A recursive algorithm proving the Strong Goldbach Conjecture

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Abstract

A Recursive Algorithm described here generates consecutive sequences of Goldbach sets

$$\left\{G_k\mathbb{P} \mid 3 \le k \le m\right\}, \text{ where } G_k\mathbb{P} = \left\{n, n' \mid n \in \mathbb{P}, n' = 2\cdot k - n \in \mathbb{P}\right\}$$

toward the proof of the Strong Goldbach Conjecture. This approach is grounded in the fundamental principle of mathematical induction and uses rather elementary set-theoretical technique.

I tried to follow the idea of Martin Aigner and Günter M. Ziegler [10] to make the content accessible to the readers with their background only in the basics of discrete mathematics.

The main idea is to develop a recursive algorithm toward building the sequence of consecutive Goldbach sets $\left\{G_k\mathbb{P} \mid 3 \le k \le m\right\}$ representing solutions to the system of Goldbach equations

$$\left\{x + y = 2\cdot k \mid 3 \le k \le m\right\} \text{ in the intervals } I_k = [3, 2\cdot k - 3],\ 3 \le k \le m.$$

Validity of the algorithm follows from the proved here recursive formula

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2\cdot(m-k)\right) \cap S_m\right] = G_m\mathbb{P} \ne \varnothing\ ,$$

given the inductive assumption that $G_k\mathbb{P} \ne \varnothing$ for all $k : 3 \le k < m$, where $S_m = I_m \cap \mathbb{P}$, and $\mathbb{P}$ is a set of all odd prime numbers. We establish a definite connection between the Goldbach function $G(2m)$ and some invariant properties for Diphantine variety of Goldbach sets.

"The most interesting facts are those which can be used several times, those which have a chance of recurring ..."

(Henry Poincaré, The Value of Science)

1. Shift invariance of Goldbach Set.

We approach here one of old classical problems in Number Theory known as the strong form of Goldbach Conjecture (SGC) [1, 5]. According to the conjecture stated by Goldbach in his letter to Euler in 1742, "every even number $2m \geq 6$ is the sum of two odd primes" [1]. Regardless numerous attempts to prove the statement, supported in our days by computer calculations up to 4×10^{18}, it remains unproven till now.

Let $\mathbb{N}$ be a set of natural numbers, and $\mathbb{P}$ a set of odd primes (all prime numbers excluding 2). The Goldbach's Conjecture (GC), as one of the oldest and notoriously known unsolved problems in Number theory, raises a question why it seems so difficult to decide whether the equation

$$p + p' = 2m, \qquad (*)$$

where p and p' are prime numbers, has at least one solution for each even number $2m \geq 6$. Indeed, occurrences of primes look very sporadic, so that it is hard to predict, that there exists a pair of primes (p, p') related by the equation (*), especially for 'big' values of m.

Notice that every solution $(n, n') = (p, p')$ in primes to the equation $p + p' = 2m$, must satisfy condition: $(n, n') \in [3, 2m-3]^2$.

We call a prime number p a G_m- prime (*Goldbach prime*) if $p' = 2m - p$ is also a prime number. Then, denote $G_m\mathbb{P}$ as set of all G_m- *primes*, and call $G_m\mathbb{P}$ *Goldbach set*.

The number of elements in set $G_m\mathbb{P}$, denoted $G(2m)$, is called Goldbach function.

Obviously, for all $m \geq 3$ we have $G_m\mathbb{P} \subset I_m = [3, 2 \cdot m - 3]$. A set $G_m\mathbb{P}$ is empty if for some $m \geq 3$ G_m- primes do not exist. Goldbach function $G(2m)$ counts the number of solutions to the equation

$$n + n' = 2m,\ (n, n') \in \mathbb{P}^2 \tag{1}$$

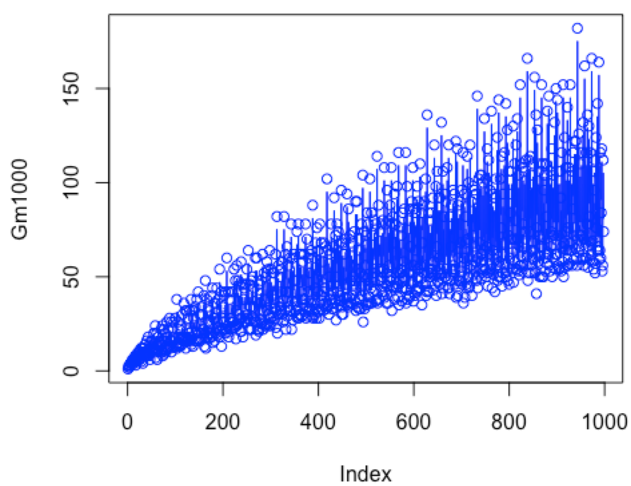
where n and n' are prime numbers, m is any integer $m \geq 3$.

Obviously, any pair (p, p') of primes greater then 2 solves (1) for $2m = p + p'$.

Due to infinity of $\mathbb{P}$, this implies that the Goldbach function has $\limsup G(2m) = \infty$.

The Strong Goldbach Conjecture states that $\min G(2m) = 1$, so that every set $G_m\mathbb{P}$ is nonempty set for all $m \geq 3$. Calculations show that $\max_{m \leq M} G(2m)$ increases with M, though $G(2m)$ is not a monotonically increasing function (Fig.1)

Goldbach function $G(2 \cdot m)$ for $m = 3,4,\ldots,1000$ (Fig.1)



We observe that each pair (n, n') which solves (1) must belong to a set $[3, 2m-3]^2 = I_m^2$, where $I_m = [3, 2m-3] = \{3,4,\ldots,2m-3\}$. Since 3 is prime, if $n' = (2 \cdot m - 3) \in \mathbb{P}$, the pair $(3, 2 \cdot n - 3)$ solves (1), so that the prime $(2 \cdot m - 3) \in G_m\mathbb{P}$, and we need to consider the case $(2 \cdot m - 3) \notin \mathbb{P}$. In general, if p is a prime number such that $(2 \cdot m - p) \in \mathbb{P}$, then $(2 \cdot m - p) \in G_m\mathbb{P}$.

Consider a *shift mapping* of an interval of integers $\theta_m : I_m \to I_m$ given by the formula

$$\theta_m(n) = 2m - n. \quad (2)$$

Denote $\mathcal{F}_m$ an algebra of all subsets of the interval $I_m = [3, 2m-3]$.

Obviously, θ_m is one-to-one and has an inverse θ_m^{-1}, so that for all $A \in \mathcal{F}_m$

we have $\theta_m(A) \in \mathcal{F}_m, \theta_m^{-1}(A) \in \mathcal{F}_m$. Denote $I_m^- = [3, m-1], I_m^0 = \{m\}, I_m^+ = [m+1, 2m-3]$.

Obviously, θ_m is idempotent: $\theta_m^2 = id$ (an identical map), that is $\theta_m^{-1} = \theta_m$.

Indeed, $\theta_m^2(n) = \theta_m(\theta_m(n)) = \theta_m\left(2 \cdot m - (2 \cdot m - n)\right) = n$.

Let S_m denote a set of prime numbers in the interval of integers $I_m = [3, 2m-3]$,

that is $S_m = I_m \cap \mathbb{P}$, and $S_m^c = I_m \setminus S_m$ its complement in I_m so that

$I_m = S_m \cup S_m^c$, $S_m \cap S_m^c = \varnothing$. While S_m stands for the set of primes in I_m,

S_m^c is the set of composite numbers in I_m.

We denote $\theta_m(S_m) = 2 \cdot m - S_m = \left\{ n' \mid n' = 2 \cdot m - n, n \in S_m \right\}$.

The Strong Goldbach Conjecture asserts that for any $m \geq 3$ the set $G_m\mathbb{P}$ is not empty:

$$G_m\mathbb{P} = \left\{ n, n' \mid n \in \mathbb{P}, n' = (2m - n) \in \mathbb{P} \right\} = (2m - S_m) \cap S_m = \theta_m(S_m) \cap S_m \neq \varnothing.$$

Lemma 1.

Golbach sets $G_m\mathbb{P}$ on intervals I_m are θ_m- shift invariant: $\theta_m(G_m\mathbb{P}) = G_m\mathbb{P}$.

Proof.

Notice that the sets $I_m, \{m\}, \{3, 2 \cdot m - 3\}$ and $\theta_m(S_m) \cap S_m$ are invariant sets of the map

$\theta_m : I_m \to I_m$ since for all $n \in I_m$ we have $\theta_m\left(\left\{n, \theta_m(n)\right\}\right) = \left\{\theta_m(n), \theta_n^2(n)\right\} = \left\{\theta_m(n), n\right\}$,

due to $\theta_m^2 = id$. θ_m-invariance of $G_m\mathbb{P}$ also follows directly from the equalities

$$\theta_m\left(G_m\mathbb{P}\right) = \theta_m\left(S_m \cap \theta_m(S_m)\right) = \theta_m(S_m) \cap \theta_m^2(S_m) = \theta_m(S_m) \cap S_m = G_m\mathbb{P}.$$

Q.E.D.

In what follows we need several recursively derived formulas.

Lemma 2.

(1) $I_m = I_{m-1} \cup \{2m-4, 2m-3\}$, where $I_m = [3, 2m-3]$ (3)

(2) $$S_m = S_{m-1} \cup \left(\mathbb{P} \cap \{2m-3\}\right) = \begin{cases} S_{m-1} \cup \{2m-3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \end{cases} \quad (4)$$

(3) $$\theta_m(S_m) = \begin{cases} \theta_m\left(S_{m-1}\right) \cup \{3\} \text{ if } \{2m-3\} \in \mathbb{P} \\ \theta_m\left(S_{m-1}\right) \text{ if } \{2m-3\} \notin \mathbb{P} \end{cases} \quad (5)$$

(4) $$G_m\mathbb{P} = \begin{cases} \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cup \{3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cap S_{m-1} \text{ if } (2m-1) \notin \mathbb{P} \end{cases} \quad (6)$$

Proof.

(1) We observe that $I_{m-1} = [3, 2\cdot(m-1)-3] = [3, 2\cdot m-5]$

so that $I_m = [3, 2m-3] = I_{m-1} \cup \{2m-4, 2m-3\}$.

(2) $\left(\{2m-4, 2m-3\} \cap \mathbb{P}\right) = \left(\{2m-3\} \cap \mathbb{P}\right)$ implies

$$S_m = I_m \cap \mathbb{P} = \left(I_{m-1} \cap \mathbb{P}\right) \cup \left(\{2m-4, 2m-3\} \cap \mathbb{P}\right) = S_{m-1} \cup \left(\mathbb{P} \cap \{2m-3\}\right),$$

Thus, $S_m = I_m \cap \mathbb{P} = S_{m-1} \cup \{2m-3\}$ if $(2m-3) \in \mathbb{P}$ and $S_m = S_{m-1}$ otherwise.

(3) $$\theta_m(S_m) = \theta_m\left(S_{m-1}\right) \cup \theta_m\left(\{2m-3\} \cap \mathbb{P}\right) = \begin{cases} \theta_m\left(S_{m-1}\right) \cup \{3\} \text{ if } \{2m-3\} \in \mathbb{P} \\ \theta_m\left(S_{m-1}\right) \text{ if } \{2m-3\} \notin \mathbb{P} \end{cases},$$

since $\theta_m(2m-3) = 2\cdot m - (2m-3) = 3$.

(4) $$G_m\mathbb{P} = \theta_m(S_m) \cap S_m = \left(\theta_m(S_{m-1}) \cup \{3\}\right) \cap S_{m-1} = \left(\theta_m(S_{m-1}) \cap S_{m-1}\right) \cup \{3\}$$

if $(2m\text{-}3) \in \mathbb{P}$, and $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1}$ if $(2m-3) \notin \mathbb{P}$, that is

$$G_m\mathbb{P} = \begin{cases} \theta_m\left(S_{m-1}\right) \cap S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \\ \left(\theta_m\left(S_{m-1}\right) \cap S_{m-1}\right) \cup \{3\} \text{ if } (2m-3) \in \mathbb{P} \end{cases}$$

Q.E.D.

Notice that (5) implies that if $(2m-3) \notin \mathbb{P}$, then $S_m = S_{m-1}$ and $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1}$. In the case when $(2\cdot m-3) \in \mathbb{P}$, we have $G_m\mathbb{P} \neq \varnothing$ since $(2m-3) \in \mathbb{P}$, so that $3 + (2m-3) = 2m$.

Thus, we need to consider the case $(2m-3) \notin \mathbb{P}$. We observe that $\theta_m(S_{m-1}) = \theta_{m-1}(S_{m-1}) + 2$, due to (7) in Lemma 3 below. This implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\ \right)\cap S_{m-1}$. If $p\in G_{m-1}\mathbb{P}\neq\varnothing$,

then $p\in S_{m-1}$ and $p\in\theta_{m-1}(S_{m-1})$. Assuming now that p is a twin prime, that is

$(p+2)\in S_m=S_{m-1}$, we have that $G_{m-1}\mathbb{P}\neq\varnothing$ implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\ \right)\cap S_{m-1}\neq\varnothing$.

See in what follows the more detailed discussion and definitions of sets of twin primes $T_1\mathbb{P}$ and t-primes $T_t\mathbb{P}$ related to the Goldbach Conjecture.

The next Lemma concerns some properties of the shift transformation θ_m $(m\geq 3)$

Lemma 3.

Consider a shift transformation

$\theta_m:\ \mathbb{Z}\to\mathbb{Z}$ such that $\theta_m(n)=2\cdot m-n,$ where $n\in\mathbb{N},\ m\in\mathbb{N}\ (m\geq 3)$.

Then, for any subset $A\subseteq\mathbb{Z}$ and integer $t\in\mathbb{N}$ the following properties of θ_m hold true:

$$\begin{aligned}\theta_{m+t}(A)&=\theta_m(A)+2\cdot t\\ \theta_m(A)&=\theta_{m-t}(A)+2\cdot t\end{aligned}\tag{7}$$

Proof.

$$\begin{aligned}\theta_{m+t}(A)&=2\cdot(m+t)-A=2\cdot m-A+2\cdot t=\theta_m(A)+2\cdot t\\ \theta_m(A)&=2\cdot m-A=2\cdot(m-t)-A+2\cdot t=\theta_{m-t}(A)+2\cdot t\end{aligned}$$

Q.E.D.

2. Twin primes, t-primes and Goldbach sets.

Let $T_1\mathbb{P}=\left\{p\mid p\in\mathbb{P}\text{ and }(p+2)\in\mathbb{P}\right\}$ stand for a set of all twin primes and consider $G_k\mathbb{P}\cap T_1\mathbb{P}$ for each $k\ (3\leq k\leq m)$. Thus, if for some $k\ (3\leq k\leq m)$ $G_k\mathbb{P}\cap T_1\mathbb{P}\neq\varnothing$, then $G_{k+1}\mathbb{P}\neq\varnothing$.

Lemma 3 implies that if for a prime $p\in G_k\mathbb{P}$ there exists a twin prime $(p+2)\in\mathbb{P}$,

then $\left(p+2\right)\in G_{k+1}\mathbb{P}$. This shows some connection between the Twin Prime Conjecture and the Strong Goldbach Conjecture (SGC), and, moreover, between the t-Prime Conjecture (de Polignac Conjecture (1849)) and SGC, as we observe below. This also shows how nonempty Goldbach sets $G_k\mathbb{P}$ can propagate further with increasing values of k.

Definition.

Denote, in general, by $T_t\mathbb{P}$ $(t \in \mathbb{N})$ a set *of t-primes* for some $t \in \mathbb{N}$, that is

$T_t\mathbb{P} = \{p \mid p \in \mathbb{P} \text{ and } (p + 2 \cdot t) \in \mathbb{P}\}$. Notice that $\bigcup_{t=0}^{\infty} T_i\mathbb{P} = T_0\mathbb{P} = \mathbb{P}$,

where $\mathbb{P}$ stands for the set of all odd prime numbers. Consider examples below:

$\{3,5,11,17,29,41\} \subset T_1\mathbb{P}$, $\{3,7,13,19,37,43\} \subset T_2\mathbb{P}$, $\{5,7,11,17,23,31,37,41\} \subset T_3\mathbb{P}$, and so on.

Propagation of nonempty $G_k\mathbb{P}$ for all $k \geq 3$ is based on the following observations.

Lemma 4.

Let $p \in G_k\mathbb{P} \neq \varnothing$ and $p \leq k$. There exist $q \in \mathbb{P}$ $(q > p)$ and $t \in \mathbb{N}$ ($1 \leq t < k - 1$) such that $q = p + 2 \cdot t \in \mathbb{P}$, and $p \in T_t\mathbb{P}$. This implies that there exists $t \in \mathbb{N}$ ($1 \leq t < k - 1$) such that $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $q = (p + 2 \cdot t) \in G_{k+t}\mathbb{P} \neq \varnothing$.

Proof.

Let $p \in G_k\mathbb{P}$, $p \leq k$. Thanks to the Bertrand's postulate [4], there exists a prime q between integers k and $2 \cdot k\, (k > 3)$. This implies that there exists $t \in \mathbb{N}$ such that this prime q can be expressed in the form $q = (p + 2 \cdot t) \in \mathbb{P}$, where $1 \leq t < k - 1$.

Indeed, we can take $t = \dfrac{q - p}{2}$, so that $q = p + 2 \cdot t \in \mathbb{P}$. Then, $p \in G_k\mathbb{P}$ implies that $p + \theta_k(p) = 2 \cdot k$ and $(p + 2 \cdot t) + \theta_k(p) = 2 \cdot k + 2 \cdot t = 2 \cdot (k + t)$.

Since $q = p + 2 \cdot t \in \mathbb{P}$ and $\theta_k(p) \in G_k\mathbb{P}$, we have $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $q = (p + 2 \cdot t) \in G_{k+t}\mathbb{P} \neq \varnothing$.

Q.E.D.

This shows how nonempty Goldbach sets $G_3\mathbb{P}, G_4\mathbb{P}, G_5\mathbb{P}, \ldots G_{12}\mathbb{P}, \ldots$ have been generated:

$$G_3\mathbb{P} = \{3\}, G_4\mathbb{P} = \{3, 3+2\}, G_5\mathbb{P} = \{3,5,5+2\}, G_6\mathbb{P} = \{5, 5+2\},$$

$$G_7\mathbb{P} = \{3,7,7+4\},\ G_8\mathbb{P} = \{3,5,11,11+2\},\ G_9\mathbb{P} = \{5,7,11,11+2\},$$

$$G_{10}\mathbb{P} = \{3,7,13,13+4\}, G_{11}\mathbb{P} = \{3,11,17,17+2\}, G_{12}\mathbb{P} = \{5,7,11,13,17+2\}\ldots$$

Let $G_k\mathbb{P} \neq \varnothing$, so that there exist $p \in G_k\mathbb{P}$ and $p' = \theta_k(p) \in G_k\mathbb{P}$, where $p' = \theta_k(p) = 2 \cdot k - p$.

Assume that q is prime and $q > p$ such that $q = p + 2 \cdot t$, where $t = \frac{q-p}{2}$.

This implies that $q + \theta_k(p) = \left(p + 2 \cdot t\right) + \theta_k(p) = \left(p + \theta_k(p)\right) + 2 \cdot t = 2 \cdot (k+t)$.

Since both q and $\theta_k(p)$ are primes and $q + \theta_k(p) = 2 \cdot (k+t)$, we have q and $\theta_k(p)$ belong to $G_{k+t}\mathbb{P} \neq \varnothing$. For instance, if for some k $(3 \le k \le m)$ we have $G_{k-1}\mathbb{P} \cap T_2\mathbb{P} \neq \varnothing$, then, due to Lemma 4, $G_{k+1}\mathbb{P} \neq \varnothing$. Consider $p = 3$ and $q = 5$, that is we start from $G_3\mathbb{P} = \{3\}$.

Then, $t = \frac{5-3}{2} = 1$ and $\theta_3(3) = 3$. Thus, we have $q + \theta_k(p) = 5 + 3 = 2 \cdot (3+1) = 8$, where 3 and 5 both belong to $G_{3+1}\mathbb{P} = G_4 = \{3,5\}$.

Then, due to Lemma 4, for each $G_k\mathbb{P} \neq \varnothing$ there exists $t \in \mathbb{N}$ such that $G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$, which implies $G_{k+t}\mathbb{P} \neq \varnothing$. This means that the occurrence of a t-prime in a non-empty set $G_k\mathbb{P}$ implies that $G_{k+t}\mathbb{P}$ is necessarily non-empty. This provides proliferation of non-empty sets $G_k\mathbb{P}$ t steps forward, so that $G_{k+t}\mathbb{P}$ is not empty for any k.

Starting from $k = 3$ and $t = 1$ the 'wave' of G_k-primes propagates forward recursively as $k \to \infty$ without gaps, supported by the existence of such t-primes.

Observe that each pair of primes (p,q) such that $p \in G_k\mathbb{P}$, $q > p$ and $q \in S_m = I_m \cap \mathbb{P}$, generates a nonempty set $G_{k+t}\mathbb{P}$, where $t = \frac{q-p}{2}$ and p is a t-prime in $G_k\mathbb{P}$.

Notice that each prime number in $G_m\mathbb{P}$ for $m \ge 3$ is a t-prime for an appropriate value of t. Our goal is to demonstrate that we can build a nonempty Goldbach set $G_m\mathbb{P}$ for every $m > 3$, given a sequence of nonempty Goldbach sets $\{G_k\mathbb{P}\}_{3 \le k \le m-1}$, by using assumption of mathematical induction. We need the following simple Lemmas.

Lemma 5.

Let $S_m = I_m \cap \mathbb{P}$, where $I_m = [3, 2 \cdot m - 3]$.

For every prime $p \in S_m$ there is $k \le m$ such that $p \in G_k\mathbb{P}$.

Proof.

Indeed, we can take $k = p$. Then, $p \in G_p\mathbb{P}$ since $p + p = 2 \cdot p \le 2 \cdot m$.

Another possibility for any prime $p \in S_m$ is to consider $k = \frac{3+p}{2}$,

since $3 + p = 2 \cdot k \le 2 \cdot m$ implies $p \in G_k\mathbb{P}$.

Q.E.D.

Lemma 6.

For all $m \ge 3$ we have

$$S_m = \bigcup_{k=3}^{m} G_k\mathbb{P} = G^{(m)}\mathbb{P}. \tag{8}$$

Proof.

For any $p \in S_m$, due to Lemma 4, there exists $k \le m$ such that $p \in G_k\mathbb{P}$, so that $p \in G^{(m)}\mathbb{P}$.

And vice versa, if $p \in G^{(m)}\mathbb{P}$, then $p \in G_k\mathbb{P}$ for some $k \le m$, so that $p \in S_m$.

The following statement concerns a recurrent formula that generates an infinite sequence of nonempty Goldbach sets $G_m\mathbb{P}$ for all $m \ge 3$.

Q.E.D.

Theorem 1.

Let $G_k\mathbb{P} \ne \varnothing$ for all $k : 3 \le k \le m-1$. If $2 \cdot m - 3 \in \mathbb{P}$, then $2 \cdot m - 3 \in G_m\mathbb{P} \ne \varnothing$.

Otherwise, if $2 \cdot m - 3 \ne G_m\mathbb{P}$, we have $S_m = S_{m-1}$, due to Lemma 2.

Then, for any $m \ge 3$ the following equality holds true:

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2 \cdot (m-k)\right) \cap S_m\right] = G_m\mathbb{P} = \varnothing \tag{9}$$

Proof.

Denote $A_{k,m} = \left(G_k\mathbb{P} + 2 \cdot (m-k)\right)$.

Then, $\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2 \cdot (m-k)\right) \cap S_m\right] = \bigcup_{k=3}^{m-1}\left[A_{k,m} \cap S_m\right] = \left(\bigcup_{k=3}^{m-1} A_{k,m}\right) \cap S_m$.

Consider $\theta_m(A_{k,m}) = 2 \cdot m - A_{k,m} = 2 \cdot m - G_k\mathbb{P} - 2 \cdot (m-k) = 2 \cdot k - G_k\mathbb{P} = \theta_k(G_k\mathbb{P}) = G_k\mathbb{P}$.

Indeed, due to Lemma 1 about θ_m - shift invariance of $G_k\mathbb{P}$, we have $\theta_k(G_k\mathbb{P}) = G_k\mathbb{P}$.

Equality $\theta_m(A_{k,m}) = G_k\mathbb{P}$ implies that

$$\theta_m\left(\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\right)=\left(\bigcup_{k=3}^{m-1}\theta_m(A_{k,m})\right)\cap\theta_m(S_m)=\left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)\cap\theta_m(S_m).$$

According to equality $S_m = \bigcup_{k=3}^{m} G_k\mathbb{P} = G^{(m)}\mathbb{P}$ (see formula (8) in Lemma 6) we find

$S_{m-1} = \left(\bigcup_{k=3}^{m-1} G_k\mathbb{P}\right) = G^{(m-1)}\mathbb{P}$. Then, $2\cdot m-3 \neq G_m\mathbb{P}$ implies $S_m = S_{m-1}$. Hence

$$\theta_m\left(\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\right)=\left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)\cap\theta_m(S_m)=S_m\cap\theta_m(S_m)=G_m\mathbb{P}.$$

We have $\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_{m-1} = G^{(m-1)}\mathbb{P}\cap S_{m-1} = G^{(m-1)}\mathbb{P} \neq \varnothing$, due to the assumption of mathematical induction. Then, from $\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_{m-1} \neq \varnothing$ it follows $\theta_m\left(\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\right) = G_m\mathbb{P} \neq \varnothing$.

Q.E.D.

The recursive formula (9) in Theorem 1 proves the Strong Goldbach Conjecture.

Lemma 4 and Theorem 1 show a definite connecion between the number of solutions to the Goldbach equation $p + p' = 2\cdot m$ in the intervals $I_m = [3, 2\cdot m - 3]$ and the number of t-primes in sets $G_k\mathbb{P}$ for $k: 3 \le k < m$. We discuss this in what follows.

3. Diophantine variety of Goldbach sets.

A seqience of Goldbach sets $\{G_k\mathbb{P} \mid 3 \le k \le m\}$ represents solutions to the system of Goldbach equations $\{x + y = 2\cdot k \mid 3 \le k \le m\}$ in the intervals $I_k = [3, 2\cdot k - 3]$.

This system is an algebraic variety given by linear equations $x + y = 2\cdot k$ $(3 \le k \le m)$, which solutions (if exist) are pairs of prime numbers $(p, p') \in \mathbb{P}^2$.

Geometrically, each Goldbach set $G_k\mathbb{P}$ is a sequence of points with coordinates

$\left(p,p'\right)\in\mathbb{P}^2$ on the segment of a straight line given by $x+y=2\cdot k$, $(x,y)\in[0,2\cdot k]$, symmetrically located on the line with respect to a point (k,k), due to invariance $\theta_k\left(G_k\mathbb{P}\right)=G_k\mathbb{P}$, where $\theta_k(x)=2k-x=y$. See below Fig. 1 and 2 representing Diophantine geometry of Goldbach sets, where dots are points with coordinates $\left(p,p'\right)\in\mathbb{P}^2$ on the corresponding lines. These dots are solutions to the Goldbach equations $x+y=2\cdot k$ $\left(3\le k\le m\right)$. The theorem below answers the question how many solutions are in each Golbach set.

Theorem 2.

The number of solutions to the Goldbach equation $p+p'=2\cdot m$ in primes $(p,p')\in\mathbb{P}^2$, where $p<m$ and m is not prime, in each ineval $I_m=[3,2\cdot m-3]$ is equal to the number of t-primes in the set $G_m\mathbb{P}$ such that $t=\dfrac{p'-p}{2}$. We have then, $p=m-t,\ p'=m+t$.

Proof.

Consider a quadratic polynomial $P_m(x)=x^2+2\cdot m\cdot x+c$ for m, c and $x\in\mathbb{Z}$.

Let a pair of primes (p,p') be a solution to the Goldbach equation $p+p'=2\cdot m$ in the interval $I_m=[3,2\cdot m-3]$. Obviously, for $c=p\cdot p'$, the pair of prime numbers $\left(p,p'\right)\in\mathbb{P}^2$ are roots of the polynomial $P_m(x)=x^2-2\cdot m\cdot x+p\cdot p'=\left(x-p\right)\cdot\left(x-p'\right)$.

Discriminant of $P_m(x)$ is $D=4\cdot(m^2-p\cdot p')=4\cdot t^2$, where t is a nonnegative integer.

Observe that $\left(m^2-p\cdot p'\right)=t^2$ implies $(m-t)\cdot(m+t)=p\cdot p'$.

Since p and p' are not equal prime numbers, the equation $(m-t)\cdot(m+t)=p\cdot p'$ for an integer t and $p\le p'$ implies: $m-t=p$ and $m+t=p'$, so that

$t=\dfrac{p'-p}{2}$ and $p'=p+2\cdot t$. This means that $p\in G_m\mathbb{P}$ is a t-prime in $G_m\mathbb{P}$, where $t=\dfrac{p'-p}{2}$.

Therefore, we have as many solutions $(p,p')\in\mathbb{P}^2$ to the equation $P_m(x)=x^2+2\cdot m\cdot x+p\cdot p'$ as there are t-primes, $t=\dfrac{p'-p}{2}$, in the set $G_m\mathbb{P}$. Assume now that $2\cdot m=p+q$, $c=p\cdot q$, where $p \in \mathbb{P}$ and $q = 2 \cdot m - p$ is an unknown integer.

Q.E.D.

For $p \in G_m\mathbb{P}$ the polynomial $P_m(x)$ takes a form: $P_m(x) = x^2 + 2 \cdot m \cdot x + p \cdot (2 \cdot m - p)$.

Its discrimimant is $D_m = 4 \cdot \left(m^2 - p \cdot (2 \cdot m - p)\right) = 4 \cdot \left(m^2 - 2 \cdot m \cdot p + p^2\right) = 4 \cdot \left(m - p\right)^2$.

The solutions to the equation $P_m(x) = 0$ are $x_{1,2} = m \pm (m - p)$, where $x_1 = p, x_2 = 2 \cdot m - p$.

For instance, let $m = 9$, $c = 45$. Then, $P_9(x) = x^2 - 18x + 45$ has 2 roots:

$x_1 = 3 \in \mathbb{P}, x_2 = 2 \cdot 9 - 3 = 15 \notin \mathbb{P}$, but they do not belong to $G_9\mathbb{P}$. Meanwhile, for $m = 9$ and $c = 65$ we have $P_9(x) = x^2 - 18x + 65$ with roots $x_1 = 5 \in \mathbb{P}, x_2 = 2 \cdot 9 - 5 = 13 \in \mathbb{P}$, so that $5 \in G_9\mathbb{P}$ and $13 \in G_9\mathbb{P}$. Notice that $3, \theta_9(3) = 15$ and $2 \cdot 9 = 18$ are not coprime numbers, while $5, \theta_9(5) = 13$ and $2 \cdot 9 = 18$ are all coprime.

Denote $[x]_p = \mathrm{mod}(x, p)$. Then, $[P_m(x)]_p = [x]_p^2 - [2 \cdot m]_p \cdot [x]_p$. The equation.

$$[P_m(x)]_p = [x]_p^2 - [2 \cdot m]_p \cdot [x]_p = [x]_p \cdot \left([x]_p - [2 \cdot m]_p\right) = 0$$

has the following different sets of solutions: $[x]_p = 0$ and $[x]_p = [2 \cdot m]_p$.

Since we are solving equation $P_m(x) = 0$ in primes within interval $I_m = [3, 2 \cdot m - 3]$, the solutions are restricted to the set $S_m = I_m \cap \mathbb{P}$. Therefore, in interval I_m equations $[x]_p = 0$ and $[x]_p = [2 \cdot m]_p$ have solutions: $x_1 = p < m$, and $x_2 = 2 \cdot m - p > m$.

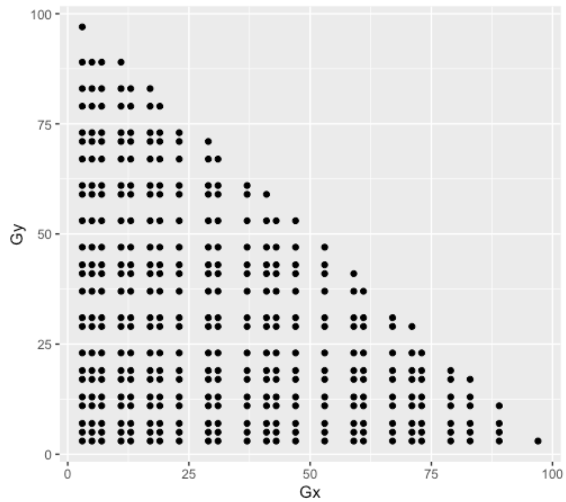
Example 1.

$G_{48}\mathbb{P} = \{7, 13, 17, 23, 29, 37, 43, 53, 59, 67, 73, 79, 83, 89\}$, $m = 48$

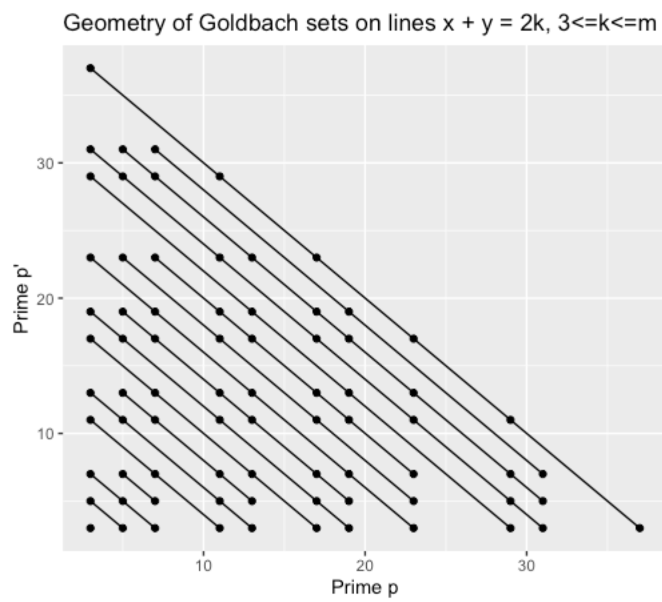
$p = m - t$	7	13	17	23	29	37	43
$p' = m + t$	89	83	79	73	67	59	53
t	41	35	31	25	19	11	5

Diophantine Geometry of Goldbach Sets (Fig.2)

$G_k\mathbb{P}\ \ (k = 3,4,\ldots,50)$ (Fig.1)



$G_k\mathbb{P}\ (k = 3,4,\ldots,40)$ (Fig.3)



Every dot in the above figure denotes a point with coordinates (p, p') such that $p + p' = 2 \cdot k$ on the line $x + y = 2 \cdot k$, where $3 \le k \le m$.

3. Recursive Algorithm generating the infinite sequence of nonempty Goldbach sets $G_m\mathbb{P} \neq \varnothing$ for all natural $m \ge 3$.

We apply now one of the most fundamental and simple proof techniques in mathematics known as *mathematical induction* [3]. Let $\mathrm{Prop}(m)$ denote a statement about a natural number m, and let m_0 be a fixed number. A proof that $\mathrm{Prop}(m)$ is true for all $m \ge m_0$ by induction requires two steps:

Basis step: Verify that $\mathrm{Prop}(m_0)$ is true.

Induction step: Assuming that $\mathrm{Prop}(k)$ is true for all k such that $k : m_0 < k \le m$, verify that $\mathrm{Prop}(m+1)$ is true.

Theorem 3.

$\mathrm{Prop}(m)$: For all integer $m \ge 3$, the set $G_m\mathbb{P}$ of solutions to the equation $n + n' = 2m$, $(n, n') \in \mathbb{P}^2$, in prime numbers is not empty: $G_m\mathbb{P} \neq \varnothing$.

The $\text{Prop}(m)$ can be equivalently stated as: $G_m\mathbb{P} = \theta_m(S_m) \cap S_m \neq \varnothing$ for all integers $m \geq 3$.

Proof.

(1) Basic step.

As we know [2], $\text{Prop}(m)$ is true for all m up to $M = 4 \cdot 10^{18}$.

Let $m_0 = 3$. Then $2 \cdot 3 = 6 = 3 + 3$.

(2) Induction step.

Assume that $G_k\mathbb{P} = \theta_k(S_k) \cap S_k \neq \varnothing$ for all integer $k : m_0 \leq k \leq m\text{-}1$.

Let $k = m$. We denote: $I_m = [3, 2 \cdot m - 3]$ and $S_m = I_m \cap \mathbb{P}$.

In Lemma 2 we proved (3) that $S_m = \begin{cases} S_{m-1} \cup \{2m-3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \end{cases}$.

In the case $(2 \cdot m - 3) \in \mathbb{P}$, the formula (5) implies $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1} \neq \varnothing$.

We can also confirm in this case directly that $G_m\mathbb{P} \neq \varnothing$, because $3 + (2 \cdot m - 3) = 2 \cdot m$

Consider now a general situation, which includes the case $(2 \cdot m - 3) \notin \mathbb{P}$.

This part of the proof consists of two steps.

On the first step we prove that $G_m\mathbb{P} = \bigcup_{k=3}^{m-1} \left[\left(G_k\mathbb{P} + 2 \cdot (m-k) \right) \cap S_m \right]$ by applying

shift transformation θ_m to both sides of the above equation and by using the property of

θ_m- invariance of Goldbach sets $G_m\mathbb{P}$ for all $m \geq 3$.

Next, we observe that $S_m = S_{m-1}$ if $(2 \cdot m - 3) \neq \mathbb{P}$. Then, due to Lemma 6, we obtain

$$G_m\mathbb{P} \subseteq S_m = S_{m-1} = \bigcup_{k=3}^{m-1} G_k\mathbb{P} = G^{(m-1)}\mathbb{P}.$$

On the second step we show that the set $\bigcup_{k=3}^{m-1} \left[G_k\mathbb{P} + 2 \cdot (m-k) \cap S_m \right]$ is not empty.

This follows from the induction assumption $G_k\mathbb{P} \neq \varnothing$ for all $k\ (3 \leq k < m)$.

Finally, Theorem 1 states the recursive formula for nonempty Goldbach sets:

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P}+2\cdot(m-k)\right)\cap S_m\right]=G_m\mathbb{P}\neq\varnothing\ (m=3,4,5,\ldots). \qquad (10)$$

This formula allows to write a computer program to generate recursively potentially infinite sequence of Goldbach sets $G_m\mathbb{P}$ for all $m\geq 3$.

Q.E.D.

See in APPENDIX the text of R-script `GenGS.R` and data lists of the calculated $G_k\mathbb{P}$ for $k:3\leq k\leq m$. An example below illustrates the above statement with some computer calculations. In this example we consider sets $G_m\mathbb{P}=\theta_m(S_m)\cap S_m$ for m from 105 to 110. Notice that many of those sets can be calculated based on the rule that if a prime $p\in G_k\mathbb{P}$ has a twin prime $(p+2)\in\mathbb{P}$, that is $t=1$ and $p\in T_1\mathbb{P}$, then $(p+2)\in G_{k+1}\mathbb{P}$.

For example, terms in $G_{106}\mathbb{P}$ are calculated with this rule by using terms in $G_{105}\mathbb{P}$. Meanwhile, terms in $G_{110}\mathbb{P}$ are calculated by using terms in $G_{108}\mathbb{P}$ for $t=2$ based on the general rule: if $p<k$ and $p\in G_k\mathbb{P}\cap T_t\mathbb{P}$, then $p\in\mathbb{P}$ and $p+2\cdot t\in\mathbb{P}$ implies $(p+2\cdot t)\in G_{k+t}\mathbb{P}$ (Lemma 4): $23+197=(19+2\cdot 2)+197=220=2\cdot 110$, since $19+197=216=2\cdot 108$.

The calculations below illustrate the conclusion of the Theorem 1 (see the data referred in Example 2). We would like to verify that $G_{110}\mathbb{P}\neq\varnothing$, by using that $G_k\mathbb{P}\neq\varnothing$ for all $k\leq 110$. Consider $G_{110}\mathbb{P}$ $(m=110,\ 2\cdot m=220)$. If we choose $t=1$ it would not work with $G_{109}\mathbb{P}$, because $G_{109}\mathbb{P}\cap T_1S_{109}=\varnothing$. We try then $G_{108}\mathbb{P}$ and $t=2$.

We have $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$ and $p=19\in G_{108}\mathbb{P}\cap T_2S_{108}$. Then, $p+2\cdot t=19+2\cdot 2=23$ should belong (due to Lemma 3) to $G_{110}\mathbb{P}$. Therefore, $2\cdot 110-23=197\in G_{110}\mathbb{P}$.

Thus, we have $23+197=2\cdot 110$, which means that $G_{110}\mathbb{P}\neq\varnothing$. Notice that in this instance $k=109$, $k+1-t=109+1-2=108$ and $(k+1-t)+t=108+2=110$, which allows us to establish that $G_{(k+1-t)+t}=G_{110}\mathbb{P}\neq\varnothing$, by using the fact that $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$.

Example 2.

Sets $G_m\mathbb{P} = \theta_m(S_m) \cap S_m$ for m from 105 to 110

$$G_{105}\mathbb{P} = \left\{ \begin{matrix} 11\ 13\ 17\ 19\ 29\ 31\ 37\ 43\ 47\ 53\ 59\ 61\ 71\ 73 \\ 79\ 83\ 97\ 101\ 103\ 107\ 109\ 113\ 127\ 131\ 137\ 139\ 149 \\ 151\ 157\ 163\ 167\ 173\ 179\ 181\ 191\ 193\ 197\ 199 \end{matrix} \right\}$$

$$G_{106}\mathbb{P} = \{13\ 19\ 31\ 61\ 73\ 103\ 109\ 139\ 151\ 181\ 193\ 199\}$$

$$G_{107}\mathbb{P} = \{3\ 17\ 23\ 41\ 47\ 83\ 101\ 107\ 113\ 131\ 167\ 173\ 191\ 197\ 211\}$$

$$G_{108}\mathbb{P} = \left\{ \begin{matrix} 5\ 17\ 19\ 23\ 37\ 43\ 53\ 59\ 67\ 79\ 89\ 103\ 107\ 109 \\ 113\ 127\ 137\ 149\ 157\ 163\ 173\ 179\ 193\ 197\ 199\ 211 \end{matrix} \right\}$$

$$G_{109}\mathbb{P} = \{7\ 19\ 37\ 61\ 67\ 79\ 109\ 139\ 151\ 157\ 181\ 199\ 211\}$$

$$G_{110}\mathbb{P} = \left\{ \begin{matrix} 23\ 29\ 41\ 47\ 53\ 71\ 83\ 89\ 107\ 113 \\ 131\ 137\ 149\ 167\ 173\ 179\ 191\ 197 \end{matrix} \right\}$$

Thus, we can predict that $G_{110}\mathbb{P} \neq \varnothing$ without explicit calculation of this set, just by using the previously calculated sets $G_{109}\mathbb{P}, G_{108}\mathbb{P}, G_{107}\mathbb{P}, \ldots$ By using the algorithm described in Lemma 5, we find that $G_{109}\mathbb{P} \cap T_1\mathbb{P} = \varnothing$, but $G_{108}\mathbb{P} \cap T_2\mathbb{P} \neq \varnothing$, since, for instance, $19 \in G_{108}\mathbb{P} \cap T_2\mathbb{P}$, and $19 + 2 \cdot 2 = 23 \in G_{110}\mathbb{P}$.

Conclusion

I tried to follow the 'natural logic' of the problem, by being more exploratory rather than artificially creative and used a computer as my permanent companion and advisor.
As to simplicity of the used methods, I recall to the point the well-known Poincaré Recurrence Theorem [7], which proof takes only a few lines of the text and uses mainly elementary set-theoretical operations. Meanwhile the significance of the Poincaré Recurrence Theorem can be hardly overestimated. Notice, by the way, that the proof of the famous Poincaré recurrence theorem is not constructive, since it does not provide a number n of iterations, after which the recurrence occurs. The Poincaré theorem states only that such number n exists. Meanwhile the statement of Theorem 1 is quite constructive since it leads to the recurrent formula (10) given above (see the calculated examples of Goldbach

set sequences and the text of R script in Appendix), which allows potentially unlimited computer calculations of consecutive nonempty Goldbach sets $\{G_k\mathbb{P} \mid 3 \le k \le m\}$ for any $m \ge 3$. This means that Strong Goldbach Conjecture holds true.

I would like to express here my acknowledgement to the peer-reviewer Dr. Dmitry Kleinbock for reading of this paper and especially for his critical and thoughtful reading of my paper [9] with the probabilistic proof of Strong Goldbach Conjecture. The spirit of friendly interaction in our numerous discussions was very crucial for me.

APPENDIX

The text of R-script for computer realization of Recursive Algorithm generating sequences of Goldbach sets $G_k\mathbb{P}$ for $k = 3,4,5,\dots,m$

```
# Function GenGS(M1,M2) recursively generates a sequence of Goldbach sets
# G(m) where M1 <= m <= M2, for an natural M1, M2 such that 5 < M1 < M2.
# Here each G(m) is calculated by using the function
# GenG(m) and the function supply:  GenGS(M1,M2) = sapply(M1:M2,GenG)
# Function GenG(m) generates sets G(m) of Goldbach primes such that
# p + p' = 2m (3 <= m <= 2m-3) for each natural m (3 <= m <= 2m-3).
# This function is based on formula (9) from Theorem 1:
# G(m) includes each p + 2t if p is a t-prime in the Goldbach set G(k)
# (3 <= k <= m-1) for t = m-k.
# Thus, G(m) is a inion of subsets tG(k) of t-primes in G(K) such that
# tG(k) = {p + 2t| p is in G(k), p + 2t is prime for each t = m - k}.
# Notice that G(m) is recursively generated from the Goldbach sets G(k),
# where 3 < k <= m-1,starting from G(3) = {3} (3+3=6).
# This is confirms non-emptiness of Goldbach sets G(m) for all natural
# m = 3,4,5,... (the Goldbach Conjucture)
# by the principle of mathematical induction.
# Needed packages: 'numbers' and 'sets'.
# Created by GMS
# Date: 06.30.21.
#
GenGS <- function (M1,M2) {
  Gen_GS <- sapply(M1:M2, GenG)
  return(Gen_GS)
}
#source('~/Documents/R/Number Theory/GenGS.R')

GenG <- function(m) {
 if (isPrime(2*m-3)){Gm <- 3 }
  else {  Gm <- NULL
  }
```

Data lists of calculated $G_k\mathbb{P}$ for $k = 3,4,5,\ldots,m$

m	Goldbach sets $G_m\mathbb{P}$ $(m = 3,4,5,\ldots,43)$
3	3
4	3 5
5	3 5 7
6	5 7
7	3 7 11
8	3 5 11 13
9	5 7 11 13
10	3 7 13 17
11	3 5 11 17 19
12	5 7 11 13 17 19
13	3 7 13 19 23
14	5 11 17 23
15	7 11 13 17 19 23
16	3 13 19 29
17	3 5 11 17 23 29 31

18	5 7 13 19 23 29 31
19	7 19 31
20	3 11 17 23 29 37
21	5 11 13 19 23 29 31 37
22	3 7 13 31 37 41
23	3 5 17 23 29 41 43
24	5 7 11 17 19 29 31 37 41 43
25	3 7 13 19 31 37 43 47
26	5 11 23 29 41 47
27	7 11 13 17 23 31 37 41 43 47
28	3 13 19 37 43 53
29	5 11 17 29 41 47 53
30	7 13 17 19 23 29 31 37 41 43 47 53
31	3 19 31 43 59
32	3 5 11 17 23 41 47 53 59 61
33	5 7 13 19 23 29 37 43 47 53 59 61
34	7 31 37 61
35	3 11 17 23 29 41 47 53 59 67
36	5 11 13 19 29 31 41 43 53 59 61 67
37	3 7 13 31 37 43 61 67 71
38	3 5 17 23 29 47 53 59 71 73
39	5 7 11 17 19 31 37 41 47 59 61 71 73
40	7 13 19 37 43 61 67 73
41	3 11 23 29 41 53 59 71 79
42	5 11 13 17 23 31 37 41 43 47 53 61 67 71 73 79
43	3 7 13 19 43 67 73 79 83

m	Goldbach sets $G_m\mathbb{P}$ $(m = 100, 101, \ldots, 128)$
100	3 7 19 37 43 61 73 97 103 127 139 157 163 181 193 197
101	3 5 11 23 29 53 71 87 101 113 149 173 179 191 197 199

102	5 7 11 13 23 31 37 41 47 53 67 73 97 101 103 107 131 137 151 157 163 167 173 191 193 197 199
103	7 13 43 67 79 97 103 109 127 139 163 193 199
104	11 17 29 41 59 71 101 107 137 149 167 179 191 197
105	11 13 17 19 29 31 37 43 47 53 59 61 71 73 79 83 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167 173 179 181 191 193 197 199
106	13 19 31 61 73 103 109 139 151 181 193 199
107	3 17 23 41 47 83 101 107 113 131 167 173 191 197 211
108	5 17 19 23 37 43 53 59 67 79 103 107 109 113 127 137 149 157 163 173 179 193 197 199 211
109	7 19 37 61 67 79 109 139 151 157 181 199 211
110	23 29 41 53 71 83 89 107 113 131 137 149 167 173 179 191 197
111	11 23 29 31 41 43 59 71 73 83 109 113 139 149 151 163 179 181 191 193 199 211
112	13 31 43 61 67 73 97 127 151 157 163 181 193 211
113	3 29 47 53 59 89 113 137 167 173 179 197 223
114	5 17 29 31 37 47 61 71 79 89 97 101 127 131 139 149 157 167 181 191 197 199 211 223
115	3 7 19 31 37 67 73 79 103 127 151 157 163 193 199 211 223 227
116	3 5 41 53 59 83 101 131 149 173 179 191 227 229
117	5 7 11 23 37 41 43 53 61 67 71 83 97 103 107 127 131 137 151 163 167 173 181 191193 197 211 223 227 229
118	3 7 13 37 43 73 79 97 109 127 139 157 163 193 199 223 229 233
119	5 11 41 47 59 71 89 101 107 131 137 149 167 179 191 197 227 233
120	7 11 13 17 29 41 43 47 59 61 67 73 83 89 101 103 109 113 127 131 137 139 151 157 167 173 179 181 193 197 199 211 223 227 229 233
121	3 13 19 31 43 61 79 103 139 163 181 199 211 223 229 239
122	3 5 11 17 47 53 71 107 113 131 137 173 191 197 227 233 239 241
123	5 7 13 17 19 23 47 53 67 73 79 83 89 97 107 109 137 139 149 157 163 167 173 179193 199 223 227 229 233 239 241

124	7 19 37 67 97 109 139 151 181 211 229 241
125	11 17 23 53 59 71 83 101 113 137 149 167 179 191 197 227 233 239
126	11 13 19 23 29 41 53 59 61 71 73 79 89 101 103 113 139 149 151 163 173 179 181 191193 199 211 223 229 233 239 241
127	3 13 31 43 61 73 97 103 127 151 157 181 193 211 223 241 251
128	5 17 23 29 59 83 89 107 149 167 173 197 227 233 239 251

References

1. Yuang Wang, *The Goldbach Conjecture*. Second Edition. World Scientific. Singapore, London., 2002
2. J. Richstein, *"Goldbach's Conjecture up to* $4 \cdot 10^{18}$. Number Theory Web, 12th November,1998.
3. V.K. Balakrishnan, *Introductory Discrete Mathematics*, Prentice-Hall International, Inc. 1991.
4. Paul Erdös, János Surányi, *Topics in the Theory of Numbers*, Springer-Verlag, 2003.
5. Richard K. Guy, *Unsolved Problems in Number Theory*. Second Edition. Springer-Verlag, 1994.
6. Song Y. Yan, *Number Theory for Computing*, Springer, 2000.
7. Harry Furstenberg, *Poincaré recurrence and Number Theory*, Bulletin of the American Mathematical Society, Volume 5, Number 3. November 1981.

8. Apostolos Doxiadis, *Uncle Petros and Goldbach's Conjecture*. Faber and Faber, London.1992,2000.

9. Gregory M. Sobko, *A Probabilistic Approach to some Additive and Multiplicative Problems of Number Theory*, 2021, **viXra:2110.0022** **pdf**

10. Martin Aigner, Günter M. Ziegler, *Proofs from The Book*, Springer-Verlag, Berlin, Heidelberg, New York, 2000.

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