

How Quantum Mechanics could be hidden within the Spacetime

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The article shows how Quantum Mechanics and General Relativity could be unified within Einstein-Cartan theory that allows the spacetime to be compressed and torqued. The compression is responsible as well-known for astronomic gravity whereas the torsion for microscopic quantum effects as the Compton wavelength appearing into the Kerr metric strongly suggests. This could lead to a better understanding of both theories and promising applications such as macroscopic quantum and gravitational science and engineering based for instance on tremendous angular momenta.

Keywords: physics, quantum, gravity, torsion, general relativity, great unified theory, string theory, geometry, AdS/CFT, particles

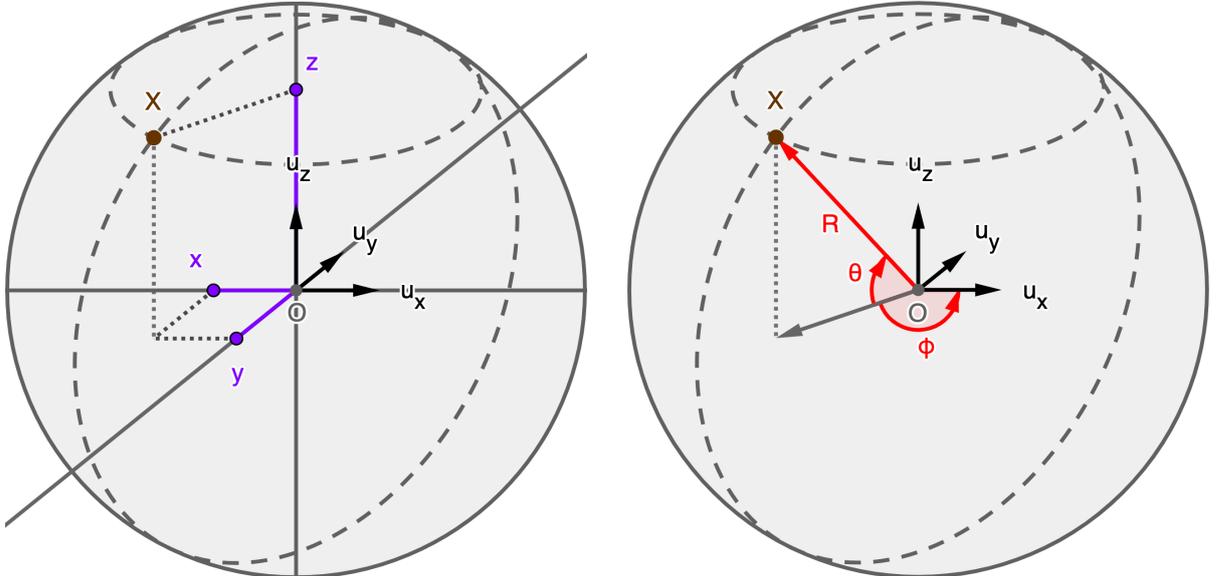
I. OUTLINE

Since the beginning of the 20th century, physicists are looking for a way to understand within a same theoretical framework General Relativity and Quantum Mechanics. Both were made from a different mathematical framework - linear algebra and differential geometry - that hardened the unification. In addition, gravity seemed to apply only on astrophysical scale only whereas quantum effects only on small particles, which means at the opposite scale. In this paper, I show a way to solve both issue, thanks to the polar

decomposition formulae and the astonishing discovery of Kerr : the Compton wavelength appears naturally within the metric of a rotating object [12]. This means that contrary to what was believed by physicists, gravity does play a key role at atomic scale.

II. THE MISSING DEGREES OF FREEDOM IN EINSTEIN EQUATION

A. The complete description of a geometry



The left scheme show how to relate a point \mathbf{x} on a sphere with Cartesian coordinates $\{x, y, z\}$ in the Cartesian unit vector base $\{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\}$. The right scheme show the same point \mathbf{x} but described now with the spherical coordinates $\{r, \theta, \phi\}$ ($r = R$ is the radius of the sphere in this case).

Figure 1. Cartesian to Spherical coordinates

A geometrical object can be described by an equation, like for instance the sphere of radius R :

$$R^2 = x^2 + y^2 + z^2 \quad (1)$$

It can be described with different set of coordinates, such as $\{x, y, z\}$ ¹ or $\{r, \theta, \phi\}$ ², both represented in figure 1, page 1, and related to each-other :

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \cos \phi \\ r \cos \theta \sin \phi \\ r \sin \theta \end{pmatrix} \\ \Leftrightarrow \mathbf{s} &= \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2 + z^2} \\ \arcsin \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \text{artan} \frac{y}{x} \end{pmatrix} \end{aligned} \quad (2)$$

The 3D sphere is of primary importance since most of the few known solutions of General Relativity like Schwarzschild or Kerr-Newman metric are based on spherical coordinates [2]. This is why it is taken as example to illustrate the subtleties leading to the result.

1. *The tangent plan contains (almost) all the information about the geometry*

One can derivate the position vector \mathbf{x} of the sphere regarding any set of coordinates (like $\{r, \theta, \phi\}$ for instance) and get the vectors tangent to the shape, like shown in figure 2, page 3 :

$$\begin{aligned} \mathbf{u}_\phi &\doteq \frac{\partial \mathbf{x}}{\partial \phi} = \begin{pmatrix} -r \cos \theta \sin \phi \\ r \cos \theta \cos \phi \\ 0 \end{pmatrix} \\ \mathbf{u}_\theta &\doteq \frac{\partial \mathbf{x}}{\partial \theta} = \begin{pmatrix} r \sin \theta \cos \phi \\ r \sin \theta \sin \phi \\ -r \cos \theta \end{pmatrix} \end{aligned} \quad (3)$$

These tangent vectors are in fact components of the Jacobean matrix of the position vector :

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \mathbf{s}} &= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = (\mathbf{u}_r \quad \mathbf{u}_\phi \quad \mathbf{u}_\theta) \\ &= \begin{pmatrix} \cos \theta \cos \phi & -r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta \sin \phi & r \cos \theta \cos \phi & r \sin \theta \sin \phi \\ \sin \theta & 0 & -r \cos \theta \end{pmatrix} \end{aligned} \quad (4)$$

This Jacobean matrix contains (almost) all the information about the geometry because the general relation with the Cartesian coordinates of equation (2) can be recovered from the Jacobean by term-by-term integration³ from the Taylor-Young expansion :

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{s}} d\mathbf{s} \quad (5)$$

Obviously, the fundamental vectorial equation (2) mapping the flat coordinates \mathbf{x} to the curved coordinates \mathbf{s} can only by integration up to a constant vector \mathbf{x}_o ⁴. This explains the *almost* :

$$\mathbf{x}(\mathbf{s}) = \int \frac{\partial \mathbf{x}}{\partial \mathbf{s}} d\mathbf{s} + \mathbf{x}_o \quad (6)$$

This understanding is key in the theory of a curved spacetime and is very different from what A. Einstein guessed [9]. Why ? Because a four dimensional (4D) Jacobean matrix \hat{J} contains $4 \times 4 = 16$ degrees of freedom while its metric contains only 10 independents parameters. In other words, Einstein's General Relativity which is based only on the metric is missing 6 degrees of freedom (at least⁵) about the 4D curved spacetime !⁶

¹ Cartesian coordinates

² Spherical coordinates

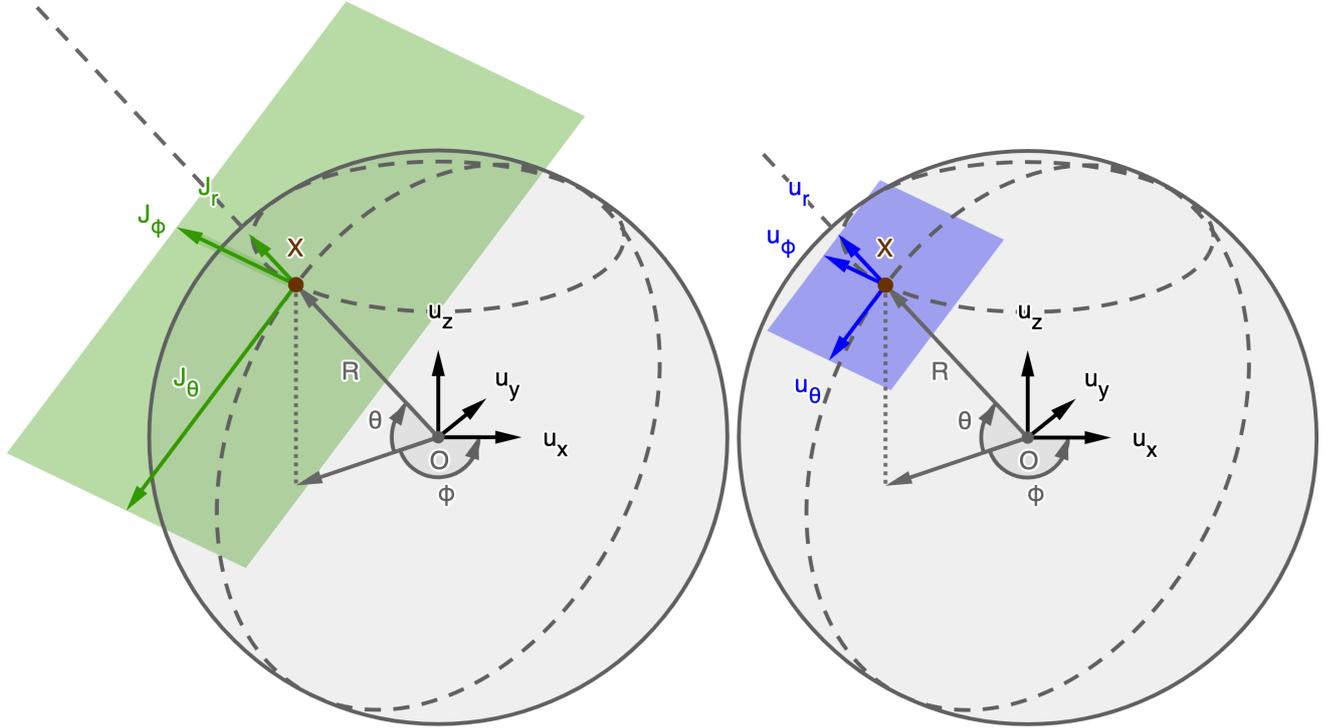
³ It is easier to integrate in the diagonal basis of the Jacobean, because variables are no longer entangled and then get back to the non-diagonal basis.

⁴ In the sphere example, the constant vector \mathbf{x}_o corresponds to the center of the object.

⁵ A 4D real Jacobean matrix contains 16 independents elements, but a complex one $2 \times 16 = 32$ independent parameters, because for every real one there is in addition an imaginary part. In this case, the hermitian metric $\hat{J}^\dagger \hat{J}$ contains $10 + 6 = 16$ in-

dependent parameters due to its hermitianity. Thus, in case of a 4D complex-valued Jacobean, Einstein's General Relativity is missing $32 - 16 = 16$ degrees of freedom !

⁶ Quantum Loop Gravity theorists came to same conclusion, that the fundamental object of the General Relativity can not be the metric but the local basis vectors, also called *Penrose tetrad* [15], but for a different reason : "The metric field $g_{\mu\nu}(x)$ can not be the fundamental field, because it does not allow fermion coupling. A better presentation of the gravitational field, compatible with the physical existence of fermions, is the tetrad formulation" [20]. We can already notice that these missing



The left scheme shows the tangent vectors of the sphere (in green) at the position \mathbf{x} (in brown), that are vectors of the Jacobian matrix \hat{J} . The tangent shows the orthogonal part $\hat{\Omega}$ of the Jacobian (obtained from its right polar decomposition) at the same point. Its vectors are represented in blue and form naturally an unit orthogonal basis. This information is precisely what is missing in Einstein's description of the metric. For instance the eigenvalues of the metric are the square of the size of the Jacobian vectors. In the general case $\{\mathbf{J}_r, \mathbf{J}_\phi, \mathbf{J}_\theta\}$ might not be aligned with $\{\mathbf{u}_r, \mathbf{u}_\phi, \mathbf{u}_\theta\}$. The information of these rotation angles between them are also contained within the metric and can also be deduced from its orthogonal eigenvectors. In fact the sphere is such a simple simple that the vectors of \hat{J} and $\hat{\Omega}$ are aligned.

Figure 2. The Jacobian tangent plan *v.s.* its orthogonal part

B. GR missing degrees of freedom are an unitary operator

What are these degrees of freedom ?

In fact it is quite straightforward to guess which are they thanks to the so-called right⁷ polar decomposition [10].

1. The polar decomposition of the spacetime Jacobian \hat{J}

Let recall that Einstein spacetime metric is :

$$\hat{G} \doteq \hat{J}^\top \hat{J} \quad (7)$$

degrees of freedom are related to fermions, thus Quantum Mechanics.

⁷ It is the right and not the left polar decomposition in order to retrieve the Minkowskian scalar product.

with $^\top$ holding for the transposition operator⁸. The polar decomposition fomulæ tells that if \hat{J} is invertible then⁹ :

$$\hat{\Omega} \doteq \hat{J} \sqrt{\hat{G}}^{-1} \quad (8)$$

exists and is an orthogonal matrix¹⁰.

In other words :

$$\hat{J} = \hat{\Omega} \sqrt{\hat{G}} \quad (9)$$

⁸ Recall : $\mathbf{a}^\top = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^\top = (a_1 \ a_2)$

⁹ Recall : the square root of a matrix can be computed by square-rooting its eigenvalues in its eigenbasis. For instance if \hat{P} is the matrix containing the eigenvectors of a matrix \hat{A} with eigenvalues λ_1 and λ_2 , then : $\sqrt{\hat{A}} \doteq \hat{P}^{-1} \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}) \hat{P}$.

¹⁰ Recall : orthogonal matrices are invertible matrices $\hat{\Omega}$ such that $\hat{\Omega}^\top = \hat{\Omega}^{-1}$.

Therefore $\hat{\Omega}$ contains the degrees of freedom that are not in the metric !

We can already notice that an orthogonal matrix is a spacial real case of a unitary matrix. Therefore Ω is solution of a Schrödinger-like equation (in the Heisenberg picture) by defining the Hamiltonian with the matrix logarithm¹¹ :

$$\begin{aligned} \hat{\Omega}(t) &\doteq e^{i \int \hat{H}(t) dt} \\ \Leftrightarrow \hat{H}(t) &= -i \frac{\partial \ln \hat{\Omega}(t)}{\partial t} \end{aligned} \quad (10)$$

Then for any ψ vector evolving between the time t and t_o :

$$\begin{aligned} -i \frac{d\psi(t)}{dt} &= \hat{H}(t)\psi(t) \\ \Leftrightarrow \psi(t) &= \hat{\Omega}(t - t_o)\psi(t_o) \end{aligned} \quad (11)$$

For a complex-valued spacetime this also works by taking the hermitian distance instead of the Riemannian one to compute the metric (use the trans-conjugate¹² \dagger instead of the transposition operator) to get the hermitian metric \hat{M} :

$$\hat{M} \doteq \hat{J}^\dagger \hat{J} \quad (12)$$

Then the polar decomposition tells that if \hat{J} is invertible :

$$\hat{U} \doteq \hat{J} \sqrt{\hat{M}}^{-1} \quad (13)$$

exists and is unitary¹³. Therefore in the complex case :

$$\hat{J} = \hat{U} \sqrt{\hat{M}} \quad (14)$$

In other words, the missing degrees of freedom in Einstein Gravity are angles¹⁴, which completes the distance information given by the metric by the way.

¹¹ Recall : similarly to square root of matrices, power series of matrices can be computed by taking the power serie of its eigen values (in the diagonal basis).

¹² Recall : $\mathbf{a}^\dagger = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^\dagger = (a_1^* \ a_2^*)$ with $*$ the complex conjugate : $z^* = (x + iy)^* = x - iy$ with x and y two real numbers and i the imaginary square root of -1 such that $i^2 = -1$.

¹³ Recall : an invertible matrix \hat{U} is unitary if and only if $\hat{U}^\dagger = \hat{U}^{-1}$.

¹⁴ Because orthogonal matrices describe rotations (or symmetries) and unitary matrices similarly describe complex-like rotations (or symmetries) like the elements of the SU(2) group.

2. Conclusion

The Jacobean of the spacetime, if invertible, can be decomposed in two parts : a metric, solution of Einstein's equation, and a unitary matrix, solution of a Schrödinger equation !

This strongly suggests that contrary to what Quantum Loop Gravity and String Theory tried to prove, Quantum Mechanics and General Relativity might be two complementary parts of a same global theory¹⁵. These mathematics of the polar decomposition are really wonderful, they explain how it is possible to unify two very different theories : Quantum Mechanics that deals with an hermitian norm and potentially complex rotations and General Relativity that deals with the local Riemannian scalar product¹⁶ with a metric that in contrary tells how distances are modified along a curved geometry. The great object of the unification is very likely to be in fact the Jacobean of the spacetime, that can be split into an Einsteinian metric and an Heisenberg unitary operator giving both Quantum and Gravity theories that appear now to be complementary.

III. GENERAL RELATIVITY TELLS US THAT THERE IS GRAVITY AT COMPTON SCALE !

The first part of the article explained how it is possible to unify at the mathematical level Quantum Mechanics and General Relativity within one global spacetime theory. But General Relativity is currently known to manifest only at astronomical scale (at the scale of the Schwarzschild radius) and not at the atomic scale (at the scale of the Compton wavelength [8]). Therefore, how this mathematical possibility could be actually achieved in Nature ? The answer was miraculously given by Kerr : the Compton wavelength does appear

¹⁵ And not GR a subpart of QM that remains to be quantified.

¹⁶ I mean by Riemannian scalar product a scalar product computed with the transposition operator : $d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{S}^\top \hat{J}^\top \hat{J} d\mathbf{S} = d\mathbf{S}^\top \hat{G} d\mathbf{S}$. It is clearly different from the hermitian scalar product used in Quantum Mechanics (within the Dirac bra-ket notation for instance), that can be generalized to complex curved spaces following the same way, just by using the trans-conjugate \dagger operator instead of the transposition \top : $d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{S}^\dagger \hat{J}^\dagger \hat{J} d\mathbf{S} = d\mathbf{S}^\dagger \hat{M} d\mathbf{S}$. A new operator then pops up, \hat{M} , the *hermitian metric*. Please also notice that these calculus works also for tangent vectors, like the velocities, "by dividing by dt ", but no longer for higher derivatives such as the acceleration vector ! Therefore please don't use Einstein's notation and metric scalar product for force or acceleration vectors, it is not correct in the general case ! Instead use the covariant derivatives.

into the Kerr metric and therefore gravity affects the dynamics at atomic scale. Not Newtonian gravity nor Einsteinian one but Cartan torsional gravity. Let's explicit this.

A. The Compton wavelength appears in Kerr's metric !

The Kerr metric is, in a symmetric matrix representation [2] :

$$\hat{G}_K = \begin{pmatrix} -\left(1 - \frac{r_s r}{\rho^2}\right) & 0 & -\frac{2r_s a r}{\rho^2} \cos \theta^2 & 0 \\ \cdot & \frac{\rho^2}{\Delta^2} & 0 & 0 \\ \cdot & \cdot & \left(r^2 + a^2 + \frac{r_s r a^2}{\rho^2} \cos \theta^2\right) \cos \theta^2 & 0 \\ \cdot & \cdot & \cdot & \rho^2 \end{pmatrix} \quad (15)$$

with :

- $\Delta \doteq r^2 + a^2 - r_s r$
- $\rho^2 \doteq r^2 + a^2 \cos^2 \theta$ the Kerr square radius
- $r_s \doteq \frac{2GM}{c^2}$ the Schwarzschild radius
- $a \doteq \frac{L}{Mc}$ the Compton wavelength

Everyone can already see that the a term is exactly the Compton wavelength for a particle with an angular momentum $L = \hbar$. This strongly suggests that gravity does play a key role at the Quantum Mechanics scale !

B. Why nobody pointed out the Compton wavelength within the Kerr metric ?

This was completely missed until now despite a lot of scientists felt that it shall be the case, such as Alexander Burinskii [6] or Kjell Rosquist [19]. Why this was not raised before ? I suggests it is due to the fact that Kerr's work was published late in 1963 [12], decades after the Quantum Mechanics was theorized. In addition, General Relativity is much harder at the mathematical level than Quantum Mechanics (a non linear *v.s.* a linear theory) which makes the QM framework more convenient and accessible for daily physics research.

C. Why the Compton wavelength appears in General Relativity ?

Why there is not the Newton constant G appearing in this a length scale ? It is because it does not come from Einstein equations that don't take correctly into account the contribution of angular momenta [4, 18] but

from Einstein-Cartan equations . Let's get the characteristic length ℓ of gravitational dynamics from a dimensional analysis on them, as it is done to estimate the Reynolds number¹⁷ :

$$\left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right] = [\kappa T_{\mu\nu}] \Rightarrow \ell \sim \kappa [E] \quad (16)$$

For the mass energy $E = Mc^2$, ℓ is proportional as expected to the Schwarzschild radius [2] :

$$\ell \sim \frac{8\pi GM}{c^2} \sim r_s \quad (17)$$

And for the electrostatic energy $E = \frac{Q^2}{4\pi\epsilon_0\ell}$ to the the Reissner-Norsdrom radius :

$$\ell \sim \sqrt{\frac{2GQ^2}{\epsilon_0 c^4}} \sim r_Q \quad (18)$$

This is why so many people believed that Gravity manifests only at astronomic length scale. But now let's do the same trick with the Einstein-Cartan set of equations¹⁸ [7, 11] :

$$[\Theta_{ac}{}^c + g_a{}^c \Theta_{bd}{}^d - g_b{}^c \Theta_{ad}{}^d] = [\kappa \sigma_a{}^{bc}] \Rightarrow \ell^2 \sim \kappa[\tau] \quad (19)$$

In Einstein-Cartan theory, the Einstein set of equations is modified to enforce the symmetrization of the energy-impulsion tensor by adding to it bounded current like for static magnets [4, 18] :

$$\begin{aligned} & \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right] \\ &= \kappa \left[T_{\mu\nu} + \frac{1}{2} \partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}) \right] \quad (20) \\ &\Rightarrow \ell \sim \kappa \left[E + \frac{\tau}{\ell} \right] \end{aligned}$$

Using the torsion length scale from equation (19) one can deduce that $\kappa \sim \frac{\ell^2}{\tau}$ and then introducing it back into equation (20) :

$$\ell \sim \left[\frac{\ell^2}{\tau} \right] \left[E + \frac{\tau}{\ell} \right] \Rightarrow \ell \sim \frac{\tau}{E} \quad (21)$$

¹⁷ Notations : $R_{\mu\nu}$ is the Ricci tensor, $g_a{}^c$ the metric tensor and $T_{\mu\nu}$ the energy-momentum tensor in Einstein notation. Λ is the so-called gravitational constant and $\kappa \doteq \frac{8\pi G}{c^4}$.

¹⁸ Notations : $\Theta_{ac}{}^c$ is the torsion tensor and $\sigma_a{}^{bc}$ is the spin tensor in Einstein notation. τ holds for the torque.

If the mass energy $E = Mc^2$ and the torque is extreme as for an object with angular momentum L spinning at the velocity c , then $\tau \sim Lc$ and ℓ is proportional to the Compton wavelength:

$$\ell \sim \frac{L}{Mc} \quad (22)$$

Indeed Kerr deduced its solution not from Einstein equations but from Petrov classification [12, 16] that was deduced from Penrose null-tetrad formalism that implicitly includes Cartan theory [12, 15]¹⁹. This is why so few physicists noticed that Kerr metric is not a solution of Einstein equations, while in fact Einstein theory does not take correctly into account spin and angular momentum due to the symmetry of the energy-momentum tensor. These remarks are really important because as you can understand it changes the characteristic scale where Gravity start to affect physics : at astronomical scale for the Einstein part and at atomic (Compton) scale for Cartan (torsion) part. In addition, Kerr metric was confirmed by Gravity Probe B experiment [24] and more recently by from gravitational waves observation [1], confirming then indirectly the Cartan extension of Einstein theory, because the Einstein one does not take properly into account rotational motions as previously recalled [4, 18]. This confirmation is great because Cartan theory suppresses singularities from black holes and the Big Bang [11, 17] and might have other interesting implications at the astronomical scale, especially since recently it was discovered that great structures within our Universe might rotate [22].

IV. DISCUSSION

A. Practical consequences

The fact that the Compton wavelength where the Quantum Mechanics start to manifest is not only at the scale of $\frac{\hbar}{mc}$ but more generally at the scale of $\frac{L}{mc}$ with L a component of the angular momentum tensor²⁰ of an object suggests that macroscopic quantum behavior shall manifest more easily in high angular momenta

systems for instance. Twisted light may be of great help to engineer such devices [21]. Such experiments could also be used to test and probe the Einstein-Cartan Quantum theory.

More generally the angular momentum tensor is a big mathematical object and contains more degrees of freedom than in the angular momentum vector. All these degrees of freedom could torque the spacetime and are candidates to describe the geometry of actual particles like the electron. Some of these components shall match with the "spin" degree of freedom of Quantum Mechanics.

B. Compatibility with other theories

Interestingly enough, when quantum mechanics is artificially added on Einstein spacetime (in a Einstein-Dirac theory) it also leads to the same conclusion : fermions can generate wormholes [5]. As shown above, Cartan theory suggests the same but in the opposite point of view : wormholes could be fermions. It also brings a new hope to J. Wheeler's theory of geometrodynamics that attempted to explain everything from a geometrical point of view²¹ [23]. In addition, as stated in section II A this theory is also compatible with Quantum Loop Gravity and the fact that particles are non-singular could also give ground to the String and M Theory assumptions about extended particles [3]. The vibrating strings or membranes could be either the event horizon or the mouth of a wormhole particle. This work can also feed the Maldacena's AdS/CFT correspondence hypothesis [13, 14].

V. CONCLUSION

First, it was shown that a complete geometrical theory of Nature shall take into account more than the metric degrees of freedom of a spacetime but also it's unitarian degrees of freedom. Second, the fact that in Einstein-Cartan theory the gravity does play a role at Compton wavelength such as proved by the Kerr metric strongly suggests that the unitary part of the Jacobean of the spacetime could explain the whole Quantum Mechanics.

Further work shall focus on :

¹⁹ Because null tetrads are sets of null vectors, in the degenerate euclidean norm used by A. Einstein in its version of General Relativity and these local null vectors are those of the spacetime Jacobean. Which means that they contains the rotational degrees of freedom that are missing in the Einstein equation, as explained previously in section II.

²⁰ It can be the angular momentum vector for instance.

²¹ But by trying to map quantum mechanics degrees of freedom (angles) into the metric (that describes distances). It could not work this way due to their complementary nature.

- Exhibit the unitary part of the Jacobean of Kerr spacetime and see if this can explain how \hbar appears in the spacetime Schrödinger equation (11).
- Look at other sources of torsional gravity from the angular momentum tensor and see how they could be related to experiments.
- Because Lorentz boost let invariant the Minkowskian metric and leads to conserved Noether currents, look at how these boosts and current are affected by a curved metric and if this can explain Higgs-like breaking mechanism.
- Deduce the Lagrangians of String Theories from Cartan particles.
- Import results from Quantum Loop Gravity theory to this one.
- Study of the potential bridges with the AdS/CFT correspondence.

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