Riemann Hypothesis proof using Balazard, Saias and Yor criterion

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Abstract
In this manuscript, we define a conformal map from the unit disc onto the semi plane. Later, we define a function $f(z) = (s-1)\zeta(s)$. We prove that $f(z)$ belongs to the Hardy space, $H^{\frac{1}{3}}(\mathbb{D})$. We apply Jensen's formula noting that the measure associated with the singular interior factor of f is zero. Finally, we get

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = 0$$

Keywords: Hardy spaces, Jensen's formula, Schwarz reflection principle, Critical strip, Critical line, Riemann zeta function, Riemann Hypothesis.
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1 Introduction

The Riemann zeta function, $\zeta(s)$ is defined as the analytic continuation of the Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges in the half plane $\Re(s) > 1$. The Riemann zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. All the non trivial zeros of the Riemann zeta function lie in the critical strip $0 < \Re(s) < 1$. Riemann Hypothesis states that all the non trivial zeros of the Riemann zeta function lies on the critical line $\Re(s) = \frac{1}{2}$.
Levinson [6], in 1974 proved that more than one third of zeros of Riemann zeta function are on the critical line. Balazard et al.[1] in 1999 proved an equivalent of the Riemann Hypothesis. Shaoji Feng [7], in 2012 proved that atleast 41.28 % of the zeros of Riemann zeta function are on the critical line. Pratt et al.[8] in 2020 proved that more than five-twelfths of the zeros are on the critical line.

2 Main Result

Let, $\sum_{\Re(\rho)>\frac{1}{2}}$ be the sum over the hypothetical zeros with real part greater than $\frac{1}{2}$ of the Riemann zeta function, $\zeta(s)$. In the sum, the zeros of multiplicity n are counted n times. Balazard et al.[1] proved that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = \sum_{\Re(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right| \tag{1}$$

and the Riemann Hypothesis is true if and only if [1],

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = 0 \tag{2}$$

The goal of this paper is to prove the following result.

Theorem 1: If $\zeta(s)$ denotes the Riemann zeta function then

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2}dt = 0$$

We start the proof of Theorem 1 as follows: Let, f be a function in the Hardy Space $H^p(\mathbb{D})$ where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $0 < p < \infty$. Denote by f^* the function defined almost everywhere on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ by,

$$f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$$

Let, $z \in \mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. For $i = \sqrt{-1}$, write

$$s = s(z) = \frac{1}{2} + \frac{i-z}{2(i+z)} = \frac{i}{i+z}$$

The formula $s(z)$ defines an injective, onto and conformal representation of unit disc $\mathbb{D}$ in the semi plane $\Re(s) > \frac{1}{2}$
By Jensen's Formula ([2, Theorem 3.61]) for $f(0) \neq 0$ and $r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta = \log|f(0)| + \sum_{|\alpha|<r, f(\alpha)=0} \log\frac{r}{|\alpha|} \tag{3}$$

where in the sum, $\sum_{|\alpha|<r, f(\alpha)=0}$, zeros of multiplicity n are counted n times.

Denote the singular interior factor of f by,

$$\exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\right\}$$

As $r \to 1, r < 1$, equation (3) becomes ([1] or [3, p. 68]),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^*(e^{i\theta})|d\theta = \log|f(0)| + \sum_{|\alpha|<1, f(\alpha)=0} \log\frac{1}{|\alpha|} + \int_{-\pi}^{\pi} d\mu(\theta) \tag{4}$$

Now we consider the function,

$$f(z) = (s-1)\zeta(s)$$

where $s = \frac{i}{i+z}$ then,

$$f(z) = -\frac{z}{i+z}\zeta\left(\frac{i}{i+z}\right)$$

Lemma 1.1: f belongs to the Hardy space, $H^{\frac{1}{3}}(\mathbb{D})$ that is $f \in H^{\frac{1}{3}}(\mathbb{D})$

Proof. $\zeta(s)$ has the following property [9, p.95],

$$|\zeta(s)| = \mathcal{O}(|s|), \quad |s| \to \infty, \ \Re(s) \geq \frac{1}{2}$$

If, $|z| < 1$ then $\Re\left(\frac{i}{i+z}\right) > \frac{1}{2}$ so we have,

$$|f(z)| = \left|\frac{z}{i+z}\zeta\left(\frac{i}{i+z}\right)\right| \leq \frac{c}{|i+z|^2}$$

for some positive constant c.

$$|f(re^{i\theta})| \leq \frac{c}{|ie^{-i\theta}+r|^2} \leq \frac{c}{\cos^2(\theta)}$$

$$\Rightarrow |f(re^{i\theta})|^{\frac{1}{3}} \leq \frac{c^{\frac{1}{3}}}{(\cos^2(\theta))^{\frac{1}{3}}}$$

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\frac{1}{3}} d\theta \leq c^{\frac{1}{3}} \int_{-\pi}^{\pi} \frac{d\theta}{(\cos^2(\theta))^{\frac{1}{3}}} = 2c^{\frac{1}{3}} \frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{3})} < \infty$$

where Γ denotes the Gamma function. Hence, $f \in H^{\frac{1}{3}}(\mathbb{D})$ □

Now using the above lemma we proceed to prove another lemma.

Lemma 1.2: Measure μ associated to the singular interior factor of f is zero.

Proof. To prove that the measure μ associated to the singular interior factor of f is zero, we adopt the method used by Bercovivi and Foias [10, Proposition 2.1]
Some theorems in Hardy space theory are ([4] and [5]),

Theorem (a): If $f \in H^p(\mathbb{D})$ where $p > 0$, then f has non tangential finite limit on the unit circle almost everywhere denoted by $f^*(e^{i\theta})$, and $\log|f(e^{i\theta})|$ is integrable unless $f(z) \equiv 0$. Also $f(e^{i\theta}) \in L^p$ [4, p.17, Theorem 2.2]

Theorem (b): Every function $f(z) \not\equiv 0$ in $H^p(\mathbb{D})$ $(p > 0)$ has a unique factorisation of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function which is determined by a positive singular measure μ and $F(z)$ is an outer function such that $F \in H^p(\mathbb{D})$ [4, p.24, Theorem 2.8]. Also, $|B(z)| < 1$ in $|z| < 1$ [4, p.19, Theorem 2.4].

Theorem (c): Let $f \in H^p(\mathbb{D})$, $p > 0$, and let Γ be an open arc on $\partial\mathbb{D}$. If $f(z)$ is analytic across Γ, then its inner factor and its outer factor are analytic across Γ. If $f(z)$ is continuous across Γ, then its outer factor is continuous across Γ [5, p.74, Theorem 6.3]

Theorem (d): If measure $\mu \not\equiv 0$, then there is a point $e^{i\theta}$ for which

$$\lim_{z \to e^{i\theta}} S(z) = 0$$

non tangentially [5, p.73, Theorem 6.2]
Moreover if

$$\lim_{h \to 0} \frac{\mu((\theta - h, \theta + h))}{h \log 1/h} = \infty,$$

then for every $n = 1, 2, ...$ [5, p.74, (6.4)]

$$\lim_{z \to e^{i\theta}} \frac{|S(z)|}{(1 - |z|^2)^n} = 0$$

Now, $f(z) = (s-1)\zeta(s)$ *where* $s = \frac{1}{1+z^2}$

We have proved earlier that $f \in H^{\frac{1}{3}}(\mathbb{D})$, so by Theorem (b), $f(z)$ has a decomposition

$$f(z) = B(z)S(z)F(z)$$

Define a set $M = \{z \in \mathbb{C} \mid |z| = 1, z \neq -i\}$
We know that $(s-1)\zeta(s)$ is analytic across the line $\Re(s) = \frac{1}{2}$. Since $f(z)$ is analytic across arc M, so by Theorem (c) its inner factor and outer factor are analytic across M. So, $f(z)$ is analytic across M.
By Theorem (d), if $\mu \not\equiv 0$ then $\lim_{z \to e^{i\theta}} \frac{|S(z)|}{(1-|z|^2)^n} = 0$

Lemma 1.2(a): If $\mu \not\equiv 0$ then $\lim_{r \to 1, r<1} f(-ir) = 0$

Proof.

$$f(-ir) = B(-ir)S(-ir)F(-ir)$$

$$|f(-ir)| = \left|(1 - r^2)^3 F(-ir) \frac{S(-ir)}{(1 - r^2)^3} B(-ir)\right|$$

By Theorem (b) above, $|B(-ir)| < 1$ for $r < 1$
By Theorem (d) above, $\lim_{r\to 1, r<1} \frac{|S(-ir)|}{(1-r^2)^3} = 0$
Since by Theorem (b), $F \in H^{\frac{1}{3}}(\mathbb{D})$ so we get [4, p.36, lemma],

$$|(1-r)^3 F(z)| \leq 8||F||_{\frac{1}{3}}$$

where $||F||_p = \lim_{r\to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}$ giving the following inequality,

$$|(1-r^2)^3 F(-ir)| \leq 8|(1-r)^3 F(-ir)| \leq 64||F||_{\frac{1}{3}}$$

Using these bounds in $|f(-ir)| = \left|(1-r^2)^3 F(-ir) \frac{S(-ir)}{(1-r^2)^3} B(-ir)\right|$ since the middle term goes to zero by Theorem (d) and the remaining two terms are bounded, so we get

$$\lim_{r\to 1, r<1} |f(-ir)| = 0$$

Since, $|.|$ is continuous function, we get , $|\lim_{r\to 1, r<1} f(-ir)| = 0$ so,

$$\lim_{r\to 1, r<1} f(-ir) = 0$$

In this case ,

$$\lim_{r\to 1, r<1} \frac{r}{1-r} \zeta\left(\frac{1}{1-r}\right) = 0$$

Let, $x = \frac{1}{1-r}$

$$\lim_{x\to\infty,\ x>0} (x-1)\zeta(x) = 0$$

which is a contradiction as the above limit is ∞.

Hence our assumption that $\mu \not\equiv 0$ is wrong. So, we must have

$$\mu \equiv 0 \tag{5}$$

□

We are ready for another lemma useful in applying Jensen's formula later.

Lemma 1.3:

$$\int_{-\pi}^{\pi} \log|f^*(e^{i\theta})| d\theta = 0$$

Proof. Let,

$$I = \int_{-\pi}^{\pi} \log|f^*(e^{i\theta})| d\theta$$

We have, $f(z) = (s-1)\zeta(s)$ where $s = \frac{i}{i+z}$

$$I = \int_{-\pi}^{\pi} \log\left|\frac{e^{i\theta}}{i+e^{i\theta}} \zeta\left(\frac{i}{i+e^{i\theta}}\right)\right| d\theta$$

Write,

$$I = K + L$$

where

$$K = \int_{-\pi}^{\pi} \log\left|\frac{e^{i\theta}}{i+e^{i\theta}}\right| d\theta$$

and

$$L = \int_{-\pi}^{\pi} \log\left|\zeta\left(\frac{i}{i+e^{i\theta}}\right)\right| d\theta$$

Lemma 1.3(b):

$$K = 0$$

Proof.

$$K = -\int_{-\pi}^{\pi} \log\left|i + e^{i\theta}\right| d\theta$$

By Jensen's formula, since $m(z) = i + z$ is analytic in $|z| \leq 1$ so we have $K = 0$

Lemma 1.3(c):

$$\int_{-\pi}^{\pi} \log\left|\zeta\left(\frac{i}{i + e^{i\theta}}\right)\right| d\theta = 0$$

Proof.

$$\frac{i}{i + e^{i\theta}} = \frac{1}{2} + \frac{i}{2}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$L = \int_{-\pi}^{\pi} \log\left|\zeta\left(\frac{i}{i + e^{i\theta}}\right)\right| d\theta = \int_{-\pi}^{\pi} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)\right| d\theta$$

Substitute $\phi = \frac{\pi}{4} - \frac{\theta}{2}$ then,

$$L = 2\int_{-\pi/4}^{3\pi/4} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\phi\right)\right| d\phi$$

$$L = 2\int_{-\pi/4}^{\pi/2} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\phi\right)\right| d\phi + 2\int_{\pi/2}^{3\pi/4} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\phi\right)\right| d\phi$$

Define

$$L_1 = 2\int_{-\pi/4}^{\pi/2} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\phi\right)\right| d\phi$$

and

$$L_2 = 2\int_{\pi/2}^{3\pi/4} \log\left|\zeta\left(\frac{1}{2} + \frac{i}{2}\tan\phi\right)\right| d\phi$$

In L_1, substitute $t = \frac{\tan\phi}{2}$ which is a valid substitution as $t = \frac{\tan\phi}{2}$ is injective on $(-\pi/4, \pi/2)$

$$L_1 = \int_{-1}^{\infty} \frac{\log\left|\zeta\left(\frac{1}{2} + it\right)\right|}{\frac{1}{4} + t^2} dt$$

In L_2, substitute $p = \frac{\tan\phi}{2}$ which is a valid substitution as $p = \frac{\tan\phi}{2}$ is injective on $(\pi/2, 3\pi/4)$

$$L_2 = \int_{\infty}^{-1} \frac{\log\left|\zeta\left(\frac{1}{2} + ip\right)\right|}{\frac{1}{4} + p^2} dp$$

$$L_2 = -\int_{-1}^{\infty} \frac{\log\left|\zeta\left(\frac{1}{2} + ip\right)\right|}{\frac{1}{4} + p^2} dp$$

Hence,

$$L = L_1 + L_2 = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} \log|f^*(e^{i\theta})| d\theta = 0 \tag{6}$$

□

Next, we proceed to another lemma.

Lemma 1.4: $f(z) = -\frac{z}{i+z}\zeta\left(\frac{i}{i+z}\right)$ is analytic in $|z| \leq r$, $r < 1$ and $\log|f(0)| = 0$

Proof. Let, $w(z) = \frac{i}{i+z}$ and define

$$h(z) = (z-1)\zeta(z)$$

$h(z)$ is entire function and $w(z)$ is analytic in $|z| \leq r$, $r < 1$ so the composition $h(w(z)) = f(z)$ is analytic in $|z| \leq r$, $r < 1$. Hence, f is continuous at zero.

$$f(0) = \lim_{z \to 0} f(z)$$

$$f(0) = \lim_{z \to 0} \frac{-z}{i+z}\zeta\left(\frac{i}{i+z}\right)$$

Let, $\eta = \frac{i}{i+z}$ then

$$f(0) = \lim_{\eta \to 1} (\eta - 1)\zeta(\eta) = 1 \tag{7}$$

$$\log|f(0)| = 0 \tag{8}$$

□

Now, we proceed to next lemma.

Since, $f(0) \neq 0$ and $f(z) = -\frac{z}{i+z}\zeta\left(\frac{i}{i+z}\right)$ so $f(\alpha) = 0$ corresponds to $\zeta\left(\frac{i}{i+\alpha}\right) = 0$.
Let, ρ denote non trivial zeros of Riemann zeta function then,

$$\rho = \frac{i}{i+\alpha}$$

Lemma 1.8:

$$\sum_{|\alpha|<1, f(\alpha)=0} \log\frac{1}{|\alpha|} = \sum_{\Re(\rho)>\frac{1}{2}, \zeta(\rho)=0} \log\left|\frac{\rho}{1-\rho}\right|$$

Proof. $\rho = \frac{i}{i+\alpha}$ gives $\alpha = i\left(\frac{1-\rho}{\rho}\right)$ so $|\alpha| < 1$ corresponds to $\Re(\rho) > \frac{1}{2}$ and $f(\alpha) = 0$ corresponds to $\zeta(\rho) = 0$. Hence we get

$$\sum_{|\alpha|<1, f(\alpha)=0} \log\frac{1}{|\alpha|} = \sum_{\Re(\rho)>\frac{1}{2}, \zeta(\rho)=0} \log\left|\frac{\rho}{1-\rho}\right| \tag{9}$$

□

Using equation (5),(6),(8) and (9) in equation (4), we get,

$$\sum_{\Re(\rho)>\frac{1}{2}, \zeta(\rho)=0} \log\left|\frac{\rho}{1-\rho}\right| = 0 \tag{10}$$

Using equation (1) and (10) gives,

$$\int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{1}{2}+it)|}{\frac{1}{4}+t^2} dt = 0$$

This proves equation (2) and completes the proof of Theorem 1. Hence the Riemann Hypothesis is true.

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