AN UPPER BOUND FOR THE ERDŐS UNIT DISTANCE PROBLEM IN THE PLANE

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Abstract. In this paper, using the method of compression, we prove a stronger upper bound for the Erdős unit distance problem in the plane by showing that

$$\#\left\{||\vec{x_j}-\vec{x_t}||:\vec{x}_t,\vec{x_j}\in\mathbb{E}\subset\mathbb{R}^2,\ ||\vec{x_j}-\vec{x_t}||=1,\ 1\leq t,j\leq n\right\}\ll_2 n^{1+o(1)}.$$

1. Introduction

Erdős posed in 1946 the problem of counting the number of unit distances that can be determined by a set of n points in the plane. It is known (see [1]) that the number of unit distances that can be determined by n points in the plane is lower bounded by

$$n^{1+\frac{c}{\log\log n}}.$$

Erdős asks if the upper bound for the number of unit distances that can be determined by n points in the plane can also be a function of this form. In other words, the problem asks if the lower bound of Erdős is the best possible. What is known currently is the upper bound (see [2]) of the form

$$n^{\frac{4}{3}}$$

due to Spencer, Szemerédi and Trotter.
In this paper we improve on this upper bound by showing that

Theorem 1.1. *The upper bound holds*

$$\#\left\{||\vec{x_j}-\vec{x_t}||:\vec{x}_t,\vec{x_j}\in\mathbb{E}\subset\mathbb{R}^2,\ ||\vec{x_j}-\vec{x_t}||=1,\ 1\leq t,j\leq n\right\}\ll_2 n^{1+o(1)}.$$

2. Preliminary results

In this section we launch the notion of compression of points in space. We study the mass of compression and its accompanied estimates. These estimates turn out to be useful for estimating the gap of compression, which we will launch in the sequel.

Definition 2.1. By the compression of scale $0<m\leq 1$ on $\mathbb{R}^n$ we mean the map $\mathbb{V}:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]=\left(\frac{m}{x_1},\frac{m}{x_2},\ldots,\frac{m}{x_n}\right)$$

Date: January 24, 2022.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. compression gap; mass of compression; compression estimates.

for $n \geq 2$ and with $x_i \neq 0$ for all $i = 1, \dots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression pushes points very close to the origin away from the origin by certain scale and similarly draws lattice points away from the origin close to the origin.

Proposition 2.1. *A compression of scale $0 < m \leq 1$ with $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map. In particular the compression $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a bijective map of order 2.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \mathbb{V}_m[(y_1, y_2, \dots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \dots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \dots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \dots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. The latter claim follows by noting that $\mathbb{V}_m^2[\vec{x}] = \vec{x}$. □

2.1. **The mass of compression estimates.**

In this section we study the mass of a compression in a given scale. We use the upper and lower estimates of the mass of compression to establish corresponding estimates for the gap of compression. These estimates will form an essential tool for establishing the main result of this paper.

Definition 2.3. By the mass of a compression of scale $0 < m \leq 1$ we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

Lemma 2.4. *The estimate holds*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $0 < m \leq 1$.

Proposition 2.2 (The mass of compression estimates)**.** *Let $(x_1, x_2, \dots, x_n) \in \mathbb{N}^n$ with $x_i \neq x_j$ for $1 \leq i, j \leq n$ with $i \neq j$, then the estimates holds*

$$m \log\left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) \ll m \log\left(1 + \frac{n-1}{\operatorname{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \geq 1$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\operatorname{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum by appealing to Lemma 2.4. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) &= m\sum_{j=1}^{n}\frac{1}{x_j}\\ &\geq m\sum_{k=0}^{n-1}\frac{1}{\sup(x_j)-k}.\end{aligned}$$

□

It is important to notice that the condition $x_i \neq x_j$ for $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)-k} \leq \mathcal{M}(\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]) \leq m\sum_{k=0}^{n-1}\frac{1}{\mathrm{Inf}(x_j)+k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1\leq i,j\leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ is such that $x_i \leq x_j$ for $1\leq i,j\leq n$.

2.2. **Compression gap estimates.** In this section we recall the notion of the gap of compression and its various estimates. We prove upper and lower bounding the gap of a point under compression of any scale.

Definition 2.6. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with $x_i\neq 0$ for all $i=1,2\ldots,n$. Then by the gap of compression of scale $1\geq m>0$ $\mathbb{V}_m$, denoted $\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]$, we mean the expression

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)] = \left|\left|\left(x_1-\frac{m}{x_1},x_2-\frac{m}{x_2},\ldots,x_n-\frac{m}{x_n}\right)\right|\right|$$

Proposition 2.3. *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ for $n\geq 2$ with $x_j\neq 0$ for $j=1,\ldots,n$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right] + m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)] - 2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2 = \mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right] - 2mn + O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right).$$

Proposition 2.3 offers us an extremely useful identity. It allows us to pass from the gap of compression on points to the relative distance to the origin. It tells us that points under compression with a large gap must be far away from the origin than points with a relatively smaller gap under compression. That is to say, the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{x}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

if and only if $||\vec{x}||<||\vec{y}||$ for $\vec{x},\vec{y}\in\mathbb{N}^n$. This important transference principle will be mostly put to use in obtaining our results.

Lemma 2.7 (Compression gap estimates)**.** *Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ for $n\geq 2$, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\mathrm{sup}(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1}-2mn.$$

Proof. The estimates follows by leveraging the estimates in Proposition 2.2 and noting that

$$n\mathrm{Inf}(x_j^2)\ll\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]\ll n\mathrm{sup}(x_j^2).$$

□

Definition 2.8. Let $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. Then by the ball induced by $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ under compression of scale m, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ we mean the inequality

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},x_2+\frac{m}{x_2},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|<\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)].$$

A point $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]}[(x_1,x_2,\ldots,x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n=2$.

Theorem 2.9. *Let $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$. Then $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\leq||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}||<||\vec{y}||$. It follows that

$$\begin{aligned}\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|&<\left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|\\&=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].\end{aligned}$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.10. *Let $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ with $x_i\neq x_j$ for all $1\leq i<j\leq n$. If $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\not\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.9 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]&>\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\&\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\&>\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.11. Theorem 2.10 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.3. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.12. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{N}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 2.13. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball. Next we show that all balls can in principle be generated by their admissible points.

Theorem 2.14. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.9, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}|| < ||\vec{y}||$ or $||\vec{y}|| < ||\vec{x}||$. By joining these points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows by using the fact that the two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}] = \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}] = \mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lies on the outer of the two indistinguishable balls and must satisfy the equality

$$\begin{aligned}\left|\left|\vec{z} - \frac{1}{2}\left(y_1 + \frac{m}{y_1}, \ldots, y_n + \frac{m}{y_n}\right)\right|\right| &= \left|\left|\vec{z} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right| \\ &= \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].\end{aligned}$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}] = \left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, \ldots, x_n + \frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. $\square$

Definition 2.15 (Translation of balls)**.** Let $\vec{x} \in \mathbb{R}^k$ and $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be the ball induced under compression. Then we denote the map

$$\mathbb{T}_{\vec{v}} : \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}] \longrightarrow \mathcal{B}^{\vec{v}}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

as the translation of the ball by the vector $\vec{v} \in \mathbb{R}^k$, so that for any $\vec{y} \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ then

$$\vec{y} + \vec{v} \in \mathcal{B}^{\vec{v}}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

3. Main theorem

In this section we leverage the estimate of the gap of compression to study the problem of determining the number of unit distances that can be formed from n points in the plane.

Theorem 3.1. *Let $\mathbb{E} \subset \mathbb{R}^2$ be a set of n points in general position and $\mathcal{I} = \left\{||\vec{x_j} - \vec{x_t}|| : \vec{x}_t, \vec{x_j} \in \mathbb{E} \subset \mathbb{R}^2,\ ||\vec{x_j} - \vec{x_t}|| = 1,\ 1 \leq t, j \leq n\right\}$, then we have*

$$\#\mathcal{I} \ll_2 n^{1+o(1)}.$$

Proof. First pick a point $\vec{x}_j \in \mathbb{R}^2$, set $\mathcal{G}\circ\mathbb{V}_1[\vec{x}_j] = 1$ and apply the compression $\mathbb{V}_1$ on $\vec{x}_j$. Next construct the ball induced under compression

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}_j]}[\vec{x}_j].$$

We remark that the ball so constructed is a ball of radius $\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}_j] = \frac{1}{2}$, so that for any admissible point $\vec{x}_k \neq \vec{x}_j$ with $\vec{x}_k \in \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_1[\vec{x}_j]}[\vec{x}_j]$ there must exists the admissible point $\mathbb{V}_1[\vec{x}_k]$ such that

$$||\vec{x}_k - \mathbb{V}_1[\vec{x}_k]|| = 1$$

so that any such $\frac{n}{2}$ pairs of admissible points determines at least $\frac{n}{2}$ unit distances. Now for any n sufficiently large such admissible points on the ball and by virtue of the restriction

$$\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j] = 1 \tag{3.1}$$

we make the optimal assignment

$$\max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2}(x_{j_s}) = n^{o(1)}, \ (n \longrightarrow \infty)$$

since points $\vec{x}_l$ far away from the origin with x_{l_s} for $1 \leq s \leq 2$ must have large compression gaps by virtue of Lemma 2.7. In particular, the point $\vec{x}_l$ must be such that $x_{l_s} = 1 + \epsilon$ with each $1 \leq s \leq 2$ for any small $\epsilon > 0$ in order to satisfy the requirement in (3.1). The number of unit distances induced by n admissible points on the ball so constructed is at most

$$\begin{aligned}
\sum_{\substack{1 \leq j \leq n \\ x_j \in \mathbb{R}^2 \\ \mathcal{G} \circ \mathbb{V}_1[\vec{x}_j] = 1}} 1 &= \sum_{\substack{1 \leq j \leq n \\ x_j \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j]}[\vec{x}_j] \bigcap \mathbb{R}^2 \\ \max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2}(x_{j_s}) = n^{o(1)}}} \mathcal{G} \circ \mathbb{V}_1[\vec{x}_j] \\
&\ll_2 \sum_{\substack{1 \leq j \leq n \\ \max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2}(x_{j_s}) = n^{o(1)}}} \sup_{1 \leq s \leq 2}(x_{j_s}) \\
&\ll_2 \sum_{\substack{1 \leq j \leq n \\ \max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2}(x_{j_s}) = n^{o(1)}}} \max_{1 \leq j \leq n} \sup_{1 \leq s \leq 2}(x_{j_s}) \\
&= n^{o(1)} \sum_{1 \leq j \leq n} 1 \\
&\ll_2 n^{1+o(1)}.
\end{aligned}$$

Now for any set of n points in general position in the plane $\mathbb{R}^2$, let us apply the translation with a fixed vector $\vec{v} \in \mathbb{R}^2$

$$\mathbb{T}_{\vec{v}} : \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j]}[\vec{x}_j] \longrightarrow \mathcal{B}^{\vec{v}}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j]}[\vec{x}_j]$$

so that the new ball $\mathcal{B}^{\vec{v}}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j]}[\vec{x}_j]$ now lives in the smallest region containing all the n points in general position. We remark that this new ball is still of radius $\frac{1}{2}$ but contains points - including admissible points - all of which are translates of points in the previous ball $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_1[\vec{x}_j]}[\vec{x}_j]$ by a fixed vector $\vec{v} \in \mathbb{R}^2$. We remark that the unit distances are all preserved so that the number of unit distances determined by the n points in general position is upper bounded by

$$\ll_2 n^{1+o(1)}$$

thereby ending the proof. □

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