42 Infinity and numerical magic

(Chapter of the book *Infinity Put to the Test* by A. León, available HERE)

Abstract.-This chapter analyzes a supertask that makes it disappear numbers from a table that contains the list of natural numbers ordered in their natural order of precedence (ω-order).

Keywords: supertask, w-order, actual infinity, potential infinity.

Making disappear a number

P1 As we will see in this chapter, it is possible to make disappear a number from a list of numbers if the list is ω-ordered, and the number in question successively exchanges its current position in the list with the number in the next position in the list, while a number in the next position in the list exists to exchange its position. This absurdity is an inevitable consequence of assuming that ω-ordered lists exist as complete totalities, even without a last element completing the corresponding list. It will also be proved these conflicting disappearances do not happen in potentially infinite lists.

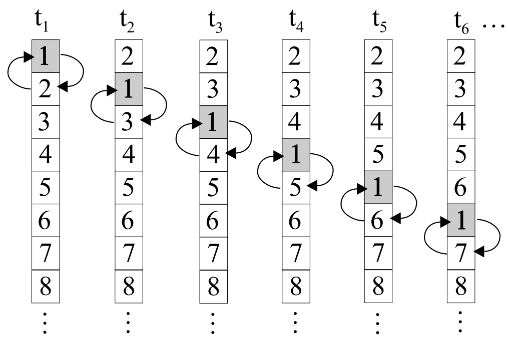

Figura 42.1 – $\langle E_{1,i} \rangle$ exchanges through the ω-ordered list of the natural numbers.

P2 Consider the ω-ordered list of all natural numbers: $\mathbb{N} = 1, 2, 3, \ldots$, and let $\langle r_i \rangle$ be the ω-ordered sequence of the rows of a table T such that $r_i = i, \forall i \in \mathbb{N}$. Assume now we exchange the number 1 with the number 2; and then the number 1 with the number 3; and then the number 1 with the number 4; and so on (Figure 42.1). In symbols:

$$E_{1,n}\begin{cases} r_n = n+1 \\ r_{n+1} = 1 \end{cases} \qquad n = 1, 2, 3, \ldots \tag{1}$$

where $E_{1,n}$ represents the exchange between the number 1 in the row r_n of T and the number $n+1$ in the row r_{n+1} of T. The purpose of the next discussion is to examine the destination of the number 1 once all possible exchanges $\langle E_{1,i}\rangle$ defined by (1) have been carried out (Principle of Execution).

P3 It is immediate to prove that for each natural number v the first v exchanges $\langle E_{1,i}\rangle_{i=1,2\ldots v}$ can be carried out. In fact, it is clear $E_{1,1}$ can be carried out because it places the number 1 in r_2 and the number 2 in r_1. Assume that, being n any natural number, the first n exchanges $\langle E_{1,i}\rangle_{i=1,2\ldots n}$ can be performed. Once performed, the number 1 will be placed in r_{n+1} and the number n+1 in r_n. Consequently, $E_{1,n+1}$ can also be performed because it places 1 in r_{n+2} and the number $n+2$ in r_{n+1}. Thus, $E_{1,1}$ can be performed, and if for any natural number n the first n exchanges $\langle E_{1,i}\rangle_{i=1,2\ldots n}$ can be performed, then the first $\langle E_{1,i}\rangle_{i=1,2\ldots(n+1)}$ exchanges can also be performed. This inductive reasoning proves that for each natural number v the first v exchanges $\langle E_{1,i}\rangle_{i=1\ldots v}$ can be carried out. We will examine the consequences of this conclusion in the following two sections by means of two independent arguments

Supertask argument

P4 Supertask theory assumes the possibility to perform infinitely many actions in a finite interval of time (see [1] for background details and Chapters 23 and 17 of this book). The short discussion that follows analyzes this assumption by mean of a supertask whose successive tasks consist just in performing the successive exchanges $\langle E_{1,i}\rangle$ defined by (1). As a consequence of those successive exchanges, the number 1, originally placed in the first row of T, will be successively placed in the 2nd, 3rd, 4th... row of T.

P5 Let $\langle t_n\rangle$ be a strictly increasing and ω-ordered sequence of instants within the real interval (t_a, t_b) whose limit is t_b. Assume each possible exchange $E_{1,i}$ is performed at the precise instants t_i of $\langle t_n\rangle$. Being t_b the limit of $\langle t_i\rangle$, the one to one correspondence between $\langle t_i\rangle$ and $\langle E_{1,i}\rangle$ defined by $f(t_i) = E_{1,i}$, proves that at the instant t_b all possible exchanges $\langle E_{1,i}\rangle$ will have been carried out (Principle of Execution P25). The problem is: in which row will be placed the number 1 at t_b?

P6 Let r_v be any row of T. Since $E_{1,v}$ places the number 1 in the row r_{v+1}, if the number 1 were in the row r_v then the first v exchanges $\langle E_{1,i}\rangle_{i=1,2,\dots v}$ would not have been carried out, which according to P3 is impossible. Thus, and being, r_v any row of T, we must conclude that at the instant t_b the number 1 has disappeared from the table. While all numbers greater than 1 remain in T, each number $n > 1$ in r_{n-1}, the number 1 has magically disappeared from T.

P7 It is worth noting the conclusion on the disappearance of the number 1 has not been deduced from the successively performed exchanges $\langle E_{1,i}\rangle$. We have simply proved that once all possible exchanges $\langle E_{1,i}\rangle$ have been carried out (Principle of Execution P25), the number 1 cannot be in any row of T, otherwise it would have to be in a certain row r_v, whatsoever it be, and then the first v exchanges $\langle E_{1,i}\rangle_{i=1,2,\dots v}$ would not have been carried out, which goes against P3.

P8 And note again, the above conclusion is not a question of indeterminacy regarding the row of T occupied by the number 1 once all possible exchanges $\langle E_{1,i}\rangle$ have bee carried out, it is a question of an actual disappearance: once all possible exchanges $\langle E_{1,i}\rangle$ have been carried out (Principle of Execution P25), the set of possible rows of T where the number 1 could be is just the empty set. In line with other arguments in this book, it is immediate the number 1 disappear from T just at t_b, an instant at which the number 1 is no longer exchanged. This is, in fact, infinitist magic. The problem is that magic is not compatible with formal sciences.

Modus Tollens argument

P9 Consider the following two propositions regarding the execution of all possible exchanges $\langle E_{1,i}\rangle$:

- p: Once performed all possible exchanges $\langle E_{1,i}\rangle$, the number 1 remains in T.
- q: Once performed all possible exchanges $\langle E_{1,i}\rangle$, the number 1 is in a certain row r_v of T.

It is quite clear that $p \Rightarrow q$ because if once performed all possible exchanges $\langle E_{1,i}\rangle$ the number 1 remains in T, then it must be in one of its rows r_v, whatever it be.

P10 We will prove now q is false. Let r_v be any row of T. If once performed all possible exchanges $\langle E_{1,i}\rangle$ the number 1 is in r_v then $E_{1,v}$ has not been carried out. But this is false because:

1) The index v in $E_{1,v}$ is a natural number.
2) According to P3, for each natural number v, it is possible to carry out the first v exchanges $\langle E_{1,i}\rangle_{i=1,2\ldots v}$.
3) All possible exchanges $\langle E_{1,i}\rangle$ have been carried out.
4) At least the first v exchanges $\langle E_{1,i}\rangle_{i=1,2\ldots v}(1)$ have been carried out.
5) $E_{1,v}$ placed the number 1 in r_{v+1}.

In consequence the number 1 is not in r_v. Therefore, and being r_v any row, we must conclude q is false.

P11 Therefore, we can write:

$$p \Rightarrow q \tag{2}$$
$$\neg q \tag{3}$$
$$\therefore \neg p \tag{4}$$

which means that once performed all possible exchanges $\langle E_{1,i}\rangle$ (Principle of Execution P25), the number 1 is no longer in the table T.

P12 Evidently, the above arguments on the disappearance of the number 1 could be applied to any other number of T. Moreover, it could be applied simultaneously to any number of numbers of T. For example, all odd (or even) numbers can disappear simultaneously from T by a sequence of exchanges similar to the above one. The reader will certainly be able to define it.

The potential infinity alternative

P13 We will end this chapter by analyzing the problem of $\langle E_{1,i}\rangle$ exchanges from the point of view of the potential infinity. From this point of view only finite totalities make sense, as large as wished but always finite. Consider, then, any finite number n and the table T_n of the first n natural numbers. $\langle E_{1,i}\rangle$ will be now defined by:

$$E_{1,i}\begin{cases} r_i = i+1 \\ r_{i+1} = 1 \end{cases} \qquad i = 1,2,3,\ldots n-1 \tag{5}$$

and then, only a finite number $n-1$ of exchanges $\langle E_{1,i}\rangle_{i=1,2,\ldots(n-1)}$ can be carried out, at the end of which the number 1 will be placed in the last row r_n of T_n.

P14 Thus, for any given natural number n the exchanges (5) in T_n are consistent. Only when they take place in the assumed complete lists T of all natural numbers they become inconsistent. In symbols:

$$E_{1,i} \begin{cases} r_i = i + 1 \\ r_{i+1} = 1 \end{cases} \quad i = 1, 2, 3, \dots n - 1 \tag{6}$$

is consistent for all $n \in \mathbb{N}$, while:

$$E_{1,i} \begin{cases} r_i = i + 1 \\ r_{i+1} = 1 \end{cases} \quad i = 1, 2, 3, \dots \tag{7}$$

is inconsistent.

Chapter References

[1] Jon Pérez Laraudogoitia, *Supertasks*, The Stanford Encyclopaedia of Philosophy (E. N. Zaltax, ed.), Standford University, URL = http://plato.stanford.edu, 2001.