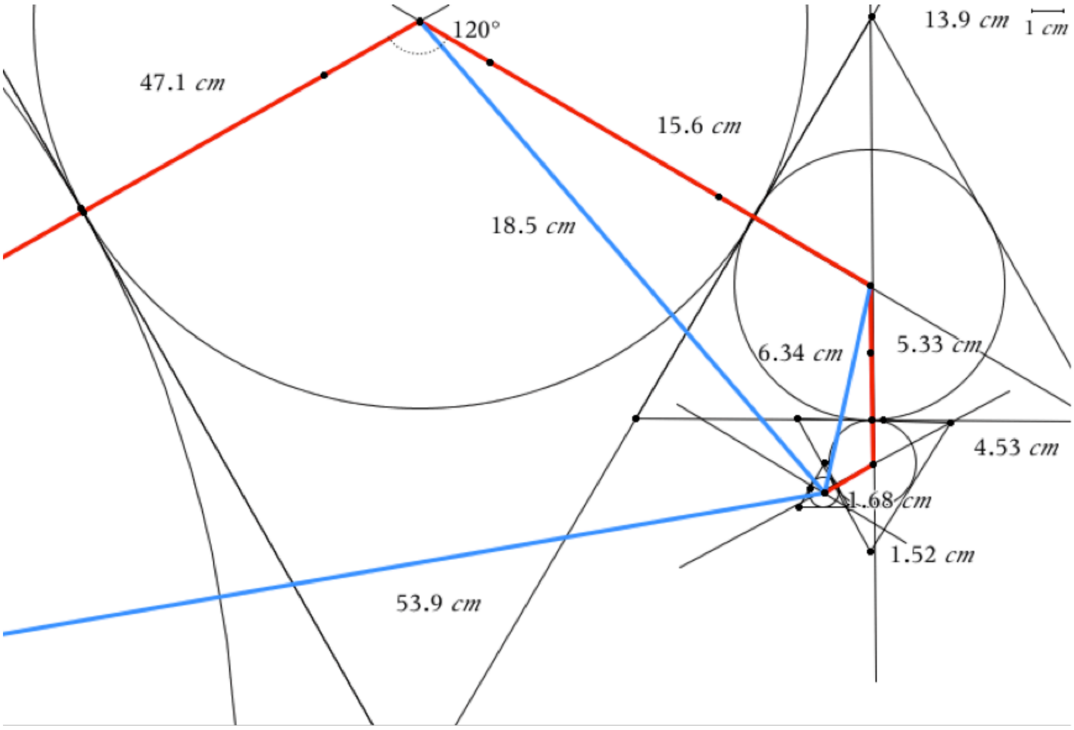
Keeping with 4.11.6, mention must be given to the distribution of measurement points made on the Hubble fractal diagram (Figure 10) and how this corresponds with and atomic like shape. Measurement points are concentrated around the origin of the curve, leaving increasing space beyond and out to the edge. This is a scale-free property of the fractal. Significance beyond this observation is outside the scope of this paper.

5 CONCLUSIONS

The in-site fractal model is an exquisite and inextricable fit to what is observed and conjectured in the cosmos. This fractal model demonstrates and addresses problems directly associated with the ΛCDM model of cosmology. From a fixed (but arbitrary) location within a (Koch snowflake) fractal set—and its beginning—the areas of triangles bits expand exponentially and marked points (on triangles) recession velocity from 'the observers' perspective also increased exponentially as a function of distance and time. This exponential expansion is a property shared by all irregular/chaotic fractal objects. The fractal model explains the conjectured 'dark energy' accelerated expansion of the universe. The model produced a cosmological constant. The fractal offers a geometric mechanism that explains the presence of the CMB and corresponds directly with conjectured expansion times of inflationary epoch expansion of the universe. There is an opportunity to further test and tie the fractal to the speed and nature of light and this (inflationary) expansion. The model demonstrated the expansion of space and reveals directly both a Hubble-Lemaitre Law and diagram. Both observations and the fractal model refute the cosmological principle. The fractal model explains and concurs with the distribution and demographics of galaxies in the observable universe—from the granted 'rough and fractal' on small cosmic scales to the old large and thin LQGs structures on large cosmic scales. As a result of the former, the model is in total violation of the cosmological principle. We do observe homogeneity, nor should we expect to in observing with a growing fractal, and the universe is therefore not isotropic. The fractal model offers a direct solution to the cosmological catastrophe, that the quantum and the cosmos are different sides of one emergent geometry. From the former, the model offers the opportunity to further our understanding of foundational quantum mechanics. The mechanism of fractal development, growth and emergence points to how quantum mechanics—the wave-particle duality of light and matter—is described by experts and this demands further exploration. Looking 'back' into the fractal corresponds with cosmological observations; looking 'forward' into the fractal from an in-situ observation point corresponds with 'the quantum'[1]. Together, fractal geometry will complete the knowledge gap. The fractal opens the door to a quantum unification, so-called quantum gravity. The model does not take away from what has already been achieved—namely General Relativity—it complements it and is a simple geometric.

By cosmological observations, the universe is behaving exactly as a growing fractal. If we had no cosmological observations, but only had fractal geometry to work from to form predictions on the structure and evolution of our universe; then based on the fractals' already universal ubiquity in our reality, we would expect the universe we currently observe. This is not the first time geometry has solved observational discrepancies or paradoxes; one only has to look at how circles and later ellipses explained and ended a paradigm. Fractals are the geometry of our time. It is time we use them.

6 APPENDIX



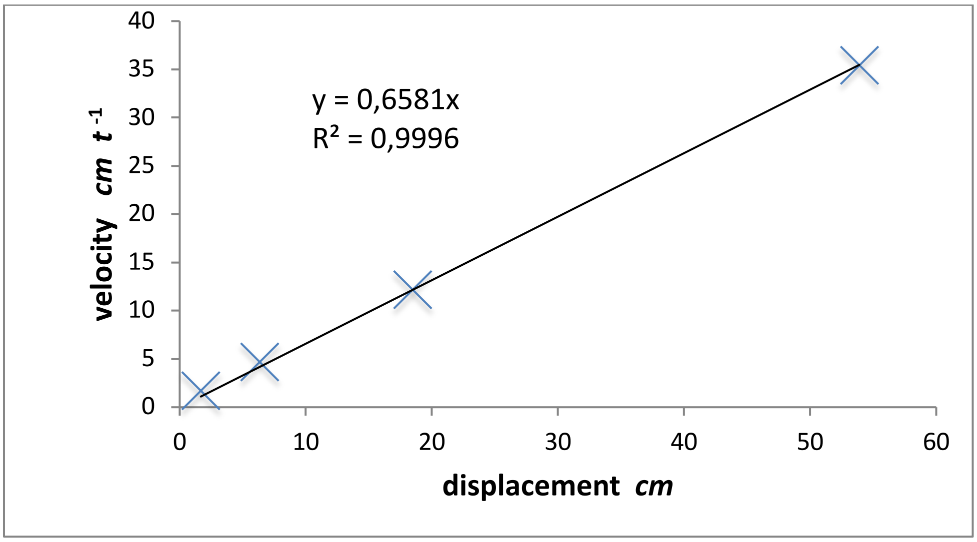
Displacement is measured between (discrete) triangle centres and used in the calculation of the fractal/Hubble constant. The red line traces the circumference (the distance) of the fractal spiral, and the blue line the displacement of the fractal spiral from an arbitrary centre of observation. cm = centimetres.

Figure 17. Displacement measurements from radii on the iterating Koch Snowflake created with TI-Nspire ™ software.

Table 1. **Displacement taken from radius measurements and calculations from the iterating Koch Snowflake fractal spiral (Appendix** Figure 17).

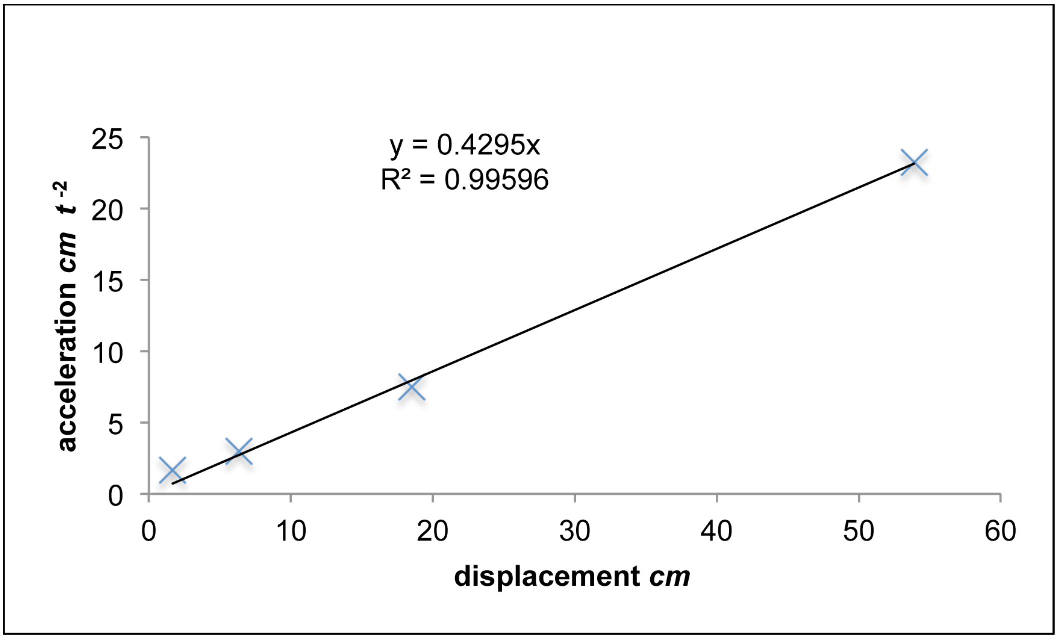
t	Displacement cm	Total Displacement : (*D*) cm	Expansion Ratio	Velocity: $cm\ i^{-1}$	Acceleration: $cm\ i^{-2}$	Acceleration Ratio
0			-			
1	1.68	1.68	-	1.68	1.68	
2	4.66	6.34	3.77	4.66	2.98	1.8
3	12.16	18.5	2.92	12.16	7.50	2.5
4	35.4	53.9	2.91	35.40	23.24	3.1

cm = centimetres.t= iteration-time.



From an arbitrary observation point on the inverted (Koch Snowflake) fractal: as the distance between triangle geometric centres points increases, the recession velocity of the points receding away increases. cm = centimetres. *t*= iteration-time.

Figure 18. The Hubble Fractal Diagram (recessional velocity vs. distance) from radius measurements (Appendix Figure 17).



From a fixed central observation point. Using radius measurements (Appendix Figure 1). As the distance between triangle geometric centres points increases, the recession acceleration of the points receding away increases. cm= centimetres. *t* = iteration-time.

Figure 19. Recessional acceleration with distance on the inverted Koch Snowflake fractal.

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