An attempt to prove the Strong Goldbach Conjecture
by the Principle of Mathematical Induction

Gregory M. Sobko

Abstract

A Recursive Algorithm described here generates consecutive sequences of Goldbach sets $\left\{G_k\mathbb{P} \mid 3 \leq k \leq m\right\}$ toward the proof of the Strong Goldbach Conjecture.

Approach suggested here is based on the fundamental principle of mathematical induction and uses rather elementary set-theoretical technique. It does not involve any sophisticated powerful tools and results of contemporary Number Theory , Algebraic Geometry, or Theory of Dynamical Systems with applications to measure preserving groups of transformations on the appropriate topological spaces [7].
This work might cause beforehand certain suspicions among specialists in this area regarding the validity of the proof (perhaps inspired by the notorious 'Uncle Petros' phenomenon [8]) . The main idea of this work is to develop a recursive algorithm toward building the sequence of consecutive Goldbach sets $\left\{G_k\mathbb{P} \mid 3 \leq k \leq m\right\}$ that represent solutions to the system of Goldbach equations $\left\{x + y = 2 \cdot k \mid 3 \leq k \leq m\right\}$ in the intervals $I_k = [3, 2 \cdot k - 3]$. Validity of the algorithm is based on the proved here recurrent formula

$$\bigcup_{k=3}^{m}\left[\left(G_k\mathbb{P} + 2 \cdot (m+1-k)\right) \cap S_m\right] = G_{m+1}\mathbb{P} \neq \varnothing \ ,$$

given the inductive assumption that $G_k\mathbb{P} \neq \varnothing$ for all $k : 3 \leq k \leq m$, where $S_m = I_m \cap \mathbb{P}$, and $\mathbb{P}$ is a set of all odd prime numbers.

> *"The most interesting facts are those which can be used several times, those which have a chance of recurring ..."*
>
> (Henry Poincaré, The Value of Science)

We approach here one of old classical problems in Number Theory known as the strong form of Goldbach Conjecture(SGC) [1, 5]. According to the conjecture stated by Goldbach in his letter to Euler in 1742, "every even number $2m \geq 6$ is the sum of two odd primes" [1].

Regardless numerous attempts to prove the statement, supported in our days by computer calculations up to 4×10^{18}, it remains unproven till now.

Let $\mathbb{N}$ be a set of natural numbers, and $\mathbb{P}$ a set of odd primes (all prime numbers excluding 2).

The Goldbach's Conjecture (GC), as one of the oldest and notoriously known unsolved problems in Number theory, raises a question why it seems so difficult to decide whether the equation

$$p + p' = 2m, \tag{*}$$

where p and p' are prime numbers, has at least one solution for each even number $2m \geq 6$. Indeed, occurrences of primes look very sporadic, so that it is hard to predict, that there exists a pair of primes (p, p') related by the equation (*), especially for 'big' values of m. Notice that every solution $(n, n') = (p, p')$ in primes to the equation $p + p' = 2m$, must satisfy the condition: $(n, n') \in [3, 2m-3]^2$. We call a prime number p a G_m- prime (a *Goldbach prime*) if $p' = 2m - p$ is also a prime number. Then, denote $G_m\mathbb{P}$ as set of all G_m*- primes*, and call $G_m\mathbb{P}$ a *Goldbach set*.

Obviously, for all $m \geq 3$ we have $G_m\mathbb{P} \subset I_m = [3, 2 \cdot m - 3]$. A set $G_m\mathbb{P}$ is empty if for some $m \geq 3$ G_m- primes do not exist. Goldbach function $G(2m)$ counts the number of solutions to the equation

$$n + n' = 2m \;, \quad (n, n') \in \mathbb{P}^2 \tag{1}$$

where n and n' are prime numbers, m is any integer $m \geq 3$.

Obviously, any pair (p, p') of primes greater then 2 solves (1) for $2m = p + p'$.

Due to infinity of $\mathbb{P}$, this implies that the Goldbach function has $\limsup G(2m) = \infty$.

The Strong Goldbach Conjecture states that $\min G(2m) = 1$, so that every set $G_m\mathbb{P}$ is a nonempty set for all $m \geq 3$. Calculations show that $\max\limits_{m \leq M} G(2m)$ increases with M, though $G(2m)$ is not a monotonically increasing function.

We observe that each pair (n, n') which solves (1) must belong to a set $[3, 2m-3]^2 = I_m^2$, where $I_m = [3, 2m-3] = \{3, 4, \ldots, 2m-3\}$. Since 3 is prime, if $n' = (2 \cdot m - 3) \in \mathbb{P}$, the pair $(3, 2 \cdot n - 3)$ solves (1), so that the prime $(2 \cdot m - 3) \in G_m\mathbb{P}$, and we need to consider the case $(2 \cdot m - 3) \notin \mathbb{P}$. In general, if p is a prime number such that $(2 \cdot m - p) \in \mathbb{P}$, then $(2 \cdot m - p) \in G_m\mathbb{P}$.

Consider a map $\theta_m : I_m \to I_m$ given by the formula $\theta_m(n) = 2m - n$.

Denote $\mathcal{F}_m$ an algebra of all subsets of the interval $I_m = [3, 2m-3]$.

Obviously, θ_m is one-to-one and has an inverse θ_m^{-1}, so that for all $A \in \mathcal{F}_m$ we have $\theta_m(A) \in \mathcal{F}, \theta_m^{-1}(A) \in \mathcal{F}$. Denote $I_m^- = [3, m-1]$, $I_m^0 = \{m\}$, $I_m^+ = [m+1, 2m-3]$.

Obviously, θ_m is idempotent: $\theta_m^2 = id$ (an identical map), that is $\theta_m^{-1} = \theta_m$.

Let S_m denote a set of prime numbers in the interval of integers $I_m = [3, 2m-3]$, that is $S_m = I_m \cap \mathbb{P}$, and $S_m^c = I_m \setminus S_m$ its complement in I_m so that $I_m = S_m \cup S_m^c$, $S_m \cap S_m^c = \varnothing$. While S_m stands for the set of primes in I_m, S_m^c is the set of composite numbers in I_m.

We denote $\theta_m(S_m) = 2 \cdot m - S_m = \{n' \mid n' = 2 \cdot m - n, n \in S_m\}$.

The Strong Goldbach Conjecture asserts that for any $m \geq 3$ the set $G_m\mathbb{P}$ is not empty:

$$G_m\mathbb{P} = \left\{ n, n' \mid n \in \mathbb{P}, n' = (2m-n) \in \mathbb{P} \right\} = (2m - S_m) \cap S_m = \theta_m(S_m) \cap S_m \neq \varnothing .$$

Lemma 1.

θ_m-invariance of Goldbach sets: $\theta_m(G_m\mathbb{P}) = G_m\mathbb{P}$.

Proof.

Notice that the sets $I_m, \{m\}, \{3, 2\cdot m-3\}$ and $\theta_m(S_m) \cap S_m$ are invariant sets of the map

$\theta_m : I_m \to I_m$ since for all $n \in I_m$ we have $\theta_m\left(\{n, \theta_m(n)\}\right) = \left\{\theta_m(n), \theta_n^2(n)\right\} = \left\{\theta_m(n), n\right\}$,

due to $\theta_m^2 = id$. θ_m-invariance of $G_m\mathbb{P}$ also follows directly from the equalities

$$\theta_m(G_m\mathbb{P}) = \theta_m\left(S_m \cap \theta_m(S_m)\right) = \theta_m(S_m) \cap \theta_m^2 S_m = \theta_m(S_m) \cap S_m = G_m\mathbb{P} .$$

Q.E.D.

In what follows we need several recursively derived formulas.

Lemma 2.

(1) $I_m = I_{m-1} \cup \left\{2m-4, 2m-3\right\}$, where $I_m = [3, 2m-3]$ (2)

(2) $$S_m = S_{m-1} \cup \left(\mathbb{P} \cap \left\{2m-3\right\}\right) = \begin{cases} S_{m-1} \cup \left\{2m-3\right\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \end{cases} \quad (3)$$

(3) $$\theta_m(S_m) = \begin{cases} \theta_m\left(S_{m-1}\right) \cup \left\{3\right\} \text{ if } \left\{2m-3\right\} \in \mathbb{P} \\ \theta_m\left(S_{m-1}\right) \text{ if } \left\{2m-3\right\} \notin \mathbb{P} \end{cases} \quad (4)$$

(4) $$G_m\mathbb{P} = \begin{cases} \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cup \left\{3\right\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cap S_{m-1} \text{ if } (2m-1) \notin \mathbb{P} \end{cases} \quad (5)$$

Proof.

(1) We observe that $I_{m-1} = [3, 2\cdot(m-1)-3] = [3, 2\cdot m-5]$

so that $I_m = [3, 2m-3] = I_{m-1} \cup \left\{2m-4, 2m-3\right\}$.

(2) $\left(\{2m-4,2m-3\}\cap\mathbb{P}\right)=\left(\{2m-3\}\cap\mathbb{P}\right)$ implies

$$S_m = I_m \cap \mathbb{P} = \left(I_{m-1}\cap\mathbb{P}\right)\cup\left(\{2m-4,2m-3\}\cap\mathbb{P}\right) = S_{m-1}\cup\left(\mathbb{P}\cap\{2m-3\}\right),$$

Thus, $S_m = I_m \cap \mathbb{P} = S_{m-1}\cup\{2m-3\}$ if $(2m-3)\in\mathbb{P}$ and $S_m = S_{m-1}$ otherwise.

(3) $\theta_m(S_m)=\theta_m\left(S_{m-1}\right)\cup\theta_m\left(\{2m-3\}\cap\mathbb{P}\right)=\begin{cases}\theta_m\left(S_{m-1}\right)\cup\{3\} \text{ if } \{2m-3\}\in\mathbb{P}\\ \theta_m\left(S_{m-1}\right) \text{ if } \{2m-3\}\notin\mathbb{P}\end{cases}$,

since $\theta_m(2m-3)=2\cdot m-(2m-3)=3$.

(4) $G_m\mathbb{P}=\theta_m(S_m)\cap S_m=\left(\theta_m(S_{m-1})\cup\{3\}\right)\cap S_{m-1}=\left(\theta_m(S_{m-1})\cap S_{m-1}\right)\cup\{3\}$

if $(2m\text{-}3)\in\mathbb{P}$, and $G_m\mathbb{P}=\theta_m(S_{m-1})\cap S_{m-1}$ if $(2m-3)\notin\mathbb{P}$, that is

$$G_m\mathbb{P}=\begin{cases}\theta_m\left(S_{m-1}\right)\cap S_{m-1} \text{ if } (2m\text{-}3)\notin\mathbb{P}\\ \left(\theta_m\left(S_{m-1}\right)\cap S_{m-1}\right)\cup\{3\} \text{ if } (2m-3)\in\mathbb{P}\end{cases}$$

Q.E.D.

Notice that (4) implies that if $(2m-3)\notin\mathbb{P}$, then $S_m = S_{m-1}$ and $G_m\mathbb{P}=\theta_m(S_{m-1})\cap S_{m-1}$.

In the case when $(2\cdot m-3)\in\mathbb{P}$, we have $G_m\mathbb{P}\neq\varnothing$

since $3\in G_m\mathbb{P}$, $(2m-3)\in\mathbb{P}$ and $3+(2m-3)=2m$.

Thus, we need to concentrate on the case $(2m-3)\notin\mathbb{P}$. We observe that $\theta_m(S_{m-1})=\theta_{m-1}(S_{m-1})+2$, due to Lemma 2 below. This implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\right)\cap S_{m-1}$. If $p\in G_{m-1}\mathbb{P}\neq\varnothing$, then $p\in S_{m-1}$ and $p\in\theta_{m-1}(S_{m-1})$. Assuming now that p is a twin prime, that is $(p+2)\in S_m = S_{m-1}$, we have that $G_{m-1}\mathbb{P}\neq\varnothing$ implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\right)\cap S_{m-1}\neq\varnothing$.

See in what follows the more detailed discussion and definitions of sets of twin primes $T_1\mathbb{P}$ and t-primes $T_t\mathbb{P}$ related to the Goldbach Conjecture.

Example 1.

Values of $I_m, S_m, \theta_m\left(S_m\right), G_m\mathbb{P}, G(2m)$ for $m = 19$:

$$I_{19} = \begin{Bmatrix} 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19 \\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31\ 32\ 33\ 34\ 35 \end{Bmatrix}$$

$$\mathrm{S}_{19} = \left\{3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\right\},\ \theta_{19}\left(S_{19}\right) = \left\{7\ 9\ 15\ 19\ 21\ 25\ 27\ 31\ 33\ 35\right\}$$

$$G_{19}\mathbb{P} = \left\{7\ 19\ 31\right\},\ G(38) = 3$$

Values of $I_m, S_m, \theta_m\left(S_m\right), G_m\mathbb{P}, G(2m)$ for $m = 20$:

$$I_{20} = \begin{Bmatrix} 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20 \\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29\ 30\ 31\ 32\ 33\ 34\ 35\ 36\ 37 \end{Bmatrix}$$

$$S_{20} = \left\{3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\right\},\ \theta_{20}\left(S_{20}\right) = \left\{3\ 9\ 11\ 17\ 21\ 23\ 27\ 29\ 33\ 35\ 37\right\}$$

$$G_{20}\mathbb{P} = \left\{3\ 11\ 17\ 23\ 29\ 37\right\},\ G(40) = 6$$

The next Lemma concerns some properties of the shift transformation θ_m $(m \ge 3)$

Lemma 3.

Define a transformation $\theta_m:\ \mathbb{Z} \to \mathbb{Z}$ such that $\theta_m(n) = 2 \cdot m - n,$ where $n \in \mathbb{N},\ m \in \mathbb{N}\ (m \ge 3)$.

Then, for any subset $A \subseteq \mathbb{Z}$ and integer $t \in \mathbb{N}$ the following properties of θ_m hold true:

$$\begin{aligned} \theta_{m+t}(A) &= \theta_m(A) + 2 \cdot t \\ \theta_m(A) &= \theta_{m-t}(A) + 2 \cdot t \end{aligned} \tag{6}$$

Proof.

$$\begin{aligned} \theta_{m+t}(A) &= 2 \cdot (m+t) - A = 2 \cdot m - A + 2 \cdot t = \theta_m(A) + 2 \cdot t \\ \theta_m(A) &= 2 \cdot m - A = 2 \cdot (m-t) - A + 2 \cdot t = \theta_{m-t}(A) + 2 \cdot t \end{aligned}$$

Q.E.D.

Let $T_1\mathbb{P} = \left\{p \mid p \in \mathbb{P} \text{ and } (p+2) \in \mathbb{P}\right\}$ stand for a set of all twin primes and consider $G_k\mathbb{P} \cap T_1\mathbb{P}$ for each $k\ (3 \le k \le m)$. Thus, if for some $k\ (3 \le k \le m)$ $G_k\mathbb{P} \cap T_1\mathbb{P} \ne \varnothing,$ then $G_{k+1}\mathbb{P} \ne \varnothing$.

Lemma 3 implies, in particular, that if a prime $p \in G_k\mathbb{P}$ has a twin prime $(p+2) \in \mathbb{P}$, then $\left(p+2\right) \in G_{k+1}\mathbb{P}$. This shows some connection between the Twin Prime Conjecture and the Strong Goldbach Conjecture (SGC), and, moreover, between the t-Prime Conjecture (de Polignac Conjecture (1849)) and SGC as we see below. This also shows how nonempty Goldbach sets $G_k\mathbb{P}$ can propagate further with increasing values of k.

Definition.

Denote, in general, by $T_t\mathbb{P}$ $(t \in \mathbb{N})$ a set *of* t*-primes* for some $t \in \mathbb{N}$, that is

$$T_t\mathbb{P} = \left\{ p \mid p \in \mathbb{P} \text{ and } (p + 2 \cdot t) \in \mathbb{P} \right\}. \text{ Notice that } \bigcup_{t=0}^{\infty} T_t\mathbb{P} = T_0\mathbb{P} = \mathbb{P}, \tag{7}$$

where $\mathbb{P}$ stands for the set of all odd prime numbers.

Consider examples below:

$\left\{3,5,11,17,29,41\right\} \subset T_1\mathbb{P}$, $\left\{3,7,13,19,37,43\right\} \subset T_2\mathbb{P}$, $\left\{5,7,11,17,23,31,37,41\right\} \subset T_3\mathbb{P}$, and so on.

Propagation of nonempty $G_k\mathbb{P}$ for all $k \geq 3$ is based on very simple observations.

Lemma 4.

Let $p \in G_k\mathbb{P} \neq \varnothing$ and $p \leq k$. There exist $q \in \mathbb{P}$ $(q > p)$ and $t \in \mathbb{N}$ ($1 \leq t < k-1$) such that $q = p + 2 \cdot t \in \mathbb{P}$, and $p \in T_t\mathbb{P}$. This implies that there exists $t \in \mathbb{N}$ ($1 \leq t < k-1$) such that $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $q = (p + 2 \cdot t) \in G_{k+t}\mathbb{P} \neq \varnothing$.

Proof.

Let $p \in G_k\mathbb{P}$, $p \leq k$. Thanks to the Bertrand's postulate [4], there exists a prime q between integers k and $2 \cdot k$ $(k > 3)$. This implies that there exists $t \in \mathbb{N}$ such that this prime q can be expressed in the form $q = (p + 2 \cdot t) \in \mathbb{P}$, where $1 \leq t < k-1$.

Indeed, we can take $t = \dfrac{q-p}{2}$, so that $q = p + 2 \cdot t \in \mathbb{P}$. Then, $p \in G_k\mathbb{P}$ implies that $p + \theta_k(p) = 2 \cdot k$ and $(p + 2 \cdot t) + \theta_k(p) = 2 \cdot k + 2 \cdot t = 2 \cdot (k + t)$.

Since $q = p + 2 \cdot p \in \mathbb{P}$ and $\theta_k(p) \in G_k\mathbb{P}$, we have $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$

and $q=(p+2\cdot t)\in G_{k+t}\mathbb{P}\neq\varnothing$.

Q.E.D.

Lemma 5.

1. For $k>3$ and $p<k$ each $p\in G_k\mathbb{P}$ is a t-prime for an appropriate value of $t>0$.
2. For any $p\in G_k\mathbb{P}$ there exist $p'=\theta_k(p)\in G_k\mathbb{P}$, $q\in\mathbb{P}, q>p$, and $t\geq 1$, such that $q=p+2\cdot t$, and both p' and q belong to $G_{k+t}\mathbb{P}\neq\varnothing$.
3. Let $G_k\mathbb{P}=\left\{p_{k,1},p_{k,2},\ldots,p_{k,n(k)-1,}\ p_{k,n(k)}\right\}$ were $p_{k,i}<p_{k,i+1}$ for all $i=1,2,\ldots,n(k)-1=G(2k)-1$. Denote $t_{ki}=\dfrac{p_{k,i+1}-p_{k,i}}{2}$.

   Then, $G_k\mathbb{P}\cap T_{t_{ki}}\mathbb{P}\neq\varnothing$ and $G_{k+t_{ki}}\mathbb{P}\neq\varnothing$ for all $i=1,2,\ldots,n(k)-1=G(2k)-1$.

4. Let for each j ($3<j\leq k$) there exist $t\geq 1$ such that $G_{j-t}\mathbb{P}\cap T_t\mathbb{P}\neq\varnothing$.

   Then $G_j\mathbb{P}\neq\varnothing$ for all j ($3<j\leq k$).

5. Let $G_j\mathbb{P}\neq\varnothing$ ($j\geq 3$) and there exists $k>j$ such that $G_j\mathbb{P}\cap T_{k-j}\mathbb{P}\neq\varnothing$. Then $G_k\mathbb{P}\neq\varnothing$
6. If for $m>3$ there exist k $(3\leq k<m)$ and $t=m-k$ such that $G_k\mathbb{P}\cap T_t\mathbb{P}\neq\varnothing$,

   then $G_m\mathbb{P}\neq\varnothing$.

Proof.

1. Indeed, for $p\in G_k\mathbb{P}$ we have $p'=\theta_k(p)=2\cdot k-p\in G_k\mathbb{P}$ and $p'=p+2t$,

   where $t=\dfrac{p'-p}{2}$, so that $p\in T_t\mathbb{P}$.

2. In general, for any $p\in G_k\mathbb{P}$ there exist $q\in\mathbb{P}, q>p$, and $t=\dfrac{q-p}{2}\geq 1$, so that $q=p+2\cdot t$ and $p+\theta_j(p)+2\cdot t=p'+q=2\cdot k+2\cdot t=2\cdot(k+t)$, which means that $p'\in G_{k+t}\mathbb{P}$ and $q\in G_{k+t}\mathbb{P}$.

3. This statement follows directly from the above statement 2.

4. Due to Lemma 4, since $G_{j-t}\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ for an appropriate $t \geq 1$, we have $G_j\mathbb{P} \neq \varnothing$ for each $j \leq k$.

5. $G_j\mathbb{P} \cap T_{k-j}\mathbb{P} \neq \varnothing$ implies that there exists $p \in G_j\mathbb{P}$ and $t = k - j$ such that $q = \left(p + 2 \cdot (k-j)\right) \in \mathbb{P}$.

   Then, $p + \theta_j(p) + 2 \cdot (k-j) = \theta_j(p) + q = 2 \cdot j + 2 \cdot (k-j) = 2 \cdot k$ and both $\theta_j(p)$ and q are primes that belong to $G_k\mathbb{P} \neq \varnothing$.

6. Due to Lemma 4, $G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $t = m - k$ imply $G_{k+t}\mathbb{P} = G_{k+(m-k)}\mathbb{P} = G_m\mathbb{P} \neq \varnothing$.

Q.E.D.

This shows how nonempty Goldbach sets $G_3\mathbb{P}, G_4\mathbb{P}, G_5\mathbb{P}, \ldots G_{12}\mathbb{P}, \ldots$
have been generated:
$G_3\mathbb{P} = \{3\}, G_4\mathbb{P} = \{3, 3+2\}, G_5\mathbb{P} = \{3, 5, 5+2\}, G_6\mathbb{P} = \{5, 5+2\},$
$G_7\mathbb{P} = \{3, 7, 7+4\},\ G_8\mathbb{P} = \{3, 5, 11, 11+2\},\ G_9\mathbb{P} = \{5, 7, 11, 11+2\},$
$G_{10}\mathbb{P} = \{3, 7, 13, 13+4\}, G_{11}\mathbb{P} = \{3, 11, 17, 17+2\}, G_{12}\mathbb{P} = \{5, 7, 11, 13, 17+2\} \ldots$

Let $p \in G_k\mathbb{P}$, $p' = \theta_k(p) \in G_k\mathbb{P}$, where $p' = \theta_k(p) = 2 \cdot k - p$. Let q be prime and $q > p$ such that $q = p + 2 \cdot t$, where $t = \dfrac{q-p}{2}$. $G_k\mathbb{P} \neq \varnothing$

This implies that $q + \theta_k(p) = \left(p + 2 \cdot t\right) + \theta_k(p) = \left(p + \theta_k(p)\right) + 2 \cdot t = 2 \cdot (k+t)$.

Since both q and $\theta_k(p)$ are primes and $q + \theta_k(p) = 2 \cdot (k+t)$, we have q and $\theta_k(p)$ belong to $G_{k+t}\mathbb{P} \neq \varnothing$. For instance, if for some k $(3 \leq k \leq m)$ we have $G_{k-1}\mathbb{P} \cap T_2\mathbb{P} \neq \varnothing$, then, due to Lemma 3, $G_{k+1}\mathbb{P} \neq \varnothing$.

Consider $p = 3$ and $q = 5$, that is we start from $G_3\mathbb{P} = \{3\}$.

Then, $t = \dfrac{5-3}{2} = 1$ and $\theta_3(3) = 3$. Thus, we have $q + \theta_k(p) = 5 + 3 = 2 \cdot (3+1) = 8$ where

3 and 5 both belong to $G_{3+1}\mathbb{P} = G_4 = \{3,5\}$.

Then, due to Lemma 4, for each $G_k\mathbb{P} \neq \emptyset$ there exists $t \in \mathbb{N}$ such that $G_k\mathbb{P} \cap T_t\mathbb{P} \neq \emptyset$, which implies $G_{k+t}\mathbb{P} \neq \emptyset$. This means that the occurrence of a t-prime in a non-empty set $G_k\mathbb{P}$ implies that $G_{k+t}\mathbb{P}$ is necessarily non-empty. This provides proliferation of a non-empty sets $G_k\mathbb{P}$ t steps forward, so that $G_{k+t}\mathbb{P}$ is not empty for any k.

Starting from $k = 3$ and $t = 1$ the 'wave' of G_k-primes propagates forward recursively as $k \to \infty$ without gaps, supported by the existence of t-primes, as we demonstrate below.

Observe that each pair of primes (p,q) such that $p \in G_k\mathbb{P}$, $q > p$ and $q \in S_m = I_m \cap \mathbb{P}$, generates a nonempty set $G_{k+t}\mathbb{P}$, where $t = \frac{q-p}{2}$ and p is a t-prime in $G_k\mathbb{P}$.

Notice that there are infinitely many t-prime numbers in $\mathbb{N}$ and at least $G(2m)-1$ t-primes in each $G_m\mathbb{P}$ for $m \geq 3$. The goal of Lemma 5 is to demonstrate that we can build a nonempty Goldbach set $G_m\mathbb{P}$ for each $m \geq 3$, given a sequence of nonempty Goldbach sets $\{G_k\mathbb{P}\}_{3\leq k\leq m-1}$, due to the assumption of mathematical induction.

We need the following simple Lemmas.

Lemma 6.

Let $S_m = I_m \cap \mathbb{P}$, where $I_m = [3, 2\cdot m - 3]$.

For every prime $p \in S_m$ there is $k \leq m$ such that $p \in G_k\mathbb{P}$.

Proof.

Indeed, we can take $k = p$. Then, $p \in G_p\mathbb{P}$ since $p + p = 2\cdot p \leq 2\cdot m$.

Another possibility is to consider $k = \dfrac{3+p}{2}$, since $3 + p \le 2 \cdot m$.

Q.E.D.

Lemma 7.

For all $m \ge 3$ we have

$$S_m = \bigcup_{k=3}^{m} G_k \mathbb{P} = G^{(m)} \mathbb{P}. \qquad (8)$$

Proof.

For any $p \in S_m$, due to Lemma 1, there exists $k \le m$ such that $p \in G_k \mathbb{P}$.

And vice versa, if $p \in G^{(m)} \mathbb{P}$, that is $S_{m-1} = \left(\bigcup_{k=3}^{m-1} G_k \mathbb{P} \right)$ $p \in G_k \mathbb{P}$ for some $k \le m$, then $p \in S_m$.

Q.E.D.

Lemma 8.

Let $G_k \mathbb{P} \ne \varnothing$ for all $k: 3 \le k \le m-1$. Assuming that $2 \cdot m - 3 \in \mathbb{P}$, we have $2 \cdot m - 3 \in G_m \mathbb{P} \ne \varnothing$.

Otherwise, if $2 \cdot m - 3 \ne G_m \mathbb{P}$, we have $S_m = S_{m-1}$. Then, for any $m \ge 3$ the equality

$$\bigcup_{k=3}^{m-1} \left[\left(G_k \mathbb{P} + 2 \cdot (m-k) \right) \cap S_m \right] = G_m \mathbb{P} = \varnothing \qquad (9)$$

holds true.

Proof.

Denote $A_{k,m} = \left(G_k \mathbb{P} + 2 \cdot (m-k) \right)$.

Then, $\bigcup_{k=3}^{m-1} \left[\left(G_k \mathbb{P} + 2 \cdot (m-k) \right) \cap S_m \right] = \bigcup_{k=3}^{m-1} \left[A_{k,m} \cap S_m \right] = \left(\bigcup_{k=3}^{m-1} A_{k,m} \right) \cap S_m$.

Consider $\theta_m(A_{k,m}) = 2 \cdot m - A_{k,m} = 2 \cdot m - G_k \mathbb{P} - 2 \cdot (m-k) = 2 \cdot k - G_k \mathbb{P} = \theta_k(G_k \mathbb{P})$.

Due to Lemma 1, we have $\theta_k(G_k \mathbb{P}) = G_k \mathbb{P}$.

This implies that $\theta_m\left(\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\right)=\left(\bigcup_{k=3}^{m-1}\theta_m(A_{k,m})\right)\cap\theta_m(S_m)=\left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)\cap\theta_m(S_m)$.

According to Lemma 7, $S_{m-1}=\left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)$. Since $S_m=S_{m-1}$ if $2\cdot m-3\neq G_m\mathbb{P}$,

we have $\theta_m\left(\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\right)=S_m\cap\theta_m(S_m)=G_m\mathbb{P}\neq\varnothing$, since $\left(\bigcup_{k=3}^{m-1} A_{k,m}\right)\cap S_m\neq\varnothing$,

due to the assumption of mathematical induction.

Q.E.D.

The proof of Lemmas 5 and 8 show a definite connecion between the number of solutions to the Goldbach equation $p+p'=2\cdot m$ in the intervals $I_m=[3,2\cdot m-3]$ and the number of t-primes in sets $G_k\mathbb{P}$ for $k:3\le k<m$.

Recursive Algorithm generating the infinite sequence of nonempty Goldbach sets $G_m\mathbb{P}\neq\varnothing$ for all natural $m\ge 3$.

Given integer $m\ge 3$, this algorithm will find all pairs (p,p') of prime numbers such that $p+p'=2\cdot m$. The algorithm works recursively and generates $G_m\mathbb{P}$, starting from $G_3\mathbb{P}=\{3\}$, by using the sequence $G_3\mathbb{P},G_4\mathbb{P},G_5\mathbb{P}\dots,G_k\mathbb{P}$ up to $k=m-1$.

1. Since $G_m\mathbb{P}$ belongs to the interval of integers $I_m=[3,2\cdot m-3]$, we verify first whether $(2\cdot m-3)\in\mathbb{P}$.

If $(2\cdot m-3)\in\mathbb{P}$, then $\{3,2\cdot m-3\}\subseteq G_m\mathbb{P}$ and we set $G_3=\{3\}$. Otherwise, $G_3\mathbb{P}=\varnothing$.

2. Then we assign $t=m-k$, $tG_k=G_k+2\cdot t$ and calculate the subsets of primes $pr\_tG_k=tG_k\cap\mathbb{P}$ (all t-primesin tG_k) to form the set $G_{k+1}=G_k\cup pr\_tG_k$, by repeating these calculations in the cycle for $k=3,4,\dots,m-1$.

3. Finally, we obtain $G_m = G_m\mathbb{P}$ as a union $\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2\cdot(m-k)\right)\cap S_m\right]$.

See in APPENDIX the text of R-script and data lists of the calculated $G_k\mathbb{P}$ for $k: 3\le k\le m$.

A seqience of Goldbach sets $\left\{G_k\mathbb{P} \mid 3\le k\le m\right\}$ represents solutions to the system

of Goldbach equations $\left\{x+y=2\cdot k \mid 3\le k\le m\right\}$ in the intervals $I_k=[3,2\cdot k-3]$.

This system is an algebraic variety given by linear equations $x+y=2\cdot k$ $\left(3\le k\le m\right)$,

which solutions (if exist) are pairs of prime numbers $\left(p,p'\right)\in\mathbb{P}^2$.

Each Goldbach set $G_k\mathbb{P}$ geometrically is a sequence of points with coordinates

$\left(p,p'\right)\in\mathbb{P}^2$ on the segment of a straight line given by $x+y=2\cdot k$, $(x,y)\in[0,2\cdot k]$,

symmetrically located on the line with respect to a point (k,k), due to invariance

$\theta_k\left(G_k\mathbb{P}\right)=G_k\mathbb{P}$, where $\theta_k(x)=2k-x=y$. See below Fig. 1 and 2 representing

Diophantine geometry of Goldbach sets, where dots are points with coordinates

$\left(p,p'\right)\in\mathbb{P}^2$ on the corresponding lines.

These dots are solutions to the Goldbach equations $x+y=2\cdot k$ $\left(3\le k\le m\right)$.

The theorem below answers the question how many solutions are in each Golbach set.

Theorem 1.

A set $G_m\mathbb{P}$ of solutions to the Goldbach equation $p+p'=2\cdot m$ in primes $(p,p')\in\mathbb{P}^2$

in each inerval $I_m=[3,2\cdot m-3]$ includes a prime $p<m$ such that both p and $p'=\theta_m(p)$

are co-primes with m, that is $gcd(m,p)=(m,p)=1$, $gcd(m,\theta_m(p))=(m,\theta_m(p))=1$.

The number of solutions to the Goldbach equation $p+p'=2\cdot m$ in primes $(p,p')\in\mathbb{P}^2$

in each inerval $I_m=[3,2\cdot m-3]$ is equal to the number of t-primes in the set $G_m\mathbb{P}$

such that $t = \dfrac{p' - p}{2}$.

Proof.

Consider a quadratic polynomial $P_m(x) = x^2 + 2 \cdot m \cdot x + c$ for integer valued m, c and $x \in \mathbb{Z}$.

Let a pair of primes (p, p') be a solution to the Goldbach equation $p + p' = 2 \cdot m$ in the interval $I_m = [3, 2 \cdot m - 3]$. Obviously, for $c = p \cdot p'$, the pair of prime numbers $\left(p, p'\right) \in \mathbb{P}^2$ are roots of the polynomial $P_m(x) = x^2 - 2 \cdot m \cdot x + p \cdot p' = \left(x - p\right) \cdot \left(x - p'\right)$.

Discriminant of $P_m(x)$ is $D = 4 \cdot (m^2 - p \cdot p') = 4 \cdot t^2$, where t is a nonnegative integer.

Observe that $\left(m^2 - p \cdot p'\right) = t^2$ follows $(m - t) \cdot (m + t) = p \cdot p'$.

Since p and p' are not equal prime numbers, the equation $(m - t) \cdot (m + t) = p \cdot p'$ for an integer t and $p \le p'$ implies: $m - t = p$ and $m + t = p'$,

so that $t = \dfrac{p' - p}{2}$ and $p' = p + 2 \cdot t$. This means that $p \in G_m\mathbb{P}$ is a t-prime in $G_m\mathbb{P}$,

where $t = \dfrac{p' - p}{2}$. Therefore, we have as many solutions to the

equation $P_m(x) = x^2 + 2 \cdot m \cdot x + p \cdot p'$ as there are t-primes, $t = \dfrac{p' - p}{2}$, in the set $G_m\mathbb{P}$.

Assume now that $2 \cdot m = p + q$, $c = p \cdot q$, where $p \in \mathbb{P}$ and $q = 2 \cdot m - p$ is an unknown integer.

Then the polynomial $P_m(x)$ takes a form: $P_m(x) = x^2 + 2 \cdot m \cdot x + p \cdot (2 \cdot m - p)$.

Its discrimimant is $D_m = 4 \cdot \left(m^2 - p \cdot (2 \cdot m - p)\right) = 4 \cdot \left(m^2 - 2 \cdot m \cdot p + p^2\right) = 4 \cdot \left(m - p\right)^2$.

The solutions to the equation $P_m(x) = 0$ are $x_{1,2} = m \pm (m - p)$, where $x_1 = p, x_2 = 2 \cdot m - p$.

For instance, let $m = 9$, $c = 45$. Then, $P_9(x) = x^2 - 18x + 45$ has 2 roots:

$x_1 = 3 \in \mathbb{P}$, $x_2 = 2 \cdot 9 - 3 = 15 \notin \mathbb{P}$, and they are not included in $G_9\mathbb{P}$. Meanwhile, for $m = 9$

and $c = 65$ we have $P_9(x) = x^2 - 18x + 65$ with roots $x_1 = 5 \in \mathbb{P}$, $x_2 = 2 \cdot 9 - 5 = 13 \in \mathbb{P}$,

so that $5 \in G_9\mathbb{P}$ and $13 \in G_9\mathbb{P}$. Notice that 3, $\theta_9(3) = 15$ and $2 \cdot 9 = 18$ are not coprime numbers,

while 5, $\theta_9(5) = 13$ and $2 \cdot 9 = 18$ are coprime. Denote $[x]_p = \mathrm{mod}(x, p)$.

Then, $[P_m(x)]_p = [x]_p^2 - [2 \cdot m]_p \cdot [x]_p$.

The equation $[P_m(x)]_p = [x]_p^2 - [2 \cdot m]_p \cdot [x]_p = [x]_p \cdot \left([x]_p - [2 \cdot m]_p\right) = 0$

has the following different sets of solutions: $[x]_p = 0$ and $\left([x]_p - [2 \cdot m - p]_p\right) = 0$.

Since we are solving equation $P_m(x) = 0$ in primes within an interval $I_m = [3, 2 \cdot m - 3]$,

the solutions are: $x_1 = p \in S_m = I_m \cap \mathbb{P}$, $x_2 = \theta_m(p) = 2 \cdot m - p \in S_m$, assuming that the numbers

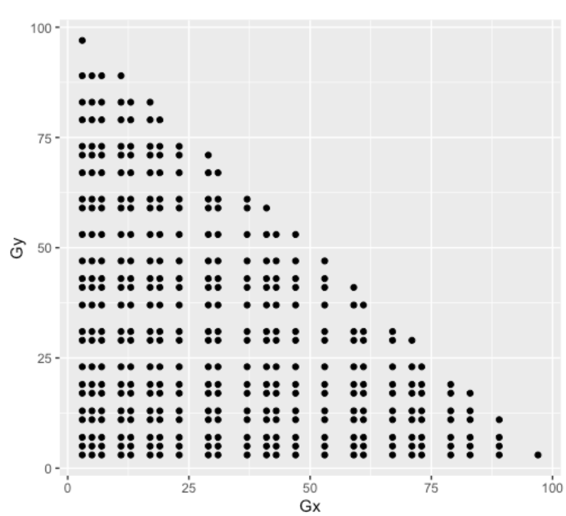
$\left(m, p\right)$ and $\left(m, \theta_m(p)\right)$ are coprime in each pair, that is, $gcd(m, p) = (m, p) = 1$

and $gcd(m, \theta_m(p)) = (m, \theta_m(p)) = 1$.

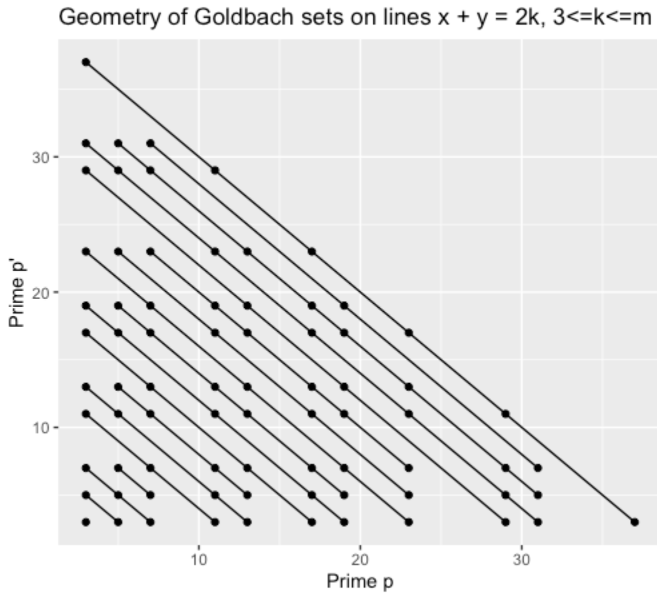
Q.E.D.

Diophantine Geometry of Goldbach Sets

$G_k\mathbb{P}$ $(k = 3, 4, \ldots, 50)$ (Fig.1)



$G_k\mathbb{P}$ $(k = 3,4,\ldots,40)$ (Fig.2)



Every dot in the above figure denotes a point with coordinates (p, p') such that

$p + p' = 2\cdot k$ on the line $x + y = 2\cdot k$, where $3 \le k \le m$.

We apply now one of the most fundamental and simple proof techniques in mathematics known as *mathematical induction* [3]. Let $\text{Prop}(m)$ denote a statement about a natural number m, and let m_0 be a fixed number. A proof that $\text{Prop}(m)$ is true for all $m \ge m_0$ by induction requires two steps:

Basis step: Verify that $\text{Prop}(m_0)$ is true.
Induction step: Assuming that $\text{Prop}(k)$ is true for all k such that $m_0 \le k \le m$, verify that $\text{Prop}(m+1)$ is true.

Theorem 2.

$\text{Prop}(m)$: For all integer $m \ge 3$, the set $G_m\mathbb{P}$ of solutions to the equation $n + n' = 2m$, $(n, n') \in \mathbb{P}^2$, in prime numbers is not empty.

The $\text{Prop}(m)$ can be equivalently stated as: $G_m\mathbb{P} = \theta_m(S_m) \cap S_m \neq \varnothing$ for all integers $m \geq 3$.

Proof.

(1) Basic step.

As we know, [2] , $\text{Prop}(m)$ is true for all m up to $M = 4 \cdot 10^{18}$.

Let $m_0 = 3$. Then $2 \cdot 3 = 6 = 3 + 3$.

(2) Induction step.

Assume that $G_k\mathbb{P} = \theta_k(S_k) \cap S_k \neq \varnothing$ for all integer $k : m_0 \leq k \leq m$.

Let $k = m+1$. In Lemma 2 we proved (3) that $S_m = \begin{cases} S_{m-1} \cup \{2m-3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \end{cases}$.

Then, we have:

$I_{m+1} = [3, 2 \cdot (m+1) - 3] = [3, 2 \cdot m - 1]$ and $S_{m+1} = \begin{cases} S_m \cup \{2m-1\} \text{ if } (2m\text{-}1) \in \mathbb{P} \\ S_m \text{ if } (2m\text{-}1) \notin \mathbb{P} \end{cases}$

In the case $(2m\text{-}1) \in \mathbb{P}$, the formula (5) implies $G_{m+1}\mathbb{P} = \left(\theta_{m+1}\left(S_m\right) \cap S_m\right) \cup \{3\} \neq \varnothing$.

We can also confirm the last equality directly, because $3 + (2m-1) = 2(m+1)$

implies that $3 \in G_{m+1}$ and $(2 \cdot m - 1) \in G_{m+1}$ if $(2 \cdot m - 1) \in \mathbb{P}$, so that $G_{m+1}\mathbb{P} \neq \varnothing$.

Consider now a general situation which includes the case $(2 \cdot m - 1) \notin \mathbb{P}$. If $(2 \cdot m - 1) \neq \mathbb{P}$,

then we have $S_{m+1} = S_m$, and due to Lemma 8, $G_{m+1}\mathbb{P} \subseteq S_{m+1} = S_m = \bigcup_{k=3}^{m} G_k\mathbb{P} = G^{(m)}\mathbb{P}$,

and we have $G_{m+1}\mathbb{P} \cap G^{(m)}\mathbb{P} \neq \varnothing$. This means that there exists $k \leq m$ such that $G_{m+1}\mathbb{P} \cap G_k\mathbb{P} \neq \varnothing$.

Therefore, $G_{m+1}\mathbb{P} \neq \varnothing$.Thus, assuming by the induction assumption, that $G_k\mathbb{P} \neq \varnothing$

for all $k \ (3 \leq k < m)$, we have $G_{m+1}\mathbb{P} \neq \varnothing$. Moreover, due to Lemma , we have a recursive formula (9) for Goldbach sets, which implies

$$\bigcup_{k=3}^{m}\left[\left(G_k\mathbb{P}+2\cdot(m+1-k)\right)\cap S_m\right]=G_{m+1}\mathbb{P}\neq\varnothing. \tag{10}$$

Q.E.D.

An example below illustrates the above statement with some computer calculations.

In this example we consider sets $G_m\mathbb{P}=\theta_m(S_m)\cap S_m$ for m from 105 to 110.

Notice that many of those sets can be calculated based on the rule that if a prime $p\in G_k\mathbb{P}$ has a twin prime $(p+2)\in\mathbb{P}$, that is $t=1$ and $p\in T_1\mathbb{P}$, then $(p+2)\in G_{k+1}\mathbb{P}$.

For example, terms in $G_{106}\mathbb{P}$ are calculated by this rule by using terms in $G_{105}\mathbb{P}$.

Meanwhile, terms in $G_{110}\mathbb{P}$ are calculated by using terms in $G_{108}\mathbb{P}$ for $t=2$ based on the general rule: if $p<k$ and $p\in G_k\mathbb{P}\cap T_t\mathbb{P}$, then $p\in\mathbb{P}$ and $p+2\cdot t\in\mathbb{P}$ implies $(p+2\cdot t)\in G_{k+t}\mathbb{P}$ (Lemma 3): $23+197=(19+2\cdot 2)+197=220=2\cdot 110$, since $19+197=216=2\cdot 108$.

The calculations below illustrate the conclusion of the Theorem

(see the data referred in Example 2). We would like to verify that $G_{110}\mathbb{P}\neq\varnothing$, by using that $G_k\mathbb{P}\neq\varnothing$ for all $k\leq 110$. Consider $G_{110}\mathbb{P}$ $(m=110,\ 2\cdot m=220)$. If we choose $t=1$ it would not work with $G_{109}\mathbb{P}$, because $G_{109}\mathbb{P}\cap T_1S_{109}=\varnothing$. We try then $G_{108}\mathbb{P}$ and $t=2$.

We have $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$ and $p=19\in G_{108}\mathbb{P}\cap T_2S_{108}$. Then, $p+2\cdot t=19+2\cdot 2=23$ should belong (due to Lemma 3) to $G_{110}\mathbb{P}$. Therefore, $2\cdot 110-23=197\in G_{110}\mathbb{P}$.

Thus, we have $23+197=2\cdot 110$, which means that $G_{110}\mathbb{P}\neq\varnothing$. Notice that in this instance $k=109$, $k+1-t=109+1-2=108$ and $(k+1-t)+t=108+2=110$ and we established that $G_{(k+1-t)+t}=G_{110}\mathbb{P}\neq\varnothing$ by using the fact that $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$.

Example 2.

Sets $G_m\mathbb{P} = \theta_m(S_m) \cap S_m$ for m from 105 to 110

$$G_{105}\mathbb{P} = \begin{Bmatrix} 11\ \ 13\ \ 17\ \ 19\ \ 29\ \ 31\ \ 37\ \ 43\ \ 47\ \ 53\ \ 59\ \ 61\ \ 71\ \ 73 \\ 79\ \ 83\ \ 97\ \ 101\ 103\ 107\ 109\ 113\ 127\ 131\ 137\ 139\ 149 \\ 151\ 157\ 163\ 167\ 173\ 179\ 181\ 191\ 193\ 197\ 199 \end{Bmatrix}$$

$$G_{106}\mathbb{P} = \{13\ \ 19\ \ 31\ \ 61\ \ 73\ 103\ 109\ 139\ 151\ 181\ 193\ 199\}$$

$$G_{107}\mathbb{P} = \{3\ \ 17\ \ 23\ \ 41\ \ 47\ \ 83\ 101\ 107\ 113\ 131\ 167\ 173\ 191\ 197\ 211\}$$

$$G_{108}\mathbb{P} = \begin{Bmatrix} 5\ \ 17\ \ 19\ \ 23\ \ 37\ \ 43\ \ 53\ \ 59\ \ 67\ \ 79\ \ 89\ 103\ 107\ 109 \\ 113\ 127\ 137\ 149\ 157\ 163\ 173\ 179\ 193\ 197\ 199\ 211 \end{Bmatrix}$$

$$G_{109}\mathbb{P} = \{7\ \ 19\ \ 37\ \ 61\ \ 67\ \ 79\ 109\ 139\ 151\ 157\ 181\ 199\ 211\}$$

$$G_{110}\mathbb{P} = \begin{Bmatrix} 23\ \ 29\ \ 41\ \ 47\ \ 53\ \ 71\ \ 83\ \ 89\ 107\ 113 \\ 131\ 137\ 149\ 167\ 173\ 179\ 191\ 197 \end{Bmatrix}$$

Thus, we can predict that $G_{110}\mathbb{P} \neq \varnothing$ without explicit calculation of this set, just by using the previously calculated sets $G_{109}\mathbb{P}, G_{108}\mathbb{P}, G_{107}\mathbb{P}, \ldots$ By using the algorithm described in Lemma 5, we find that $G_{109}\mathbb{P} \cap T_1\mathbb{P} = \varnothing$, but $G_{108}\mathbb{P} \cap T_2\mathbb{P} \neq \varnothing$, since, for instance, $19 \in G_{108}\mathbb{P} \cap T_2\mathbb{P}$, and $19 + 2 \cdot 2 = 23 \in G_{110}\mathbb{P}$.

Conclusion

I tried to follow the 'natural logic' of the problem, by being more exploratory rather than artificially creative and used a computer as my permanent companion and an advisor.

As to simplicity of the used methods, I recall to the point the well-known Poincaré Recurrence Theorem [7], which proof takes only a few lines of the text and is based mainly on elementary set-theoretical operations. Meanwhile the significance of the Poincaré Recurrence Theorem can be hardly overestimated.

Notice that the proof in Lemma 5 that there exists $t \geq 1$ such that $G_{k+1-t}\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$, which immediately implies that $G_{k+1}\mathbb{P} \neq \varnothing$ for all k $(3 \leq k \leq m)$ is not constructive, since it does not provide a formula but outlines an algorithm for finding the number t. This is a typical "existence" theorem. Notice, by the way, that the proof of the famous Poincaré recurrence theorem is not constructive as well, since it does not provide a number n of iterations, after which the recurrence occurs. The Poincaré theorem states only that such number n exists. Meanwhile the proof in Lemma 8 is quite constructive since it is based on the recursive formula (10) given above (see the calculated examples of Goldbach set sequences in the Appendix).

I would like to express here my acknowledgement to the peer-reviewer Dr. Dmitry Kleinbock for his critical and thoughtful reading of many versions of this paper, valuable advice and support. The spirit of friendly interaction in our numerous discussions was very crucial for me.

APPENDIX

The text of R-script for computer realization of Recursive Algorithm generating sequences of Goldbach sets $G_k\mathbb{P}$ for $k = 3,4,5,\ldots,m$

```
# Function GenG(m) generates sets G(m) of Goldbach primes such that p + p' =
2m (3 <= m <= 2m-3)
# for each natural m (3 <= m <= 2m-3). This function is based on Lemma3
algorithm:
# G(m) includes each p + 2t if p is a t-prime in the Goldbach set G(k) (3 <= k
<= m-1) for t = m-k.
# Thus, G(m) is a inion of subsets tG(k) of t-primes in G(K) such that
# tG(k) = {p + 2t| p is in G(k), p + 2t is prime for each t = m - k}.
# Notice that G(m) is recurrently generated from the Godbach sets G(k), where
3 <= k <= m-1,
# starting from G(3) = {3} (3+3=6). This confirms by the principle of
mathematical induction
# non-emptiness of Goldbach sets G(m) for all natural m = 3,4,5,... (the
Goldbach Conjucture).

# Needed packages: 'numbers' and 'sets'. Needed function: GmR.
# Created by GMS
# Date: 06.30.21.
#
GenG <- function(m) {
 if (isPrime(2*m-3)){
   Gm <- 3
 }
  else {
    Gm <- NULL
  }
  for (k in (3: m-1)) {
    Gk <- Gm(k)
    t <- m - k
    tGk <- Gk + 2*t
    pr_tGk <- tGk[isPrime(tGk)]
    Gm <- union(Gm, pr_tGk)
  }
  return(sort(Gm))
}

#source('~/Documents/R/Number Theory/GenG.R')
```

Data lists of calculated $G_k\mathbb{P}$ for $k = 3,4,5,\ldots,m$

m	Goldbach sets $G_m\mathbb{P}$ $(m = 3,4,5,\ldots,43)$
3	3
4	3 5
5	3 5 7
6	5 7
7	3 7 11
8	3 5 11 13
9	5 7 11 13
10	3 7 13 17
11	3 5 11 17 19
12	5 7 11 13 17 19
13	3 7 13 19 23
14	5 11 17 23
15	7 11 13 17 19 23
16	3 13 19 29
17	3 5 11 17 23 29 31
18	5 7 13 19 23 29 31
19	7 19 31
20	3 11 17 23 29 37
21	5 11 13 19 23 29 31 37
22	3 7 13 31 37 41
23	3 5 17 23 29 41 43
24	5 7 11 17 19 29 31 37 41 43
25	3 7 13 19 31 37 43 47
26	5 11 23 29 41 47
27	7 11 13 17 23 31 37 41 43 47
28	3 13 19 37 43 53
29	5 11 17 29 41 47 53
30	7 13 17 19 23 29 31 37 41 43 47 53
31	3 19 31 43 59
32	3 5 11 17 23 41 47 53 59 61
33	5 7 13 19 23 29 37 43 47 53 59 61
34	7 31 37 61
35	3 11 17 23 29 41 47 53 59 67
36	5 11 13 19 29 31 41 43 53 59 61 67
37	3 7 13 31 37 43 61 67 71
38	3 5 17 23 29 47 53 59 71 73
39	5 7 11 17 19 31 37 41 47 59 61 71 73
40	7 13 19 37 43 61 67 73
41	3 11 23 29 41 53 59 71 79
42	5 11 13 17 23 31 37 41 43 47 53 61 67 71 73 79
43	3 7 13 19 43 67 73 79 83

m	Goldbach sets $G_m\mathbb{P}$ ($m = 100,101,\ldots,128$)
100	3 7 19 37 43 61 73 97 103 127 139 157 163 181 193 197
101	3 5 11 23 29 53 71 87 101 113 149 173 179 191 197 199
102	5 7 11 13 23 31 37 41 47 53 67 73 97 101 103 107 131 137 151 157 163 167 173 191 193 197 199
103	7 13 43 67 79 97 103 109 127 139 163 193 199
104	11 17 29 41 59 71 101 107 137 149 167 179 191 197
105	11 13 17 19 29 31 37 43 47 53 59 61 71 73 79 83 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167 173 179 181 191 193 197 199
106	13 19 31 61 73 103 109 139 151 181 193 199
107	3 17 23 41 47 83 101 107 113 131 167 173 191 197 211
108	5 17 19 23 37 43 53 59 67 79 103 107 109 113 127 137 149 157 163 173 179 193 197 199 211
109	7 19 37 61 67 79 109 139 151 157 181 199 211
110	23 29 41 53 71 83 89 107 113 131 137 149 167 173 179 191 197
111	11 23 29 31 41 43 59 71 73 83 109 113 139 149 151 163 179 181 191 193 199 211
112	13 31 43 61 67 73 97 127 151 157 163 181 193 211
113	3 29 47 53 59 89 113 137 167 173 179 197 223
114	5 17 29 31 37 47 61 71 79 89 97 101 127 131 139 149 157 167 181 191 197 199 211 223
115	3 7 19 31 37 67 73 79 103 127 151 157 163 193 199 211 223 227
116	3 5 41 53 59 83 101 131 149 173 179 191 227 229
117	5 7 11 23 37 41 43 53 61 67 71 83 97 103 107 127 131 137 151 163 167 173 181 191193 197 211 223 227 229
118	3 7 13 37 43 73 79 97 109 127 139 157 163 193 199 223 229 233
119	5 11 41 47 59 71 89 101 107 131 137 149 167 179 191 197 227 233
120	7 11 13 17 29 41 43 47 59 61 67 73 83 89 101 103 109 113 127 131 137 139 151 157 167 173 179 181 193 197 199 211 223 227 229 233
121	3 13 19 31 43 61 79 103 139 163 181 199 211 223 229 239
122	3 5 11 17 47 53 71 107 113 131 137 173 191 197 227 233 239 241
123	5 7 13 17 19 23 47 53 67 73 79 83 89 97 107 109 137 139 149 157 163 167 173 179193 199 223 227 229 233 239 241
124	7 19 37 67 97 109 139 151 181 211 229 241
125	11 17 23 53 59 71 83 101 113 137 149 167 179 191 197 227 233 239
126	11 13 19 23 29 41 53 59 61 71 73 79 89 101 103 113 139 149 151 163 173 179 181 191193 199 211 223 229 233 239 241
127	3 13 31 43 61 73 97 103 127 151 157 181 193 211 223 241 251
128	5 17 23 29 59 83 89 107 149 167 173 197 227 233 239 251

References

1. Yuang Wang, *The Goldbach Conjecture*. Second Edition. World Scientific. Singapore, London., 2002
2. J. Richstein, *Goldbach's Conjecture up to* $4 \cdot 10^{18}$. Number Theory Web, 12th November,1998.
3. V.K. Balakrishnan, *Introductory Discrete Mathematics*, Prentice-Hall International, Inc. 1991.
4. Paul Erdös, János Surányi, *Topics in the Theory of Numbers*, Springer-Verlag, 2003.
5. Richard K. Guy, *Unsolved Problems in Number Theory*. Second Edition. Springer-Verlag, 1994.
6. Song Y. Yan, *Number Theory for Computing*, Springer, 2000.
7. Harry Furstenberg, *Poincaré recurrence and Number Theory*, Bulletin of the American Mathematical Society, Volume 5, Number 3. November 1981.
8. Apostolos Doxiadis, Uncle Petros and Goldbach's Conjecture.

   Faber and Faber, London.1992,2000.

July 5, 2021.
San Diego.