Enomoto's problem in Wasan geometry

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Abstract. We consider Enomoto's problem involving a chain of circles touching two parallel lines and three circles with collinear centers. Generalizing the problem, we unexpectedly get a generalization of a property of the power of a point with respect to a circle.

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Japanese mathematics developed in Edo era is called Wasan. In this note we consider a problem in Wasan geometry appeared in a sangaku, which is a framed wooden board with geometric problems written on it. The figures of the problems were beautifully drawn in color and the board was dedicated to a shrine or a temple. Today, sangaku is an iconic word for Wasan geometry. For a brief introduction of Wasan geometry, see [4]. In this note, we consider the sangaku problem proposed by Enomoto (榎本信房) in 1807 [3], which is stated as follows (see Figure 1):

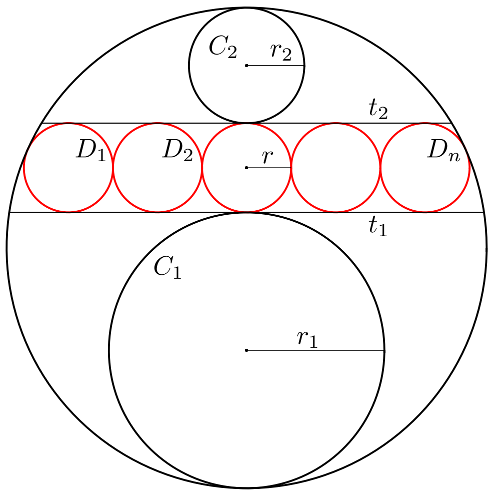

Figure 1: $4r_1r_2 = (n-1)^2r^2$.

Problem 1. Let D_1, D_2, $\cdots$, D_n be a chain of circles of radius r touching two parallel lines t_1 and t_2. A circle C_i of radius r_i touches t_i from the side opposite to t_j for $\{i, j\} = \{1, 2\}$ so that the line joining the centers of C_1 and C_2 is the perpendicular bisector of the segment joining the centers of D_1 and D_n. If a circle touches C_1, C_2, D_1 and D_n internally, then show that the following relation holds:

$$4r_1r_2 = (n-1)^2r^2. \tag{1}$$

The relation (1) shows that the product r_1r_2 is constant if the circles D_1, D_2, $\cdots$, D_n are fixed. Problem 1 is generalized as follows.

Theorem 1. *For a segment DH and a circle δ of center H, let γ be a semicircle of diameter AB for points A and B lying on the perpendicular to DH at D. If the two tangents of δ parallel to DH meet AB in points E and F so that $\overrightarrow{AB}$ and*

$\overrightarrow{EF}$ *have the same direction, then the following statements hold.*
(i) *If δ touches γ internally, then $|AE||BF| = |DH|^2$.*
(ii) *If δ touches γ externally, then $|AF||BE| = |DH|^2$.*

Proof. Assume that $r > 0$ and the points A, B, E and F have coordinates $(-r, 0)$, $(r, 0)$, $(2e, 0)$ and $(2f, 0)$, respectively, and C is the center of γ, i.e., the origin. Then D has coordinates $(e + f, 0)$, $|CD| = |e + f|$ and δ has radius $f - e$. If δ touches γ internally (see Figures 2 and 3), we get $|CH| = |r - (f - e)|$ and

$$|AE||BF| = |-r - 2e||r - 2f| = |CH|^2 - |CD|^2 = |DH|^2$$

by the right triangle CHD. This proves (i). The part (ii) is proved similarly, where we use $|CH| = |r + (f - e)|$ (see Figure 4). □

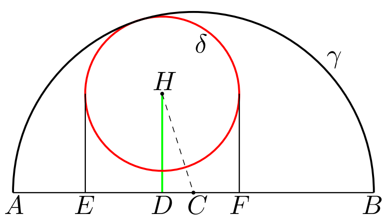

Figure 2: $|AE||BF| = |DH|^2$.

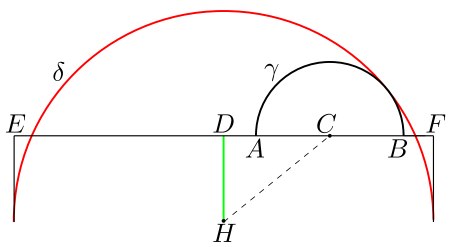

Figure 3: $|AE||BF| = |DH|^2$.

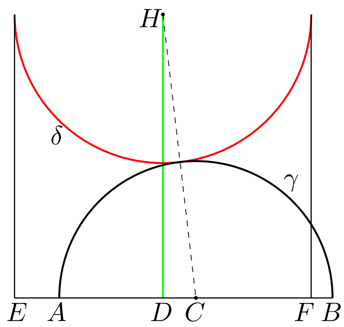

Figure 4: $|AF||BE| = |DH|^2$.

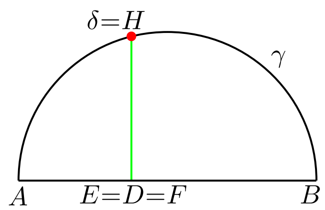

Figure 5: $|AD||BD| = |DH|^2$.

The theorem shows that the products $|AE||BF|$ and $|AF||BE|$ are constant if the segment CH and the circle δ are fixed while the points A and B vary. Problem 1 and its solution (1) are obtained if $|DH| = (n-1)r$ in (i). The solution of the same problem cited in [1] (Problem 4.9.2) and [2] (Problem 8.9.3) states $r_1 r_2 = ((2n-1)/2)^2 r^2$, which is incorrect by (1). If the circle δ degenerates to the point H, we get the relation $|AD||BD| = |DH|^2$, which shows the unsigned power of the point D with respect to the circle γ (see Figure 5). Therefore Theorem 1 is also a generalization of this relation.

References

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