Geometry of the Ellipse and the Ellipsoid

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Abstract: It is a chapter that concerns the geometry of the ellipse and the ellipsoid of revolution. We give the formulas of the 2D plane coordinates and the 3D Cartesian coordinates (X, Y, Z) in function of the geodetic coordinates (φ, λ, he). A section is devoted to the geodesic lines of the ellipsoid of revolution. We give the proofs of the differential equations of the geodesic lines and their integration.

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Chapter 1

Geometry of The Ellipse and The Ellipsoid

1.1 Geometry of the ellipse

1.1.1 Definitions

Definition 1.1 *The ellipse is the locus of the points whose sum of the distances to two distinct fixed points (foci) is constant:*

$$\boxed{MF + MF' = constant = 2a} \tag{1.1}$$

*where a is called the semi-major axis of the ellipse (**Fig. 1.1**).*

Definition 1.2 *An ellipse is the transform by affinity of a circle in the ratio b/a where b is the semi-minor axis (**Fig. 1.2**).*

To the point $M' \in$ circle $\Longrightarrow M \in$ ellipse with :

$$HM = \frac{b}{a} HM' \tag{1.2}$$

Let ψ be the angle $\widehat{HOM'}$, ψ is called the parametric latitude or reduced latitude, then the coordinates of M':

$$x = OH = OM' cos\psi$$
$$y = OL = OM' sin\psi$$

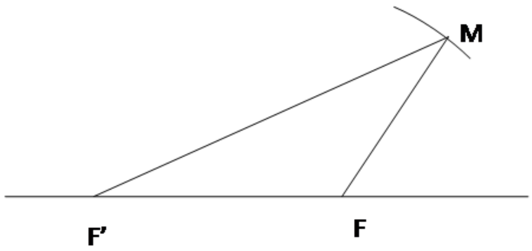

Fig. 1.1 Definition of the ellipse

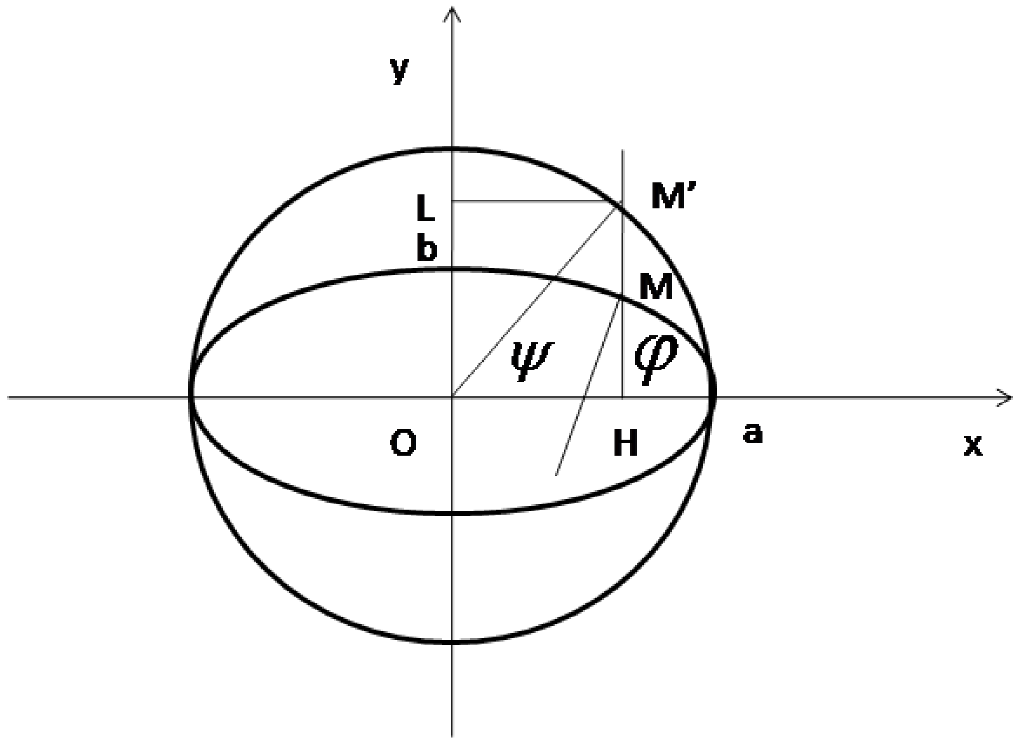

Fig. 1.2 The affinity transformation

Hence, the coordinates of M on the ellipse are:

$$\begin{cases} x = OH = acos\psi \\ y = OL = \dfrac{b}{a}HM' = \dfrac{b}{a}asin\psi = bsin\psi \end{cases} \tag{1.3}$$

In the system of axes Ox, Oy, the equation of the ellipse is written:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We call respectively the flattening, the square of the first eccentricity and the square of the second eccentricity the quantities:

$$\boxed{\alpha = \frac{a-b}{a}, \quad e^2 = \frac{a^2-b^2}{a^2}, \quad e'^2 = \frac{a^2-b^2}{b^2}} \tag{1.4}$$

1.2 PARAMETRIC EQUATIONS OF AN ELLIPSE

The equations (1.3) represent the parametric equations of an ellipse in function of the latitude ψ. We will express these equations as a function of the angle φ of the normal at M with the axis Ox.

Let TM' be the tangent at M' on the circle of radius a, the point T is the intersection of this tangent with the axis Ox. The transform of this tangent by affinity of ratio b/a of this tangent is the line tangent to the ellipse at the point M and it goes through T (**Fig. 1.3**).

In the triangle MHT, we have:

$$tg\varphi = \frac{HT}{MH}$$

and in the triangle $M'HT$:

$$tg\psi = \frac{HT}{M'H}$$

so :

$$\frac{tg\psi}{tg\varphi} = \frac{HT}{M'H}\frac{MH}{HT} = \frac{MH}{M'H} = \text{ratio of the affinity} = \frac{b}{a}$$

It follows:

$$\boxed{tg\psi = \frac{b}{a}tg\varphi} \tag{1.5}$$

From (1.5), we write $cos\psi$ and $sin\psi$ in function of the angle φ, then:

$$\frac{1}{cos^2\psi} = 1 + tg^2\psi = 1 + (b/a)^2 tg^2\varphi = \frac{a^2cos^2\varphi + b^2sin^2\varphi}{a^2cos^2\varphi}$$

and :

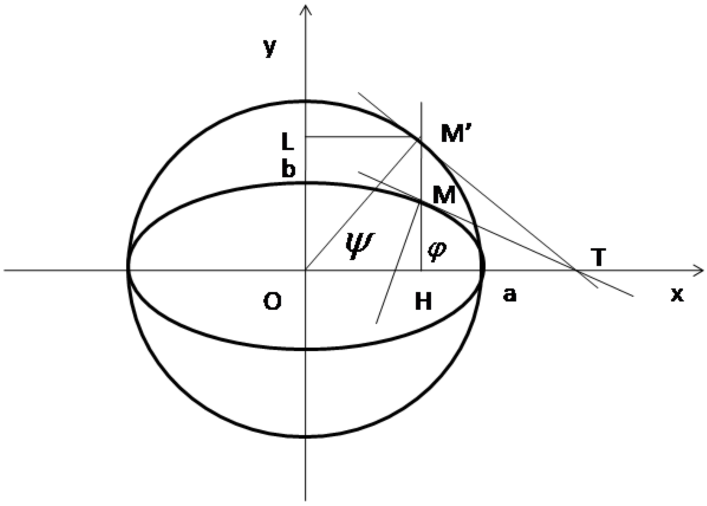

Fig. 1.3 The relation between φ and ψ

$$cos^2\psi = \frac{a^2cos^2\varphi}{a^2cos^2\varphi + b^2sin^2\varphi}$$

Let:

$$W^2 = \frac{a^2cos^2\varphi + b^2sin^2\varphi}{a^2} = 1 - e^2sin^2\varphi \tag{1.6}$$

then:

$$W = \frac{cos\varphi}{cos\psi} \tag{1.7}$$

We calculate in the same way $sin\psi$:

$$sin^2\psi = 1 - cos^2\psi = 1 - \frac{a^2cos^2\varphi}{a^2cos^2\varphi + b^2sin^2\varphi}$$

we obtain:

$$sin^2\psi = \frac{b^2sin^2\varphi}{a^2cos^2\varphi + b^2sin^2\varphi} \tag{1.8}$$

Let :

$$V^2 = \frac{a^2cos^2\varphi + b^2sin^2\varphi}{b^2} = 1 - e'^2cos^2\varphi \tag{1.9}$$

where e' is the second eccentricity, then:

$$V = \frac{sin\varphi}{sin\psi} = \frac{a}{b}W \tag{1.10}$$

It follows the parametric equations of the ellipse as functions of φ as:

$$X = acos\psi = a\frac{cos\varphi}{W} = a\frac{cos\varphi}{\sqrt{1 - e^2sin^2\varphi}} \tag{1.11}$$

$$Y = bsin\psi = \frac{bsin\varphi}{V} = \frac{b^2 sin\varphi}{a\sqrt{1-e^2sin^2\varphi}} = a(1-e^2)\frac{sin\varphi}{\sqrt{1-e^2sin^2\varphi}} \tag{1.12}$$

let:

$$\boxed{\begin{array}{l} X = a\dfrac{cos\varphi}{\sqrt{1-e^2sin^2\varphi}} \\ Y = a(1-e^2)\dfrac{sin\varphi}{\sqrt{1-e^2sin^2\varphi}} \end{array}} \tag{1.13}$$

1.2.1 Differential Relations between φ and ψ

From the relation (1.5), we obtain:

$$\frac{d\psi}{cos^2\psi} = \frac{bd\varphi}{acos^2\varphi} \Longrightarrow \frac{d\psi}{d\varphi} = \frac{bcos^2\psi}{acos^2\varphi} \tag{1.14}$$

and using (1.7), we have:

$$\boxed{\frac{d\psi}{d\varphi} = \frac{b}{aW^2} = \frac{b}{a(1-e^2sin^2\varphi)}} \tag{1.15}$$

1.3 THE EXPRESSION OF THE CALCULATION OF THE GREAT NORMAL

We call the *great normal* the lenght of JM. The vector $\boldsymbol{JM}$ is normal to the ellipse at M, let $\boldsymbol{l} = \dfrac{\boldsymbol{JM}}{||\boldsymbol{JM}||}$, its components are: $(cos\varphi, sin\varphi)$ (**Fig. 1.4**).

Then the Cartesian equation of the normal is:

$$\frac{X - X_M}{cos\varphi} = \frac{Y - Y_M}{sin\varphi} \tag{1.16}$$

We obtain the ordinate of J in taking $X = 0$ in (1.16), then:

$$\frac{-X_M}{cos\varphi} = \frac{Y - Y_M}{sin\varphi} \Longrightarrow Y_J = Y_M - X_M tg\varphi$$

It follows the expression of the distance MJ equal to :

$$MJ = \sqrt{(Y_J - Y_M)^2 + X_M^2} = \sqrt{X_M^2 tg^2\varphi + X_M^2} = X_M\sqrt{1+tg^2\varphi} = \frac{X_M}{cos\varphi}\frac{X_M}{cos\varphi}$$

But:

$$X_M = \frac{acos\varphi}{W} \Longrightarrow MJ = \frac{acos\varphi}{Wcos\varphi} = \frac{a}{W} = \frac{a}{\sqrt{1-e^2sin^2\varphi}}$$

Let :

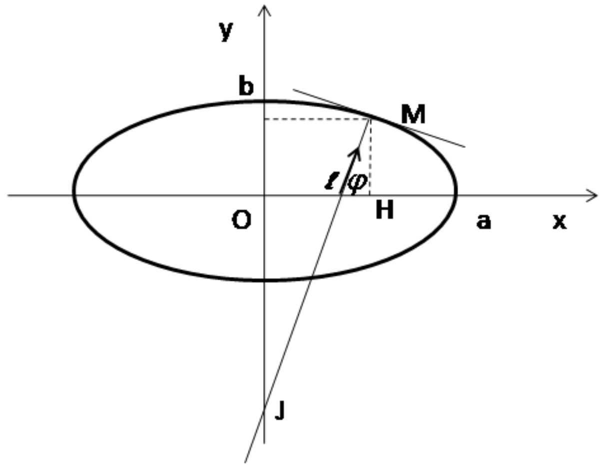

Fig. 1.4 The great normal

$$\boxed{N(\varphi) = MJ = \frac{a}{\sqrt{1-e^2 sin^2\varphi}}} \tag{1.17}$$

N is called *the great normal*.

The parametric equations of an ellipse (1.3) become:

$$X = acos\psi = a\frac{cos\varphi}{W} = a\frac{cos\varphi}{\sqrt{1-e^2 sin^2\varphi}} = N(\varphi)cos\varphi$$
$$Y = bsin\psi = b\frac{sin\varphi}{V} = a(1-e^2)\frac{sin\varphi}{\sqrt{1-e^2 sin^2\varphi}} = (1-e^2)N(\varphi)sin\varphi$$

Finally:

$$\boxed{\begin{array}{l} X = a\dfrac{cos\varphi}{\sqrt{1-e^2 sin^2\varphi}} = N(\varphi)cos\varphi \\ Y = a(1-e^2)\dfrac{sin\varphi}{\sqrt{1-e^2 sin^2\varphi}} = (1-e^2)N(\varphi)sin\varphi \end{array}} \tag{1.18}$$

1.3.1 Elementary arc ds and Radius of Curvature of an Ellipse

The elementary arc ds of an ellipse is determined from the parametric equations as:

$$ds^2 = dX^2 + dY^2 = a^2 sin^2\psi d\psi^2 + b^2.cos^2\psi d\psi^2$$
$$\text{or } ds^2 = (a^2 sin^2\psi + b^2.cos^2\psi)d\psi^2$$

Using the equations (1.7) and (1.10), we obtain:

$$ds = \frac{b}{W}.d\psi$$

Replacing $d\psi$ using (1.15), we find:

$$ds = a(1-e^2)\frac{d\varphi}{(1-e^2sin^2\varphi)^{3/2}}$$

The length of the meridian arc counted from the equator is:

$$\boxed{s(\varphi) = \int_0^{\varphi} ds = a(1-e^2)\int_0^{\varphi}\frac{dt}{(1-e^2sin^2t)^{3/2}}} \tag{1.19}$$

The integration is obtained from a limited development of $(1-e^2sin^2t)^{-3/2}$ (Voir plus loin). The radius of curvature of an ellipse ρ is given from ds as:

$$\boxed{\rho = \frac{ds}{d\varphi} = \frac{b^2}{aW^3} = \frac{a(1-e^2)}{(1-e^2sin^2\varphi)^{3/2}}} \tag{1.20}$$

1.4 GEOMETRY OF THE ELLIPSOID OF REVOLUTION

We will study the properties of the ellipsoid of revolution obtained by the rotation of an ellipse around the semi-minor axis as shown in the figure below (**Fig. 1.5**):

1.4.1 The Geographic Coordinates

The geographic coordinates defined on the ellipsoid of revolution are:

- the longitude λ: angle of the meridian plane of point M with the origin meridian plane, in our case, the origin plane is the XOZ plane, - the latitude φ: angle of the direction of the normal at point M with the equatorial plane;

- the ellipsoid altitude *he*, if the point is on the ellipsoid, then $he = 0$.

In the plane *ROZ* (**Fig. 1.6**), with $\boldsymbol{r}$ and $\boldsymbol{k}$ the unit vectors of the axis *OR* and *OZ*, we can write:

$$\boldsymbol{OM} = acos\psi\boldsymbol{r} + bsin\psi\boldsymbol{k}$$
$$\text{and } \boldsymbol{r} = cos\lambda\boldsymbol{i} + sin\lambda\boldsymbol{j}$$

Then:

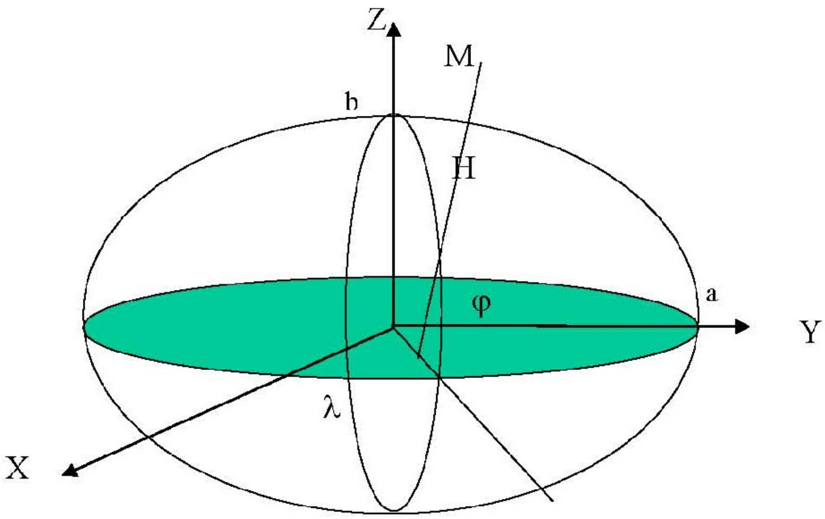

Fig. 1.5 The Ellipsoid of revolution: Reference ellipsoid

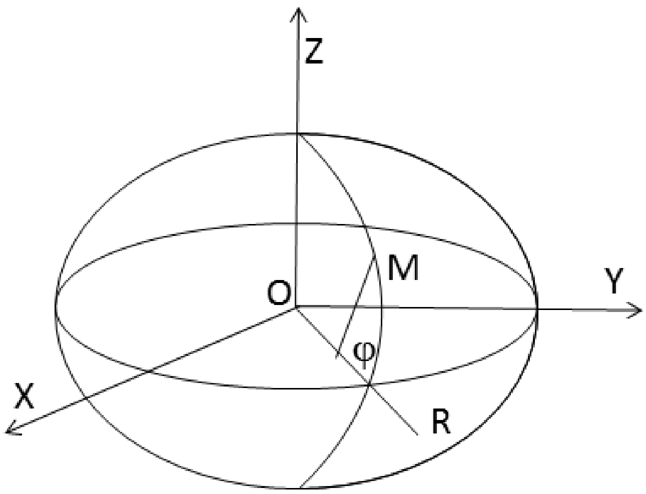

Fig. 1.6 Geodetic coordinates

$$\boldsymbol{OM} = acos\psi cos\lambda \boldsymbol{i} + acos\psi sin\lambda \boldsymbol{j} + bsin\psi \boldsymbol{k}$$

It follows the parametric equations of the point M are :

$$X = acos\psi cos\lambda$$
$$Y = acos\psi sin\lambda$$
$$Z = bsin\psi$$

Replacing ψ in function of φ, we obtain:

$$X = acos\psi cos\lambda = \frac{acos\varphi}{W}cos\lambda = Ncos\varphi cos\lambda$$
$$Y = acos\psi sin\lambda = Ncos\varphi sin\lambda$$
$$Z = bsin\psi = \frac{b^2 sin\varphi}{a\sqrt{1-e^2sin^2\varphi}} = N(1-e^2)sin\varphi$$

Let :

$$\boxed{\begin{array}{l} X = Ncos\varphi cos\lambda \\ Y = Ncos\varphi sin\lambda \\ Z = N(1-e^2)sin\varphi \end{array}} \tag{1.21}$$

If $he \neq 0$, then the coordinates of M are:

$$\boxed{\begin{array}{l} X = (N+he)cos\varphi cos\lambda \\ Y = (N+he)cos\varphi sin\lambda \\ Z = (N(1-e^2)+he)sin\varphi \end{array}} \tag{1.22}$$

1.4.2 *Passage from three-dimensional coordinates* (X,Y,Z) *to* (φ,λ,he) *coordinates*

From the first two equations of (1.22) and ignoring the special case $(X=0)$, we get:

$$\boxed{tg\lambda = \frac{Y}{X} \Longrightarrow \lambda = Arctg\frac{Y}{X}} \tag{1.23}$$

We denote :

$$r = \sqrt{X^2+Y^2} = (N+he)cos\varphi$$

From (1.22), can write:

$$Z = (N+he)sin\varphi - Ne^2 sin\varphi \tag{1.24}$$

or:

$$Z = Z' - Ne^2 sin\varphi$$

with:

$$Z' = (N+he)sin\varphi \tag{1.25}$$

The calculation of φ is done by iterations:

- 1^{st} iteration:

$$Z' = Z \Rightarrow tg\varphi = \frac{Z'}{r} \Rightarrow \varphi_1 = Arctg\frac{Z'}{r} \tag{1.26}$$

- 2^{nd} iteration: $N = a(1-e^2 sin^2\varphi_1)^{-1/2}, \quad Z' = Z + Ne^2.sin\varphi_1$

and : $\varphi_2 = Arctg(Z'/r)$.

- 3^{rd} iteration: $N = a(1-e^2 sin^2\varphi_2)^{-1/2} \quad Z' = Z + Ne^2.sin\varphi_2$

and : $\varphi_3 = Arctg(Z'/r)$. In general, 3 to 4 iterations are sufficient and we obtain:

$$\boxed{\varphi = \varphi_3} \tag{1.27}$$

As a result, we can determine the geodetic altitude *he* :

$$\boxed{he = \frac{r}{cos\varphi} - N(\varphi)} \tag{1.28}$$

1.5 Calculation of the Geodesic Lines of the Ellipsoid of Revolution

" Alongside the main difficulty, that which lies at the very bottom of things, there are a host of secondary difficulties which further complicate the task of the researcher. It would therefore be advantageous to study first a problem where one would encounter this main difficulty, but where one would be freed from all the secondary difficulties. This problem is obvious, it is that of the **geodesic lines** of a surface; it is still a problem of dynamics, so that the main difficulty remains; but it is the simplest of all dynamic problems. "

(**H. Poincaré**[1], 1905)

We define the geodesic lines of a surface then we establish the geodesic equations for a given surface. As an application, we detail those of the ellipsoid of revolution. We will integrate these equations.

1.5.1 Introduction and Notations

Let (S) be one surface defined by the parameters (u,v) with $(u,v) \in \mathscr{D}$ one domain $\subset \mathbb{R}^2$. One point $M \in (S)$ verifies :

[1] **Henri Poincaré** (1854-1912): French mathematician, among the greatest of the 19th century.

$$\boldsymbol{OM} = \boldsymbol{OM}(u,v)\left|\begin{array}{l} x(u,v) \\ y(u,v) \\ z(u,v) \end{array}\right. \tag{1.29}$$

We introduce the usual notations :

$$\begin{aligned} E &= \frac{\partial \boldsymbol{M}}{\partial u}.\frac{\partial \boldsymbol{M}}{\partial u} = \left\|\frac{\partial \boldsymbol{M}}{\partial u}\right\|^2 \\ F &= \frac{\partial \boldsymbol{M}}{\partial u}.\frac{\partial \boldsymbol{M}}{\partial v} \\ G &= \frac{\partial \boldsymbol{M}}{\partial v}.\frac{\partial \boldsymbol{M}}{\partial v} = \left\|\frac{\partial \boldsymbol{M}}{\partial v}\right\|^2 \end{aligned} \tag{1.30}$$

From the equations (1.30), we obtain the equations :

$$\begin{aligned} \frac{\partial E}{\partial u} &= 2\frac{\partial \boldsymbol{M}}{\partial u}.\frac{\partial^2 \boldsymbol{M}}{\partial u^2} \\ \frac{\partial E}{\partial v} &= 2\frac{\partial \boldsymbol{M}}{\partial u}.\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v} \\ \frac{\partial F}{\partial u} &= \frac{\partial^2 \boldsymbol{M}}{\partial u^2}.\frac{\partial \boldsymbol{M}}{\partial v} + \frac{\partial \boldsymbol{M}}{\partial u}.\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v} \\ \frac{\partial F}{\partial v} &= \frac{\partial^2 \boldsymbol{M}}{\partial v^2}.\frac{\partial \boldsymbol{M}}{\partial u} + \frac{\partial \boldsymbol{M}}{\partial v}.\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v} \\ \frac{\partial G}{\partial u} &= 2\frac{\partial \boldsymbol{M}}{\partial v}.\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v} \\ \frac{\partial G}{\partial v} &= 2\frac{\partial \boldsymbol{M}}{\partial v}.\frac{\partial^2 \boldsymbol{M}}{\partial v^2} \end{aligned} \tag{1.31}$$

Let $\boldsymbol{n}$ be the unit vector at $M(u,v)$ on the surface (S), $\boldsymbol{n}$ is given by :

$$\boldsymbol{n} = \frac{\dfrac{\partial \boldsymbol{M}}{\partial u} \wedge \dfrac{\partial \boldsymbol{M}}{\partial u}}{H} \tag{1.32}$$

with:

$$H = \left\|\frac{\partial \boldsymbol{M}}{\partial u} \wedge \frac{\partial \boldsymbol{M}}{\partial u}\right\| \tag{1.33}$$

then :

$$ds^2 = E.du^2 + 2.F.du.dv + G.dv^2 \tag{1.34}$$

The equation (1.34) represents the square of the infinitesimal length of the arc.

Let (Γ) be a curve traced on (S) and $\boldsymbol{N}$ is the unit vector of the principal normal along (Γ).

Definition 1.3 *A curve (Γ) is a geodesic line on the surface (S) if and only if the vectors $\boldsymbol{n}$ and $\boldsymbol{N}$ are collinear.*

We prove by the calculation of variations that the geodesic line between two points of a surface (S) when it exists is the curve of minimum length joining the two points.

1.5.2 Differential Equations of Geodesic Lines

We calculate $\boldsymbol{N}$, we get :

$$\boldsymbol{N} = R\frac{d\boldsymbol{T}}{ds}$$

but:

$$\boldsymbol{T} = \frac{d\boldsymbol{M}}{ds} = \frac{\partial \boldsymbol{M}}{\partial u}\frac{du}{ds} + \frac{\partial \boldsymbol{M}}{\partial v}\frac{dv}{ds}$$

then:

$$\frac{d\boldsymbol{T}}{ds} = \frac{\partial^2 \boldsymbol{M}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial \boldsymbol{M}}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial \boldsymbol{M}}{\partial v}\frac{d^2u}{ds^2} + \frac{\partial^2 \boldsymbol{M}}{\partial v^2}\left(\frac{dv}{ds}\right)^2$$

The condition $\boldsymbol{n}$ // $\boldsymbol{N}$ can be written as :

$$\boldsymbol{N} \wedge \boldsymbol{n} = 0$$

so:

$$R\frac{d\boldsymbol{T}}{ds} \wedge \left(\frac{\dfrac{\partial \boldsymbol{M}}{\partial u} \wedge \dfrac{\partial \boldsymbol{M}}{\partial u}}{H}\right) = 0 \tag{1.35}$$

Using the formula of the cross product:

$$\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = (\boldsymbol{A}.\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}.\boldsymbol{B})\boldsymbol{C} \tag{1.36}$$

we obtain:

$$\left(\frac{d\boldsymbol{T}}{ds}.\frac{\partial \boldsymbol{M}}{\partial v}\right)\frac{\partial \boldsymbol{M}}{\partial u} - \left(\frac{d\boldsymbol{T}}{ds}.\frac{\partial \boldsymbol{M}}{\partial u}\right)\frac{\partial \boldsymbol{M}}{\partial v} = 0$$

But $\dfrac{\partial \boldsymbol{M}}{\partial u}$ and $\dfrac{\partial \boldsymbol{M}}{\partial v}$ form a base of the tangent plane in M, it follows the two conditions:

$$\frac{d\boldsymbol{T}}{ds}.\frac{\partial \boldsymbol{M}}{\partial v} = 0 \quad \text{and} \quad \frac{d\boldsymbol{T}}{ds}.\frac{\partial \boldsymbol{M}}{\partial u} = 0 \tag{1.37}$$

It gives two differential equations of second order:

$$\frac{\partial^2 \boldsymbol{M}}{\partial u^2}.\frac{\partial \boldsymbol{M}}{\partial v}\left(\frac{du}{ds}\right)^2 + F\frac{d^2u}{ds^2} + 2\frac{\partial^2 \boldsymbol{M}}{\partial u \partial v}.\frac{\partial \boldsymbol{M}}{\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2 \boldsymbol{M}}{\partial v^2}.\frac{\partial \boldsymbol{M}}{\partial v}\left(\frac{dv}{ds}\right)^2 + G\frac{d^2v}{ds^2} = 0 \tag{1.38}$$

and:

$$\frac{\partial^2 \boldsymbol{M}}{\partial v^2}.\frac{\partial \boldsymbol{M}}{\partial u}\left(\frac{dv}{ds}\right)^2+F\frac{d^2v}{ds^2}+2\frac{\partial^2 \boldsymbol{M}}{\partial u\partial v}.\frac{\partial \boldsymbol{M}}{\partial u}\frac{du}{ds}\frac{dv}{ds}+\frac{\partial^2 \boldsymbol{M}}{\partial u^2}.\frac{\partial \boldsymbol{M}}{\partial u}\left(\frac{du}{ds}\right)^2+E\frac{d^2u}{ds^2}=0 \tag{1.39}$$

We denote:

$$\begin{aligned} E'_u=\frac{\partial E}{\partial u};\quad E'_v=\frac{\partial E}{\partial v};\quad F'_u=\frac{\partial F}{\partial u}\\ F'_v=\frac{\partial F}{\partial v};\quad G'_u=\frac{\partial G}{\partial u};\quad G'_v=\frac{\partial G}{\partial v} \end{aligned} \tag{1.40}$$

and we use the equations (1.31), (1.38) et 1.39), these last 2 equations can be written :

$$\boxed{(F'_u-\frac{E'_v}{2})\left(\frac{du}{ds}\right)^2+F\frac{d^2u}{ds^2}+G'_u\frac{du}{ds}\frac{dv}{ds}+\frac{G'_v}{2}\left(\frac{dv}{ds}\right)^2+G\frac{d^2v}{ds^2}=0} \tag{1.41}$$

$$\boxed{(F'_v-\frac{G'_u}{2})\left(\frac{dv}{ds}\right)^2+F\frac{d^2v}{ds^2}+E'_v\frac{dv}{ds}\frac{du}{ds}+\frac{E'_u}{2}\left(\frac{du}{ds}\right)^2+E\frac{d^2u}{ds^2}=0} \tag{1.42}$$

1.5.3 Determination of the Geodesic Lines of the Ellipsoid of Revolution

We now consider as surface the ellipsoid of revolution which we parametrize as follows:

$$\begin{aligned} X&=N.cos\varphi cos\lambda\\ Y&=Ncos\varphi sin\lambda\\ Z&=N(1-e^2)sin\varphi \end{aligned} \tag{1.43}$$

then :

$$N=\frac{a}{\sqrt{1-e^2sin^2\varphi}}=aW^{-1/2}$$

is the radius of curvature of the great normal with:

$$W=1-e^2sin^2\varphi$$

Let :

$$r=Ncos\varphi$$

be the radius of the parallel of latitude φ and ρ the radius of curvature of the meridian given by :

$$\rho=\frac{a(1-e^2)}{(1-e^2sin^2\varphi)\sqrt{1-e^2sin^2\varphi}}=a(1-e^2)W^{-3/2}$$

Then, the first fundamental form is written as :

$$ds^2=\rho^2d\varphi^2+r^2d\lambda^2 \tag{1.44}$$

Considering as variables $u=\varphi$ and $v=\lambda$, we obtain:

$$E=E(\varphi)=\rho^2, \quad F=0, \quad G=r^2 \tag{1.45}$$

$$E'_\varphi=2\rho\rho', E'_\lambda=0, F'_\varphi=F'_\lambda=0, G'_\varphi=2rr'=-2r\rho sin\varphi, G'_\lambda=0 \tag{1.46}$$

Then, the equations (1.41) and (1.42) become :

$$-2r\rho sin\varphi\frac{d\varphi}{ds}\frac{d\lambda}{ds}+r^2\frac{d^2\lambda}{ds^2}=0 \tag{1.47}$$

$$r\rho sin\varphi\left(\frac{d\lambda}{ds}\right)^2+\rho\rho'\left(\frac{d\varphi}{ds}\right)^2+\rho^2\frac{d^2\varphi}{ds^2}=0 \tag{1.48}$$

We write the first equation as:

$$\frac{d}{ds}\left(r^2\frac{d\lambda}{ds}\right)=0 \tag{1.49}$$

its integration gives :

$$r^2\frac{d\lambda}{ds}=C=constant \tag{1.50}$$

Then, we find Clairaut's relation :[2]

$$\boxed{r.sinAz=constant=C=a.sinAze} \tag{1.51}$$

where Az is the azimuth of the geodesic line at the point M and Aze its initial azimuth at the point M_0 on the equator.

The equation (1.48) is written:

$$\rho\left(rsin\varphi\left(\frac{d\lambda}{ds}\right)^2+\rho'\left(\frac{d\varphi}{ds}\right)^2+\rho\frac{d^2\varphi}{ds^2}\right)=0 \tag{1.52}$$

It gives:

- $\rho=0$ the point M is on the equator: $\varphi=0$ and $r=a$ the semi-major axis of the ellipsoid and the equation (1.47) becomes:

$$\frac{d^2\lambda}{ds^2}=0 \tag{1.53}$$

its integration gives:

$$\lambda-\lambda_0=l(s-s_0) \tag{1.54}$$

the point M describe the equator and the geodesic line is the great circle of radius a.

- $\rho\neq 0$, the point M is not on the equator, the equation (1.48) is written as :

$$\rho\frac{d^2\varphi}{ds^2}+\rho'\left(\frac{d\varphi}{ds}\right)^2+rsin\varphi\left(\frac{d\lambda}{ds}\right)^2=0 \tag{1.55}$$

To integrate (1.55), we will use a new function, let :

[2] **Alexis Claude de Clairaut** (1713-1765): French mathematician, astronome and geophysist.

$$Z = \frac{d\lambda}{d\varphi} \tag{1.56}$$

be the new function. From (1.50), we obtain :

$$\frac{d\varphi}{ds} = \frac{d\varphi}{d\lambda}\frac{d\lambda}{ds} = \frac{C}{r^2}\frac{d\varphi}{d\lambda} = \frac{C}{r^2 Z}$$

so :

$$\frac{d\varphi}{ds} = \frac{C}{r^2 Z} \tag{1.57}$$

We calculate now the second derivative $d^2\varphi/ds^2$:

$$\frac{d^2\varphi}{ds^2} = \frac{d}{ds}\left(\frac{d\varphi}{ds}\right) = \frac{d}{d\varphi}\left(\frac{d\varphi}{ds}\right)\frac{d\varphi}{ds} = \frac{1}{2}\frac{d}{d\varphi}\left(\frac{d\varphi}{ds}\right)^2 \tag{1.58}$$

Using (1.50) and (1.58), the equation (1.55) is written :

$$\frac{\rho}{2}\frac{d}{d\varphi}\left[\left(\frac{d\varphi}{ds}\right)^2\right] + \rho'\left(\frac{d\varphi}{ds}\right)^2 + sin\varphi\left(\frac{C^2}{r^3}\right) = 0 \tag{1.59}$$

Let :

$$U = \left(\frac{d\varphi}{ds}\right)^2 \tag{1.60}$$

The equation (1.59) becomes:

$$\frac{\rho}{2}\frac{dU}{d\varphi} + \rho' U = -\frac{C^2 sin\varphi}{r^3} \tag{1.61}$$

The equation (1.61) is a first order linear differential equation with second member. Its resolution without a second member gives :

$$U = \frac{k}{\rho^2} \tag{1.62}$$

Using the second member of (1.61), we consider that k is a function of φ, then we obtain :

$$U = \frac{1}{\rho^2}\left(k_0 - \frac{C^2}{r^2}\right) = \frac{k_0 r^2 - C^2}{\rho^2 r^2} \tag{1.63}$$

with k_0 the constant of integration. U is a positive function, we must obtain :

$$k_0 r^2 - C^2 > 0 \tag{1.64}$$

The equation (1.60) becomes :

$$U = \left(\frac{d\varphi}{ds}\right)^2 = \frac{k_0 r^2 - C^2}{\rho^2 r^2} \tag{1.65}$$

We use the equations (1.57) and (1.65), we obtain :

$$\left(\frac{d\varphi}{ds}\right)^2 = \frac{k_0 r^2 - C^2}{\rho^2 r^2} = \left(\frac{C}{r^2 Z}\right)^2 = \frac{C^2}{r^4 Z^2} = \frac{C^2}{r^4}\left(\frac{d\varphi}{d\lambda}\right)^2 \tag{1.66}$$

it gives :

$$\left(\frac{d\lambda}{d\varphi}\right)^2 = \frac{\rho^2}{r^2}\frac{C^2}{k_0 r^2 - C^2} \tag{1.67}$$

To determine the value of k_0, we write $\frac{d\lambda}{ds}$ using the equations (1.50) and (1.67). The term ds^2 is :

$$ds^2 = \rho^2 d\varphi^2 + r^2 d\lambda^2 = \frac{r^2(k_0 r^2 - C^2)}{C^2} d\lambda^2 + r^2 d\lambda^2$$

soit:

$$ds^2 = \frac{r^4 k_0}{C^2} d\lambda^2 \Rightarrow \left(\frac{d\lambda}{ds}\right)^2 = \frac{C^2}{k_0 r^4} \tag{1.68}$$

But from (1.50) :

$$\left(\frac{d\lambda}{ds}\right)^2 = \frac{C^2}{r^4}$$

then $k_0 = 1$, it follows :

$$\left(\frac{d\lambda}{d\varphi}\right)^2 = \frac{\rho^2}{r^2}\frac{C^2}{r^2 - C^2} \tag{1.69}$$

To be able to integrate the preceding equation, we express $r^2 - C^2$, then :

$$r^2 - C^2 = N^2 cos^2\varphi - C^2 = \frac{a^2 cos^2\varphi}{1 - e^2 sin^2\varphi} - C^2 =$$

$$\frac{(a^2 - C^2)\left(1 - \frac{a^2 - C^2 e^2}{a^2 - C^2} sin^2\varphi\right)}{W} \tag{1.70}$$

Let :

$$k^2 = \frac{a^2 - C^2 e^2}{a^2 - C^2} \tag{1.71}$$

then :

$$r^2 - C^2 = (a^2 - C^2)(1 - k^2 sin^2\varphi)/W \tag{1.72}$$

We notice that the coefficient k is greater than 1, therefore the geodesic latitude φ remains lower than the latitude φ_1 defined by $sin\varphi_1 = 1/k$.

Then, the equation (1.69) is written :

$$\left(\frac{d\lambda}{d\varphi}\right)^2 = \frac{(1-e^2)^2 C^2}{(a^2 - C^2)cos^2\varphi(1 - e^2 sin^2\varphi)(1 - k^2 sin^2\varphi)} \tag{1.73}$$

Hence by replacing C by $a.sin(Aze)$ and since $tg(Aze)$ has the same sign as $(d\lambda/d\varphi)$, then, we can write:

$$\frac{d\lambda}{d\varphi} = \frac{(1-e^2)tg(Aze)}{cos\varphi\sqrt{(1 - e^2 sin^2\varphi)(1 - k^2 sin^2\varphi)}} \tag{1.74}$$

Integrating between 0 and φ:

$$\lambda - \lambda_e = \int_0^{\varphi} \frac{(1-e^2)tg(Aze)}{cost\sqrt{(1 - e^2 sin^2 t)(1 - k^2 sin^2 t)}} dt =$$

$$(1-e^2)tg(Aze)\int_0^{\varphi}\frac{dt}{cost\sqrt{(1-e^2sin^2t)(1-k^2sin^2t)}}$$

or :

$$\lambda-\lambda_e=(1-e^2)tg(Aze)\int_0^{\varphi}\frac{dt}{cost\sqrt{(1-e^2sin^2t)(1-k^2sin^2t)}} \tag{1.75}$$

Taking as variable $w = sint$, the integral (1.75) becomes:

$$\boxed{\lambda-\lambda_e=(1-e^2)tg(Aze)\int_0^{sin\varphi}\frac{dw}{(1-w^2)\sqrt{(1-e^2w^2)(1-k^2w^2)}}} \tag{1.76}$$

Now, we seek to express the curvilinear abscissa s as a function of φ. The expression of ds^2 is equal to:

$$ds^2=\rho^2d\varphi^2+r^2d\lambda^2=\rho^2d\varphi^2+\frac{C^2}{r^2}ds^2$$

or:

$$ds^2=\frac{r^2\rho^2d\varphi^2}{r^2-C^2}=\frac{a^2(1-e^2)^2cos^2\varphi d\varphi^2}{cos^2(Aze)(1-e^2sin^2\varphi)^3(1-k^2sin^2\varphi)} \tag{1.77}$$

Then:

$$ds=\frac{a(1-e^2)cos\varphi d\varphi}{cos(Aze)(1-e^2sin^2\varphi)\sqrt{(1-k^2sin^2\varphi)(1-e^2sin^2\varphi)}} \tag{1.78}$$

By taking $t = sin\varphi$ as a new variable, the integral of (1.78) gives by taking as the origin of the curvilinear abscissa s a point on the equator:

$$\boxed{s=\frac{a(1-e^2)}{cosAze}\int_0^{sin\varphi}\frac{dt}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}}} \tag{1.79}$$

The integrals (1.76) and (1.79) are called elliptic integrals of the third kind.

1.6 APPLICATIONS TO DIRECT AND INVERSE PROBLEMS OF THE COMPUTATION OF GEODESIC LINES

In this second part, we will deal numerically the application of the preceding formulas in the resolution of the problems called respectively direct and inverse of the computation of the geodesic lines.

1.6.1 The Direct Problem

We give :

- the coordinates (φ_1,λ_1) of a point M_1;

- the length s of the geodesic line from M_1 to M_2;

- the geodetic azimuth Az_1 of the geodesic line from M_1 to M_2.

We ask to calcultate:

- the geodetic coordinates (φ_2,λ_2) of the point M_2;

- the geodetic azimuth Az_2 at M_2.

Solution: 1. Calculate the constant C, $C = N(\varphi_1).cos\varphi_1.sinAz_1 = a.sin(Aze)$, then Aze and k.

2. Determination of φ_2 from :

$$\Delta s = \frac{a(1-e^2)}{cosAze}\frac{cos\varphi_1\Delta\varphi}{(1-e^2sin^2\varphi_1)\sqrt{(1-k^2sin^2\varphi_1)(1-e^2sin^2\varphi_1)}}$$

with $\Delta\varphi = \varphi_2 - \varphi_1$.

3. Knowing φ_2, we calculate λ_2 using :

$$\lambda_2 - \lambda_1 = (1-e^2)tg(Aze)\int_{sin\varphi_1}^{sin\varphi_2}\frac{dw}{(1-w^2)\sqrt{(1-e^2w^2)(1-k^2w^2)}}$$

4. Calculate Az_2 by the formula $sin(Az_2) = C/r(\varphi_2)$.

1.6.2 The Inverse Problem

We give the coordinates (φ_1,λ_1) and (φ_2,λ_2) of two points M_1 and M_2. We ask to calculate:

- the length s of the geodesic line from M_1 to M_2;

- the geodetic azimuth Az_1 at M_1;

- the geodetic azimuth Az_2 at M_2.

Solution:

1. We must calculate the constant C: from the equation (1.69), we can write :

$$\left(\frac{\Delta\lambda}{\Delta\varphi}\right)^2 = \frac{\rho^2(\varphi_1)}{r^2(\varphi_1)}\frac{C^2}{(r^2(\varphi_1)-C^2)} = \frac{(\lambda_2-\lambda_1)^2}{(\varphi_2-\varphi_1)^2}$$

that gives C:

$$C = \frac{\frac{r^2}{\rho}\frac{\Delta\lambda}{\Delta\varphi}}{\sqrt{1+\frac{r^2}{\rho^2}\left(\frac{\Delta\lambda}{\Delta\varphi}\right)^2}}$$

Considering the azimuth between 0 and π, therefore Az is positive, C is positive. By calculating it for φ_1 and φ_2, we get C by the mean value:

$$C = \frac{C_1(\varphi_1)+C_2(\varphi_2)}{2}$$

2. Then, we obtain the value of k by (1.71):

$$k = \frac{a^2-C^2e^2}{a^2-C^2}$$

3. Having the value of C, from (1.51), we obtain Az_1 and Az_2 :

$$sinAz_1 = \frac{C}{r(\varphi_1)} \quad \text{and} \quad sinAz_2 = \frac{C}{r(\varphi_2)}$$

4. Then, we obtain also Az_e: $sinAz_e = \dfrac{C}{a}$

5. Finally, the equation (1.79) gives s.

We retire the process.

1.6.3 Computation of the term (1.79)

In this paragraph, we calculate in detail :

$$s = \frac{a(1-e^2)}{cosAz_e}\int_0^{sin\varphi}\frac{dt}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}}$$

For $|x|<1$, we have the following limited developments:

$$\frac{1}{(1+x)^{3/2}} = 1-\frac{3}{2}x+\frac{15}{8}x^2-\frac{35}{16}x^3+\frac{315}{128}x^4+... \tag{1.80}$$

$$\frac{1}{\sqrt{1-x}} = 1+\frac{x}{2}+\frac{3x^2}{8}+\frac{5x^3}{16}+\frac{35x^4}{128}+... \tag{1.81}$$

Taking $x=-e^2t^2$ and $x=k^2t^2$, we obtain:

$$\frac{1}{(1-e^2t^2)^{3/2}} = 1+\frac{3}{2}e^2t^2+\frac{15}{8}e^4t^4+\frac{35}{16}e^6t^6+\frac{315}{128}e^8t^8+...$$

$$\frac{1}{\sqrt{1-k^2t^2}} = 1+\frac{k^2t^2}{2}+\frac{3k^4t^4}{8}+\frac{5k^6t^6}{16}+\frac{35k^8t^8}{128}+... \tag{1.82}$$

Then:

$$\frac{1}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}}=1+\frac{k^2+3e^2}{2}t^2+\frac{3k^4+6e^2k^2+15e^4}{8}t^4+$$
$$\frac{5k^6+9k^4e^2+15k^2e^4+35e^6}{16}t^6+\frac{35k^8+60k^6e^2+90k^4e^4+140k^2e^6+315e^8}{128}t^8+... \quad (1.83)$$

or to the order 4 :

$$\frac{1}{(1-e^2t^2)\sqrt{(1-k^2t^2)(1-e^2t^2)}}=1+mt^2+nt^4+... \quad (1.84)$$

with:

$$m=\frac{k^2+3e^2}{2}; \quad n=\frac{3k^4+6e^2k^2+15e^4}{8}$$

1.6.4 Calculation of the expression (1.76)

We have:

$$\lambda-\lambda_e=(1-e^2)tg(Aze)\int_0^{sin\varphi}\frac{dw}{(1-w^2)\sqrt{(1-e^2w^2)(1-k^2w^2)}}$$

that becomes in our case:

$$\lambda_2-\lambda_1=(1-e^2)tg(Aze)\int_{sin\varphi_1}^{sin\varphi_2}\frac{dt}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}}$$

But from (1.81):

$$\frac{1}{\sqrt{1-e^2t^2}}=1+\frac{1}{2}e^2t^2+\frac{3}{8}e^4t^4+\frac{5}{16}e^6t^6+\frac{35}{128}e^8t^8+...$$

and :

$$\frac{1}{\sqrt{1-k^2t^2}}=1+\frac{k^2t^2}{2}+\frac{3k^4t^4}{8}+\frac{5k^6t^6}{16}+\frac{35k^8t^8}{128}+...$$

and for $(1-t^2)^{-1}$, we obtain:

$$\frac{1}{1-t^2}=1+t^2+t^4+t^6+t^8+...$$

Then :

$$\frac{1}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}}=1+\frac{2+k^2+e^2}{2}t^2+$$
$$\frac{8+4k^2+4e^2+3k^4+2e^2k^2+3e^4}{8}t^4+$$
$$\frac{16+8k^2+8e^2+6k^4+4e^2k^2+6e^4+5k^6+3k^4e^2+3k^2e^4+5e^6}{16}t^6+...$$

that we write under the form:

$$\frac{1}{(1-t^2)\sqrt{(1-e^2t^2)(1-k^2t^2)}}=1+\alpha t^2+\beta t^4+\gamma t^6+... \quad (1.85)$$

with:

$$\begin{cases} \alpha = \dfrac{2+k^2+e^2}{2} \\ \beta = \dfrac{8+4k^2+4e^2+3k^4+2e^2k^2+3e^4}{8} \\ \gamma = \dfrac{16+8k^2+8e^2+6k^4+4e^2k^2+6e^4+5k^6+3k^4e^2+3k^2e^4+5e^6}{16} \end{cases} \tag{1.86}$$