A Proof of the Erdös-Straus Conjecture

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Abstract

In this article, we classify positive integers step by step, and use the formulation to represent a certain class therein until all classes.

First, divide all integers ≥2 into 8 kinds, and formulate each of 7 kinds therein into a sum of 3 unit fractions.

For the unsolved kind, again divide it into 3 genera, and formulate each of 2 genera therein into a sum of 3 unit fractions.

For the unsolved genus, further divide it into 5 sorts, and formulate each of 3 sorts therein into a sum of 3 unit fractions.

For two unsolved sorts i.e. $4/(49+120c)$ and $4/(121+120c)$ where $c\geq 0$, we use an unit fraction plus a proper fraction to replace each of them, then take out the unit fraction as $1/x$. After that, we take out an unit fraction from the proper fraction and regard the unit fraction as $1/y$, and finally, prove that the remainder can be identically converted to $1/z$.

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1. Introduction

The Erdös-Straus conjecture relates to Egyptian fractions. In 1948, Paul Erdös conjectured that for any integer $n \geq 2$, there are invariably $4/n=1/x+1/y+1/z$, where x, y and z are positive integers; [1].

Later, Ernst G. Straus further conjectured that x, y and z satisfy $x \neq y$, $y \neq z$ and $z \neq x$, because there are the convertible formulas $1/2r+1/2r=1/(r+1)+1/r(r+1)$ and $1/(2r+1)+1/(2r+1)=1/(r+1)+1/(r+1)(2r+1)$ where $r \geq 1$; [2].

Thus, the Erdös conjecture and the Straus conjecture are equivalent from each other, and they are called the Erdös-Straus conjecture collectively.

As a general rule, the Erdös-Straus conjecture states that for every integer $n \geq 2$, there are positive integers x, y and z, such that $4/n=1/x+1/y+1/z$. Yet it remains a conjecture that has neither is proved nor disproved; [3].

2. Divide integers≥2 into 8 kinds and formulate 7 kinds therein

First, divide integers≥2 into 8 kinds, i.e. 8k+1with k≥1, and 8k+2, 8k+3, 8k+4, 8k+5, 8k+6, 8k+7, 8k+8, where k≥0, and arrange them as follows:

K\n:	8k+1,	8k+2,	8k+3,	8k+4,	8k+5,	8k+6,	8k+7,	8k+8
0,	①,	2,	3,	4,	5,	6,	7,	8,
1,	9,	10,	11,	12,	13,	14,	15,	16,
2,	17,	18,	19,	20,	21,	22,	23,	24,
3,	25,	26,	27,	28,	29,	30,	31,	32,
…,	…,	…,	…,	…,	…,	…,	…,	…,

Excepting n=8k+1, formulate each of other 7 kinds into 1/x+1/y+1/z:

(1) For n=8k+2, there are 4/(8k+2)=1/(4k+1)+1/(4k+2)+1/(4k+1)(4k+2);

(2) For n=8k+3, there are 4/(8k+3)=1/(2k+2)+1/(2k+1)(2k+2)+1/(2k+1)(8k+3);

(3) For n=8k+4, there are 4/(8k+4)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+1)(2k+2);

(4) For n=8k+5, there are 4/(8k+5)=1/(2k+2)+1/(8k+5)(2k+2)+1/(8k+5)(k+1);

(5) For n=8k+6, there are 4/(8k+6)=1/(4k+3)+1/(4k+4)+1/(4k+3)(4k+4);

(6) For n=8k+7, there are 4/(8k+7)=1/(2k+3)+1/(2k+2)(2k+3)+1/(2k+2)(8k+7);

(7) For n=8k+8, there are 4/(8k+8)=1/(2k+4)+1/(2k+2)(2k+3)+1/(2k+3)(2k+4).

By this token, n as above 7 kinds of integers be suitable to the conjecture.

3. Divide the unsolved kind into 3 genera and formulate 2 genera therein

For the unsolved kind n=8k+1 with k≥1, divide it by 3 and get 3 genera: (1) the remainder is 0 when k=1+3t; (2) the remainder is 2 when k=2+3t; (3) the remainder is 1 when k=3+3t, where t≥0, and *ut infra*.

k:	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15, ...
8k+1:	9,	17,	25,	33,	41,	49,	57,	65,	73,	81,	89,	97,	105,	113,	121, ...
The remainder:	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1,	0,	2,	1, ...

Excepting the genus (3), we formulate other 2 genera as follows:

(8) For (8k+1)/3 per the remainder=0, there are 4/(8k+1)=1/(8k+1)/3+ 1/(8k+2)+1/(8k+1)(8k+2).

Due to k=1+3t and t≥0, there are (8k+1)/3= 8t+3, so we confirm that

(8k+1)/3 in the preceding equation is an integer.

(9) For (8k+1)/3 per the remainder=2, there are 4/(8k+1)=1/(8k+2)/3+ 1/(8k+1)+1/(8k+1)(8k+2)/3.

Due to k=2+3t and t≥0, there are (8k+2)/3= 8t+6, so we confirm that (8k+2)/3 in the preceding equation is an integer.

4. Divide the unsolved genus into 5 sorts and formulate 3 sorts therein

For the unsolved genus (8k+1)/3 per the remainder=1 when k=3+3t and t≥0, i.e. 8k+1=25, 49, 73, 97, 121 etc. we divide them into 5 sorts: 25+120c, 49+120c, 73+120c, 97+120c and 121+120c where c≥0, and *ut infra*.

C\n:	25+120c,	49+120c,	73+120c,	97+120c,	121+120c,
0,	25,	49,	73,	97,	121,
1,	145,	169,	193,	217,	241,
2,	265,	289,	313,	337,	361,
…,	…,	…,	…,	…,	…,

Excepting n=49+120c and n=121+120c, formulate other 3 sorts as follows:

(10) For n=25+120c, there are 4/(25+120c)=1/(25+120c)+1/(50+240c)+1/(10+48c);

(11) For n=73+120c, there are 4/(73+120c)=1/(73+120c)(10+15c)+1/(20+30c)+ 1/(73+120c)(4+6c) ;

(12) For n=97+120c, there are 4/(97+120c)=1/(25+30c)+1/(97+120c)(50+60c)+ 1/(97+120c)(10+12c).

For each of listed above 12 equations which express 4/n=1/x+1/y+1/z, please each reader self to make a check respectively.

5. Prove the sort 4/(49+120c)=1/x+1/y+1/z

For a proof of the sort 4/(49+120c), it means that when c is equal to each of positive integers plus 0, there always are 4/(49+120c)=1/x+1/y+1/z. After c is given any value, 4/(49+120c) can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below:

4/(49+120c)

= 1/(13+30c) + 3/(13+30c)(49+120c)

= 1/(14+30c) + 7/(14+30c)(49+120c)

= 1/(15+30c) + 11/(15+30c)(49+120c)

…

= 1/(13+α+30c) + (3+4α)/(13+α+30c)(49+120c), where α≥0 and c≥0

…

As listed above, we can first let 1/(13+α+30c)=1/x, then go to prove (3+4α)/(13+α+30c)(49+120c) = 1/y +1/z, where c≥0 and α≥0, *ut infra*.

***Proof*·** First, we analyse 3+4α on the place of numerator, it is not hard to see, except 3+4α as one numerator, it can also be expressed as the sum of an even number plus an odd number to act as two numerators, i.e. (4α+3), (4α+2)+1, (4α+1)+2, (4α)+3, (4α-1)+4, (4α-2)+5, (4α-3)+6, ...

If there are two addends on the place of numerator, then these two addends are regarded as two matching numerators, and that two matching numerators are denoted by ψ and φ, also there is $\psi>\varphi$.

In numerators with the same denominator, largest ψ is denoted as ψ_1. It is obvious that ψ_1 matches with smallest φ, and $\psi_1=4\alpha+2$ and smallest $\varphi=1$.

And then let us think about the denominator (13+α+30c)(49+120c), actually just 13+α+30c is enough, while reserve 49+120c for later.

In the fraction (4α+3)/(13+α+30c), let each α be assigned a value for each time, according to the order α= 0, 1, 2, 3,... So the denominator 13+α+30c can be assigned into infinite more consecutive positive integers.

As the value of α goes up, accordingly numerators are getting more and more, and newly- added numerators are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, the denominators 13+α+30c and the numerators 4α+3, ψ and φ are listed below.

13+α+30c,	α,	(4α+3),	(4α+2)+1,	(4α+1)+2,	(4α)+3,	(4α-1)+4,	(4α-2)+5,	(4α-3)+6, ...	
13+30c,	0,	3,	2+1,	1+2					
14+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6	
15+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...	
16+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...	
17+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...	
...,	...,	...,	...,	...,	...,	...,	...,	...,	...

As can be seen from the list above, every denominator (13+α+30c) corresponds with two special matching numerators ψ_1 and 1, from this, we get the unit fraction $1/(13+\alpha+30c)$.

For the unit fraction $1/(13+\alpha+30c)$, multiply its denominator by $49+120c$ reserved, then we get the unit fraction $1/(13+\alpha+30c)(49+120c)$, and let $1/(13+\alpha+30c)(49+120c) = 1/y$.

After that, we start to prove that $\psi_1/(13+\alpha+30c)$ i.e. $(4\alpha+2)/(13+\alpha+30c)$ is an unit fraction.

Since the numerator $4\alpha+2$ is an even number, such that the denominator $(13+\alpha+30c)$ must be an even numbers. Only in this case, it can reduce the fraction, so α in the denominator $13+\alpha+30c$ is only an odd number.

After α is assigned to odd numbers 1, 3, 5 and otherwise, and the fraction $(4\alpha+2)/(13+\alpha+30c)$ after the values assignment divided by 2, then the fraction $(4\alpha+2)/(13+\alpha+30c)$ is turned into the fraction $(3+4t)/(k+15c)$ identically, where $c\geq 0$, $t\geq 0$ and $k\geq 7$.

The point above is that $3+4t$ and $k+15c$ after the values assignment make up a fraction, they are on the same order of taking values of t and k, according to the order from small to large, i.e. $(3+4t)/(k+15c)= 3/(7+15c)$, $7/(8+15c)$, $11/(9+15c)$, ...

Such being the case, let the numerator and denominator of the fraction $(3+4t)/(k+15c)$ divided by $3+4t$, then we get a temporary indeterminate unit fraction, and its denominator is $(k+15c)/(3+4t)$, and its numerator is 1.

Thus, be necessary to prove that the denominator $(k+15c)/(3+4t)$ is able to become a positive integer in which case $t \geq 0$, $k \geq 7$ and $c \geq 0$.

In the fraction $(k+15c)/(3+4t)$, due to $k \geq 7$, the numerator $k+15c$ after the values assignment are infinite more consecutive positive integers, while the denominator $3+4t = 3, 7, 11$ and otherwise positive odd numbers.

The key above is that each value of $3+4t$ after the values assignment can seek its integral multiples within infinite more consecutive positive integers of $k+15c$, in which case $t \geq 0$, $k \geq 7$ and $c \geq 0$.

As is known to all, there is a positive integer that contains the odd factor $2n+1$ within $2n+1$ consecutive positive integers, where $n=1, 2, 3, \ldots$

Like that, there is a positive integer that contains the odd factor $3+4t$ within $3+4t$ consecutive positive integers of $k+15c$, no matter which odd number that $3+4t$ is equal to, where $t \geq 0$, $k \geq 7$ and $c \geq 0$. It is obvious that a fraction that consists of such a positive integer as the numerator and $3+4t$ as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer.

That is to say, $(k+15c)/(3+4t)$ as the denominator of the aforesaid temporary indeterminate unit fraction can become a positive integer, and the positive integer is represented by μ, and thus in this case the fraction $(3+4t)/(k+15c)$ is exactly $1/\mu$.

For the unit fraction $1/\mu$, multiply its denominator by $49+120c$ reserved, then we get the unit fraction $1/\mu(49+120c)$, and let $1/\mu(49+120c)=1/z$.

If $3+4\alpha$ serve as one numerator, we get $(3+4\alpha)/(13+\alpha+30c)(49+120c)=1/y$ likewise by the method of proving $\psi_1/(13+\alpha+30c)(49+120c)=1/z$.

When $3+4\alpha$ serve as one numerator and from this get an unit fraction, we can multiply the denominator of the unit fraction by 2 to make a sum of two identical unit fractions, then convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r = 1/(r+1)+1/r(r+1)$.

Thus it can be seen, the fraction $(3+4\alpha)/(13+\alpha+30c)(49+120c)$ is surely able to be expressed into a sum of two each other's -distinct unit fractions, where $c\geq0$ and $\alpha\geq0$.

Overall, there are $4/(49+120c)=1/(13+\alpha+30c)+1/(13+\alpha+30c)(49+120c) + 1/\mu(49+120c)$, where $\alpha\geq0$, μ is an integer and $\mu=(k+15c)/(3+4t)$, $t\geq0$, $k\geq7$ and $c\geq0$.

In other words, we have proved $4/(49+120c)=1/x+1/y+1/z$.

6. Prove the sort $4/(121+120c)=1/x+1/y+1/z$

The proof in this section is exactly similar to that in the section 5. Namely, for a proof of the sort $4/(121+120c)$, it means that when c is equal to each of positive integers plus 0, there always are $4/(121+120c)=1/x+1/y+1/z$.

After c is given any value, $4/(121+120c)$ can be substituted by each of infinite more a sum of an unit fraction plus a proper fraction, and that these fractions are different from one another, as listed below.

$4/(121+120c)$

$= 1/(31+30c) + 3/(31+30c)(121+120c)$,

$= 1/(32+30c) + 7/(32+30c)(121+120c)$,

$= 1/(33+30c) + 11/(33+30c)(121+120c)$,

…

$= 1/(31+\alpha+30c) + (3+4\alpha)/(31+\alpha+30c)(121+120c)$, where $\alpha \geq 0$ and $c \geq 0$.

…

As listed above, we can first let $1/(31+\alpha+30c)=1/x$, then go to prove $(3+4\alpha)/(31+\alpha+30c)(121+120c) =1/y +1/z$, where $c \geq 0$ and $\alpha \geq 0$, *ut infra*.

***Proof*·** First, we analyse $3+4\alpha$ on the place of numerator, it is not hard to see, except $3+4\alpha$ as one numerator, it can also be expressed as the sum of an even number and an odd number to act as two numerators, i.e. $(4\alpha+3)$, $(4\alpha+2)+1$, $(4\alpha+1)+2$, $(4\alpha)+3$, $(4\alpha-1)+4$, $(4\alpha-2)+5$, $(4\alpha-3)+6$, ...

If there are two addends on the place of numerator, then these two addends are regarded as two matching numerators, and that two matching numerators are denoted by ψ and φ, also there is $\psi>\varphi$.

In numerators with the same denominator, largest ψ is denoted as ψ_1. It is obvious that ψ_1 matches with smallest φ, and $\psi_1=4\alpha+2$ and smallest $\varphi=1$.

And then let us think about the denominator $(31+\alpha+30c)(121+120c)$, actually just $31+\alpha+30c$ is enough, while reserve $121+120c$ for later.

In the fraction $(4\alpha+3)/(31+\alpha+30c)$, let each α be assigned a value for each time, according to the order α= 0, 1, 2, 3, ...So the denominator 31+α+30c can be assigned into infinite more consecutive positive integers.

As the value of α goes up, accordingly, numerators are getting more and more, and newly- added numerators are getting bigger and bigger.

When α =0, 1, 2, 3 and otherwise, the denominators 31+α+30c and the numerators 4α+3, ψ and φ are listed below.

31+α+30c,	α,	(4α+3),	(4α+2)+1,	(4α+1)+2,	(4α)+3,	(4α-1)+4,	(4α-2)+5,	(4α-3)+6, ...	
31+30c,	0,	3,	2+1,	1+2					
32+30c,	1,	7,	6+1,	5+2,	4+3,	3+4 ,	2+5,	1+6	
33+30c,	2,	11,	10+1,	9+2,	8+3,	7+4,	6+5,	5+6, ...	
34+30c,	3,	15,	14+1,	13+2,	12+3,	11+4,	10+5,	9+6, ...	
35+30c,	4,	19,	18+1,	17+2,	16+3,	15+4,	14+5,	13+6,...	
...,	...,	...,	...,	...,	...,	...,	...,	...,	...

As can be seen from the list above, every denominator (31+α+30c) corresponds with two special matching numerators ψ_1 and 1, from this, we get the unit fraction 1/(31+α+30c).

For the unit fraction 1/(31+α+30c), multiply its denominator by 121+120c reserved, then we get the unit fraction 1/(31+α+30c)(121+120c), and let 1/(31+α+30c)(121+120c) =1/y.

After that, we start to prove that ψ_1/(31+α+30c) i.e. (4α+2)/(31+α+30c) is an unit fraction.

Since the numerator 4α+2 is an even number, such that the denominator

(31+α+30c) must be an even numbers. Only in this case, it can reduce the fraction, so α in the denominator 31+α+30c is only an odd number.

After α is assigned to odd numbers 1, 3, 5 and otherwise, and the fraction (4α+2)/(31+α+30c) after the values assignment divided by 2, then the fraction (4α+2)/(31+α+30c) is turned into the fraction (3+4t)/(m+15c) identically, where c≥0, t≥0 and m≥16.

The point above is that 3+4t and m+15c after the values assignment make up a fraction, they are on the same order of taking values of t and k, according to the order from small to large, i.e. (3+4t)/(m+15c)= 3/(16+15c), 7/(17+15c), 11/(18+15c), ...

Such being the case, let the numerator and denominator of the fraction (3+4t)/(m+15c) divided by 3+4t, then we get a temporary indeterminate unit fraction, and its denominator is (m+15c)/(3+4t), and its numerator is 1.

Thus, be necessary to prove that the denominator (m+15c)/(3+4t) is able to become a positive integer in which case t≥0, m≥16 and c≥0.

In the fraction (m+15c)/(3+4t), due to m≥16, the numerator m+15c after the values assignment are infinite more consecutive positive integers, while the denominator 3+4t=3, 7, 11 and otherwise positive odd numbers.

The key above is that each value of 3+4t after the values assignment can seek its integral multiples within infinite more consecutive positive integers of m+15c, in which case t≥0, m≥16 and c≥ 0.

As is known to all, there is a positive integer that contains the odd factor 2n+1 within 2n+1 consecutive positive integers, where n=1, 2, 3, ...

Like that, there is a positive integer that contains the odd factor 3+4t within 3+4t consecutive positive integers of m+15c, no matter which odd number that 3+4t is equal to, where $t\geq 0$, $m\geq 16$ and $c\geq 0$. It is obvious that a fraction that consists of such a positive integer as the numerator and 3+4t as the denominator is an improper fraction.

Undoubtedly, every such improper fraction that is found in this way, via the reduction, it is surely a positive integer.

That is to say, (m+15c)/(3+4t) as the denominator of the aforesaid temporary indeterminate unit fraction can become a positive integer, and the positive integer is represented by λ, and thus in this case, the fraction (3+4t)/(m+15c) is exactly $1/\lambda$.

For the unit fraction $1/\lambda$, multiply its denominator by 121+120c reserved, then we get the unit fraction $1/\lambda(121+120c)$, and let $1/\lambda(121+120c)=1/z$.

If $3+4\alpha$ serve as one numerator, we get $(3+4\alpha)/(31+\alpha+30c)(121+120c)=1/y$ likewise by the method of proving $\psi_1/(31+\alpha+30c)(121+120c)=1/z$.

When $3+4\alpha$ serve as one numerator and from this get an unit fraction, we can multiply the denominator of the unit fraction by 2 to make a sum of two identical unit fractions, then convert them into the sum of two each other's -distinct unit fractions by the formula $1/2r+1/2r=1/(r+1)+1/r(r+1)$.

Thus it can be seen, the fraction (3+4α)/(31+α+30c)(121+120c) is surely able to be expressed into a sum of two each other's -distinct unit fractions, where c≥0 and α≥0.

Overall, there are 4/(121+120c)=1/(31+α+30c)+1/(31+α+30c)(121+120c) + 1/λ(121+120c), where α≥0, λ is an integer and λ=(m+15c)/(3+4t), t≥0, m≥16, and c≥0.

In other words, we have proved 4/(121+120c)=1/x+1/y+1/z.

The proof was thus brought to a close. As a consequence, the Erdös-Straus conjecture is tenable.

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