Magic Squares with Centrally Embedded Squares: A Construction Method

Clarence Gipbsin and Lamarr Widmer

Abstract

We present a method which modifies a magic square of order n and then adds two outer rows and two outer columns to produce a magic square of order $n + 2$. The modification of the original square will preserve the equality of sums of the rows, columns, and main diagonals. This modified square will be centrally embedded in the magic square of order $n + 2$.

Definitions

For the purposes of this paper, a **magic square** of order n shall mean an $n \times n$ arrangement of the integers 1 through n^2 such that the sums of each row, each column and both main diagonals all equal the magic sum $S = \frac{n^2(n^2+1)}{2n} = \frac{n}{2}\,(n^2 + 1)$. An **embedded square** of order n shall mean an $n \times n$ arrangement of distinct positive integers such that each row, each column and both main diagonals have the same sum. Figure 1 shows a magic square of order 8 with centrally embedded squares of order 4 and 6.

13	64	2	62	4	8	56	51
59	19	21	16	50	48	41	6
55	20	28	39	38	25	45	10
54	47	33	30	31	36	18	11
53	43	29	34	35	32	22	12
5	42	40	27	26	37	23	60
7	24	44	49	15	17	46	58
14	1	63	3	61	57	9	52

Figure 1

Construction Method

Our method may be carried out with the aid of a calculator for the necessary arithmetic and a spreadsheet to check the results. Each stage of our construction starts with a magic square, modifies it and adds two outer rows and two outer columns to arrive at a magic square of higher order. The original square will become an embedded square at the center of the larger one. We will signal those steps of this process which involve trial-and-error experimentation. One example will serve to illustrate our process.

Our example begins with the order-11 square shown in Figure 2, which will become the embedded square within a magic square of order 13. The new square will contain integers 1 through 169. For this reason, we add 24 to each entry, giving us the values in Figure 3. The columns, rows and diagonals of this array all sum to 935. The numbers in the two new rows and columns will be integers 1 through 24 and 146 through 169. They form 24 pairs, each summing to 170, as listed in Figure 4. So, when any such pair is added at the ends of an existing row, column or diagonal, the new row, column, or diagonal will consist of thirteen integers whose sum is 1105. If the two new rows and two new columns also sum to 1105, we will indeed have a magic square of order 13.

114	19	18	16	12	15	113	115	117	118	14
10	28	101	30	31	35	98	97	93	36	112
6	22	76	84	83	45	44	47	48	100	116
3	23	80	52	73	55	71	54	42	99	119
2	96	37	66	60	65	58	56	85	26	120
109	89	79	69	59	61	63	53	43	33	13
111	90	40	50	64	57	62	72	82	32	11
1	27	41	68	49	67	51	70	81	95	121
105	88	74	38	39	77	78	75	46	34	17
102	86	21	92	91	87	24	25	29	94	20
108	103	104	106	110	107	9	7	5	4	8

Figure 2

138	43	42	40	36	39	137	139	141	142	38
34	52	125	54	55	59	122	121	117	60	136
30	46	100	108	107	69	68	71	72	124	140
27	47	104	76	97	79	95	78	66	123	143
26	120	61	90	84	89	82	80	109	50	144
133	113	103	93	83	85	87	77	67	57	37
135	114	64	74	88	81	86	96	106	56	35
25	51	65	92	73	91	75	94	105	119	145
129	112	98	62	63	101	102	99	70	58	41
126	110	45	116	115	111	48	49	53	118	44
132	127	128	130	134	131	33	31	29	28	32

Figure 3

Here is where our trial-and-error experimentation begins. We first choose two of our 24 pairs to occupy the corners of our new order-13 square as shown in Figure 5. We illustrate by choosing the pairs (4,166) and (24,146).

pair		difference	
1	169	168	x
2	168	166	
3	167	164	x
4	166	162	x
5	165	160	
6	164	158	x
7	163	156	x
8	162	154	
9	161	152	
10	160	150	x
11	159	148	x
12	158	146	
13	157	144	
14	156	142	
15	155	140	x
16	154	138	x
17	153	136	
18	152	134	
19	151	132	
20	150	130	x
21	149	128	x
22	148	126	
23	147	124	x
24	146	122	x

Figure 4

24												166
	138	43	42	40	36	39	137	139	141	142	38	
	34	52	125	54	55	59	122	121	117	60	136	
	30	46	100	108	107	69	68	71	72	124	140	
	27	47	104	76	97	79	95	78	66	123	143	
	26	120	61	90	84	89	82	80	109	50	144	
	133	113	103	93	83	85	87	77	67	57	37	
	135	114	64	74	88	81	86	96	106	56	35	
	25	51	65	92	73	91	75	94	105	119	145	
	129	112	98	62	63	101	102	99	70	58	41	
	126	110	45	116	115	111	48	49	53	118	44	
	132	127	128	130	134	131	33	31	29	28	32	
4												146

Figure 5

The magic sum for an order-13 magic square is 1105. Thus, we determine that the remaining 11 numbers in our top row must sum to 915 and the remaining numbers in the rightmost column must sum to 793. One choice for completion of the rightmost column is

$$793 = 7 + 147 + 149 + 150 + 154 + 155 + 1 + 3 + 6 + 10 + 11$$

At this point, 13 of our 24 pairs have been used. They are marked in Figure 4. Here is the moment of truth: can we achieve the needed sum of 915 with the remaining pairs? If we add the larger numbers from each of these pairs, we have 1731. We need 915 and $1731 - 915 = 816$. Therefore, if we can find numbers in the column of differences that add to 816, they will indicate where we should choose the smaller number of the associated pair. In this case, we have $816 = 126 + 132 + 134 + 136 + 142 + 146$,

and then $915 = 168 + 165 + 162 + 161 + 12 + 157 + 14 + 17 + 18 + 19 + 22$.

Our completed square is in Figure 6.

If we had not been able to achieve the needed sum of 816 in the column of differences, then we would change our choice at one of the earlier steps and try again. We may change our choice either for the pairs occupying the corners or for the numbers summing to 793.

Observations and Suggestions for Further Investigation

Our method shows that there are a staggering number of such squares. In the example shown above, the eleven numbers completing the top row and the eleven numbers completing the rightmost column can be inserted in any order. This alone accounts for $(11!)^2 \approx 1.593 \times 10^{15}$ distinct squares. With this flexibility, we may pose other challenges. For example, the order-19 square in Figure 7 has the property that all entries above the central number, 181, end in the digit 5 while all entries below it end in the digit 7. The reader will notice a similar property for numbers in the tenth row. The order-20 square in Figure 8 exhibits a similar property along the main diagonals.

24	168	165	162	161	12	19	157	14	17	18	22	166
23	138	43	42	40	36	39	137	139	141	142	38	147
21	34	52	125	54	55	59	122	121	117	60	136	149
20	30	46	100	108	107	69	68	71	72	124	140	150
16	27	47	104	76	97	79	95	78	66	123	143	154
15	26	120	61	90	84	89	82	80	109	50	144	155
163	133	113	103	93	83	85	87	77	67	57	37	7
169	135	114	64	74	88	81	86	96	106	56	35	1
167	25	51	65	92	73	91	75	94	105	119	145	3
164	129	112	98	62	63	101	102	99	70	58	41	6
160	126	110	45	116	115	111	48	49	53	118	44	10
159	132	127	128	130	134	131	33	31	29	28	32	11
4	2	5	8	9	158	151	13	156	153	152	148	146

Figure 6

30	18	19	20	24	28	29	34	36	25	348	349	350	351	352	354	355	357	360
1	68	324	322	320	318	317	314	312	65	51	52	57	58	60	61	62	316	361
3	67	96	292	291	290	286	281	278	85	83	86	89	91	92	93	282	295	359
4	66	280	120	264	261	258	257	108	115	253	110	113	114	118	262	82	296	358
6	64	95	119	234	139	138	136	132	135	233	235	237	238	134	243	267	298	356
9	63	94	117	130	148	221	150	151	155	218	217	213	156	232	245	268	299	353
15	59	87	116	126	142	196	204	203	165	164	167	168	220	236	246	275	303	347
16	55	90	112	123	143	200	172	193	175	191	174	162	219	239	250	272	307	346
331	56	88	111	122	216	157	186	180	185	178	176	205	146	240	251	274	306	31
339	309	289	259	229	209	199	189	179	181	183	173	163	153	133	103	73	53	23
329	325	79	265	231	210	160	170	184	177	182	192	202	152	131	97	283	37	33
330	323	293	263	121	147	161	188	169	187	171	190	201	215	241	99	69	39	32
336	321	288	260	225	208	194	158	159	197	198	195	166	154	137	102	74	41	26
335	319	287	256	222	206	141	212	211	207	144	145	149	214	140	106	75	43	27
341	315	285	255	228	223	224	226	230	227	129	127	125	124	128	107	77	47	21
340	313	284	100	98	101	104	105	254	247	109	252	249	248	244	242	78	49	22
345	308	80	70	71	72	76	81	84	277	279	276	273	271	270	269	266	54	17
327	46	38	40	42	44	45	48	50	297	311	310	305	304	302	301	300	294	35
2	344	343	342	338	334	333	328	326	337	14	13	12	11	10	8	7	5	332

Figure 7

364	371	382	383	386	387	389	391	393	20	17	11	9	7	36	35	29	24	13	363
1	334	361	357	46	354	351	51	52	53	54	56	343	60	335	69	71	329	333	400
2	39	304	74	76	82	84	94	96	100	102	318	322	323	328	302	300	303	362	399
3	360	75	274	104	106	116	118	120	122	124	294	292	290	288	286	273	326	41	398
397	359	324	126	254	256	260	265	271	150	146	137	139	142	133	253	275	77	42	4
396	43	321	276	129	234	244	242	246	166	164	162	158	156	233	272	125	80	358	5
395	356	81	278	131	250	224	221	222	181	182	175	176	223	151	270	123	320	45	6
385	55	316	121	269	152	227	214	210	192	190	184	213	174	249	132	280	85	346	16
380	57	86	119	267	153	229	218	194	199	206	203	183	172	248	134	282	315	344	21
22	342	314	284	266	247	232	212	205	204	193	200	189	169	154	135	117	87	59	379
23	61	88	287	138	241	170	185	195	198	207	202	216	231	160	263	114	313	340	378
376	339	89	112	140	161	171	186	208	201	196	197	215	230	240	261	289	312	62	25
26	63	311	291	258	163	173	188	191	209	211	217	187	228	238	143	110	90	338	375
27	337	310	108	257	236	178	180	179	220	219	226	225	177	165	144	293	91	64	374
373	65	92	105	149	168	157	159	155	235	237	239	243	245	167	252	296	309	336	28
370	352	93	298	148	145	141	136	130	251	255	264	262	259	268	147	103	308	49	31
32	48	306	128	297	295	285	283	281	279	277	107	109	111	113	115	127	95	353	369
33	331	98	327	325	319	317	307	305	301	299	83	79	78	73	99	101	97	70	368
367	68	40	44	355	47	50	350	349	348	347	345	58	341	66	332	330	72	67	34
38	30	19	18	15	14	12	10	8	381	384	390	392	394	365	366	372	377	388	37

Figure 8

Examination of the modulo-4 values of our integers proved particularly interesting. For an order-11 square our method dictates that the center square be congruent to 1. The magic sum 671 is congruent to 3. Careful consideration of our choices throughout the construction process produced the square in Figure 9, where the coloring reveals the modulo-4 value of the entries.

8	2	112	6	4	107	12	102	108	106	104
17	36	98	96	94	29	30	32	34	100	105
19	95	48	46	44	43	80	82	84	27	103
13	101	85	56	70	53	54	72	37	21	109
11	31	83	55	60	59	64	67	39	91	111
121	97	45	73	65	61	57	49	77	25	1
119	99	47	71	58	63	62	51	75	23	3
117	33	81	50	52	69	68	66	41	89	5
115	35	38	76	78	79	42	40	74	87	7
113	22	24	26	28	93	92	90	88	86	9
18	120	10	116	118	15	110	20	14	16	114

Figure 9

All the trial-and-error choices mentioned above involve a finite number of possibilities. As such, they are resolvable by an exhaustive search. This is a project for a recreational programmer. The authors are recreational mathematicians; we have no interest in carrying out such a project but would certainly be interested in the results.

The first option in each stage of our method is the choice of two pairs of numbers to occupy the corner positions. Experience at the lower orders shows that certain choices at this step leave positions that cannot be successfully completed by any of the choices available for subsequent options to complete the outer rows and columns. A characterization of these futile choices would be interesting and avoiding them would be a significant improvement in our method.

About the authors:

Mr. Clarence Gipbsin is a student of divine mathematics who resides in California.

Dr. Lamarr Widmer is Emeritus Professor of Mathematics at Messiah University in Pennsylvania. He may be contacted at widmer@messiah.edu .