ON THE SHORTEST ADDITION CHAINS OF NUMBERS OF SPECIAL FORMS

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Abstract. In this paper we study the shortest addition chains of numbers of special forms. We obtain the crude inequality

$$\iota(2^n-1)\leq n+1+G(n)$$

for some function $G:\mathbb{N}\longrightarrow \mathbb{N}$. In particular we obtain the weaker inequality

$$\iota(2^n-1)\leq n+1+\left\lfloor \frac{n-2}{2}\right\rfloor$$

where $\iota(n)$ is the length of the shortest addition chain producing n.

1. Introduction

An addition chain producing $n\geq 3$, roughly speaking, is a sequence of numbers of the form $1,2,s_3,s_4,\ldots,s_{k-1},s_k=n$ where each term is the sum of two earlier terms in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The number of terms in the sequence excluding n is the length of the chain. There are quite a number of addition chains producing a fixed number n. Among them the shortest is regarded as the shortest or optimal addition chain producing n. Nonetheless minimizing an addition chain can be an arduous endeavour, given that there are currently no efficient method for obtaining the shortest addition producing a given number. This makes the theory of addition chains an interesting subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing n, Arnold scholz conjectured the inequality

Conjecture 1.1 (Scholz)**.** The inequality holds

$$\iota(2^n-1)\leq n-1+\iota(n).$$

It has been shown computationally that the conjecture holds for all $n\leq 5784688$ and in fact it is an equality for all $n\leq 64$ [2]. Alfred Brauer proved the scholz conjecture for the star addition chain, an addition chain where each term obtained by summing uses the immediately subsequent number in the chain. By denoting the shortest length of the star addition chain by $\iota^*(n)$, it is shown that (See,[1])

Theorem 1.1. *The inequality holds*

$$\iota^*(2^n-1)\leq n-1+\iota^*(n).$$

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In this paper we study short addition chains producing numbers of the form $2^n - 1$ and the scholz conjecture. We obtain some crude and much more weaker inequalities related to the scholz conjecture.

2. **Sub-addition chains**

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n \geq 3$, then by the addition chain of length $k-1$ producing n we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i th **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiners** and r_i the **regulator** of the i th generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$.

Definition 2.2. Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then we call the sub-sequence (s_{j_m}) for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

2.1. **Equivalent addition chains.**

In this section we introduce and study the notion of **equivalence** of an addition chain producing a given number. We launch the following languages.

Definition 2.3. Let $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1+1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

where a_i is the **determiner** and r_i the **regulator** of the i th generator of the chain. Also let $1, 2, \ldots, u_{l-1}, u_l = m$ be an addition chain producing m with the corresponding sequence of partition

$$2 = 1+1, \ldots, u_{l-1} = g_{l-1} + h_{l-1}, u_l = g_l + h_l = m$$

where g_i is the **determiner** and h_i the **regulator** of the i th generator of the chain. Then we say the addition chain $1, 2, \ldots, s_{k-1}, s_k = n$ is **equivalent** to the addition chain $1, 2, \ldots, u_{l-1}, u_l = m$ if there exist a complete sub-addition chain (u_{i_t}) of the chain (u_i) such that for each determiner g_i and the corresponding regulator h_i in the sub-addition chain there exists some $v_i, d_i \geq 0$ - not necessarily distinct - with $v_i, d_i \in \mathbb{Z}$ such that $g_i = a_i - v_i$ and $h_i = r_i - d_i$. We call each v_i and d_i the **stabilizer** of the determiner and the regulator of the i th generator of the chain. We denote the equivalence by $(s_j) \Rightarrow (u_i)$.

Proposition 2.1. *Let $1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} = n$ and $1, 2, \ldots, u_{\delta(m)}, u_{\delta(m)+1} = m$ be addition chains producing n and m with length $\delta(n)$ and $\delta(m)$, respectively. If $(s_j) \Rightarrow (u_i)$, then $\delta(n) \leq \delta(m)$.*

Proof. Let $1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} = n$ and $1, 2, \ldots, u_{\delta(m)}, u_{\delta(m)+1} = m$ be addition chains producing n and m with length $\delta(n)$ and $\delta(m)$, respectively. Suppose $(s_j) \Rightarrow (u_i)$, then by Definition 2.3 there must exists a complete sub-addition chain (u_{i_t}) of the chain (u_i) such that for each determiner g_i and the corresponding regulator h_i in the sub-addition chain there exists some stabilizers $v_i, d_i \geq 0$ - not necessarily distinct - with $v_i, d_i \in \mathbb{Z}$ such that $g_i = a_i - v_i$ and $h_i = r_i - d_i$. Since the length of the complete sub-addition chain (u_{i_t}) is at most the length of the addition (u_i), and the number of terms in the complete sub-addition chain corresponds to the number of terms in the chain (s_j), it follows that $\delta(n) \leq \delta(m)$, thereby ending the proof. □

2.2. **Equivalent addition chains in a fixed base.** In this section we introduce the notion of equivalence of an addition chain in fixed base.

Definition 2.4. Let (u_i) be an addition chain producing m and (s_j) be an addition chain producing n. Then we say the addition chain (s_j) is equivalent to the addition chain (u_i) in base n if there exists a complete sub-addition chain (s_{j_m}) of the chain (s_j) such that $(s_{j_m}) \Rightarrow (u_i)$. We denote the length of the chain (u_i) in **base** n with $\delta_n(m)$ and the length of the shortest of all such chains in **base** n with $\iota_n(m)$.

3. Addition chains of numbers of special forms

In this section we study addition chains of numbers of special forms. We examine ways of minimizing the length of addition chains for numbers of the forms $2^n, 2^n - 1$ and $2^n + 1$. For addition chains producing 2^n and $2^n + 1$ the process is natural and trivial as opposed to those producing $2^n - 1$. We launch the following primary results.

Proposition 3.1. *Let $\iota(n)$ denotes the length of the* ***shortest*** *addition chain producing n. Then $\iota(2^n) = n$ and*

$$\iota(2^n + 1) = \iota(2^n) + 1 = n + 1.$$

Proof. It suffices to construct the corresponding sequence of partition generating the shortest addition producing 2^n. Let us choose the corresponding sequence of partitions

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n + 1$, so that the choice of the regulator $r_i = 2^{i-2}$ minimizes the length of the chain and hence generates the shortest length of the chain $\iota(2^n) = n$, since there are n regulators counting multiplicity in the chain. The corresponding shortest addition chain producing $2^n + 1$ is also obtained with the corresponding sequence of partition by adjoining the term $2^n + 1$ to the last term in the corresponding sequence of partition for the chain producing 2^n, since every addition chain starts with 1, so that we have the shortest length

$$\iota(2^n + 1) = \iota(2^n) + 1 = n + 1.$$

□

Remark 3.1. We now prove an important result that will have significant impact on the main result in this paper. One could consider this result as a weaker version and a first step to affirming the scholz conjecture.

Theorem 3.2. *Let $\iota(n)$ denotes the length of the shortest addition chain producing n. Then there exists some $G : \mathbb{N} \longrightarrow \mathbb{N}$ such that*

$$\iota_{2^n}(2^n - 1) = \iota(2^n) + 1 + G(n) = n + 1 + G(n).$$

Proof. First, let us construct the shortest addition chain producing 2^n as $1, 2, 2^2, \ldots, 2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n+1$, where a_i and r_i denotes the determiner and the regulator of the i th generator of the chain. Next we construct a sub-addition chain of some equivalent addition chain by choosing the stabilizers $v_3 = 0$ and $d_3 = 1$ for the determiner a_3 and the regulator r_3, respectively, and choose new regulators with determiners $g_i = h_i = 2^{i-2} - 2^{i-4}$ for all $4 \leq i \leq n + 1$. Then we obtain a complete sub-addition chain $1, 2, 3, 6, \ldots, 2^{n-2} - 2^{n-4}, 2^{n-1} - 2^{n-3}, 2^n - 2^{n-2}$ of some equivalent addition chain producing $2^n - 1$ with the corresponding sequence of partition

$$2 = 1 + 1, 2 + (2 - 1) = 2^2 - 1, (2^2 - 1) + (2^2 - 1) = 2^3 - 2$$
$$\ldots,$$
$$(2^{n-1} - 2^{n-3}) = (2^{n-2} - 2^{n-4}) + (2^{n-2} - 2^{n-4}), (2^n - 2^{n-2}) = (2^{n-1} - 2^{n-3}) + (2^{n-1} - 2^{n-3}).$$

Appealing to Proposition 2.1 we obtain the relation

$$\iota(2^n) + 1 < \iota_{2^n}(2^n - 1)$$

so that there exists some $G := G(n) \in \mathbb{N}$ such that we can write $\iota_{2^n}(2^n - 1) = \iota(2^n) + 1 + G(n) = n + 1 + G(n)$. □

Corollary 3.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing $2^n - 1$. Then

$$\iota(2^n - 1) \leq n + 1 + G(n)$$

where $G : \mathbb{N} \longrightarrow \mathbb{N}$.

Proof. By invoking Theorem 3.2, we can write

$$\iota(2^n - 1) \leq \iota_{2^n}(2^n - 1) = n + 1 + G(n)$$

with $G : \mathbb{N} \longrightarrow \mathbb{N}$. □

Corollary 3.1 could be considered as a weaker and a crude form of the inequality relating the length of the shortest addition producing $2^n - 1$ to the length of the shortest addition chain producing n. In particular, the scholz conjecture is the claim that

$$G(n) \leq \iota(n) - 2.$$

To this end minimizing the function $G(n)$ is of ultimate goal, which also requires choosing a suitably short addition chain that completes the addition chain producing $2^n - 1$. We prove a much weaker upper bound for the function $G(n)$ with a certain construction for the addition chain completing the addition chain in base 2^n.

Theorem 3.3. *The inequality holds*

$$\iota_{2^n}(2^n - 1) \leq n + 1 + \left\lfloor \frac{n-2}{2} \right\rfloor .$$

Proof. By invoking Proposition 3.2 we can write

$$\iota_{2^n}(2^n - 1) = n + 1 + G(n)$$

where $G : \mathbb{N} \longrightarrow \mathbb{N}$ under the complete sub-addition chain $1, 2, 3, 6, \ldots, 2^{n-2} - 2^{n-4}, 2^{n-1} - 2^{n-3}, 2^n - 2^{n-2}$ of some addition chain with corresponding sequence of partitions

$$\begin{aligned} 2 = 1 + 1, 2 + (2-1) = 2^2 - 1, (2^2 - 1) + (2^2 - 1) = 2^3 - 2 \\ \ldots, \\ (2^{n-1} - 2^{n-3}) = (2^{n-2} - 2^{n-4}) + (2^{n-2} - 2^{n-4}), (2^n - 2^{n-2}) = (2^{n-1} - 2^{n-3}) \\ + (2^{n-1} - 2^{n-3}). \end{aligned}$$

We note that we can extend the complete sub-addition chain to the addition chain producing $2^n - 1$ in the following manner

$$\begin{aligned} 1, 2, 3, 6, \ldots, 2^{n-2} - 2^{n-4}, 2^{n-1} - 2^{n-3}, 2^n - 2^{n-2}, 2^n - 2^{n-4}, 2^n - 2^{n-6}, \ldots, 2^n - 2^0 \\ = 2^n - 1 \end{aligned}$$

by adjoining the corresponding sequence of partition to the sequence of partition of the complete sub-addition chain

$$(2^n - 2^{n-2}) + (2^{n-2} - 2^{n-4}), (2^n - 2^{n-4}) + (2^{n-4} - 2^{n-6}), \ldots (2^n - 2^2) + (2^2 - 1)$$

we obtain

$$G(n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor$$

where $G(n)$ counts the number of terms adjoined to the complete sub-addition chain before $2^n - 1$, thereby proving the inequality. □

[1].

References

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