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Pythagorean Common Prime Factor Conjecture

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Abstract

In this paper, the author proposes and proves a conjecture to be called the Pythagorean common prime factor conjecture, This conjecture states that if $A^2 + B^2 = C^2$, where A, B, and C are positive integers, then A, B and C may have a common prime factor. The approach used in this paper is exactly the same as the approach used in proving the Beal conjecture (viXra:2012.0120 & viXra:2104.0098). To prove the Pythagorean common prime factor conjecture, one would let r, s and t be prime factors of A, B and C, respectively, such that $A = Dr$, $B = Es$, and $C = Ft$, where D, E and F are positive integers. Then, the equation $A^2 + B^2 = C^2$ becomes $D^2r^2 + E^2s^2 = F^2t^2$. The proof would be complete after proving that $r^2 = t^2$ and $s^2 = t^2$, which would imply that $r = s = t$. The proofs of the above equalities would also involve showing that the ratio, $(r^2/t^2) = 1$ and the ratio, $(s^2/t^2) = 1$. Of the two numerical examples, $3^2 + 4^2 = 5^2$, and $6^2 + 8^2 = 10^2$, of the Pythagorean equation, the three terms of the first equation have no common prime factor; but the terms of the second equation have the common prime factor, 2. Perhaps, if there had been a Pythagorean common prime factor conjecture and its proof 24 years ago, Beal conjecture would have been proved 23 years ago. The main principle for obtaining relationships between the prime factors on the left side of the equation and the prime factor on the right side of the equation is that the power of each prime factor on the left side of the equation equals the same power of the prime factor on the right side of the equation. High school students can learn and prove this conjecture for a bonus question on a final class exam.

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Option 1
Introduction

Two numerical examples of the Pythagorean equation are $3^2+4^2=5^2$, and $6^2+8^2=10^2$. The three terms of the first equation have no common prime factor; but the terms of the second equation have the common prime factor, 2. To prove the Pythagorean common prime factor conjecture, one will let r, s and t be prime factors of A, B and C, respectively, such that $A=Dr$, $B=Es$, and $C=Ft$, where D, E and F are positive integers, Then, the equation, $A^2+B^2=C^2$ becomes $D^2r^2+E^2s^2=F^2t^2$. The proof would be complete after showing that $r=s=t$. Since one would like to prove equalities from the equation, $D^2r^2+E^2s^2=F^2t^2$, one will need equalities between the powers of the prime factors on the left side of the equation and the power of the prime factor on the right side of the equation. Two approaches will be covered in finding these equalities.

Approach 1: Common Sense Approach

One can identify the powers involved with respect to the prime factors, r, s, t, as r^2, s^2, and t^2. and one would like to have equalities involving r^2, t^2, and s^2, The possible equalities between the powers of the prime factors on the left side and the power of the prime factor on the right side of the equation, $D^2r^2+E^2s^2=F^2t^2$, are $r^2=t^2$, and $s^2=t^2$, which on inspection, would lead to the conclusion, $r=t$, $s=t$, and $r=s=t$.Therefore, one conjectures the equalities, $r^2=t^2$ and $s^2=t^2$. These conjectures will be proved in the Pythagorean common prime factor conjecture proof. . Two main steps are involved in the proof. In the first step, one will determine how r and t are related, and in the second step, one will determine how s and t are related.

Approach 2: Factorization Approach

In approach 2, one would be guided by the properties of factored numerical Pythagorean equations.

Illustration of the equalities $r^2=t^2$ and $s^2=t^2$ of factored Pythagorean equations

For the factorization with respect to r^2:

$\boxed{r^2=t^2}$ $\quad D^2r^2+E^2s^2=F^2t^2$

$$\underbrace{r^2}_{K}[\underbrace{D^2+E^2s^2\bullet r^{-2}}_{L}]=\underbrace{t^2}_{M}\underbrace{t^{2-2}F^2}_{P} \qquad (K=M)$$

Example 1:

$$6^2+8^2=10^2$$
$$2^2\bullet 3^2+2^6=2^2\bullet 5^2$$
$$2^2(3^2+2^4)=2^2\bullet 5^2$$
$$\underbrace{2^2}_{k}(\underbrace{3^2+2^4}_{L})=\underbrace{2^2}_{M}\bullet\underbrace{5^2}_{P}$$

For the factorization with respect to s^2:

$\boxed{s^2=t^2}$ $\quad D^2r^2+E^2s^2=F^2t^2$

$$\underbrace{s^2}_{K}[\underbrace{E^2+D^2r^2\bullet s^{-2}}_{L}]=\underbrace{t^2}_{M}\underbrace{t^{2-2}F^2}_{P} \qquad (K=M)$$

Example 2:

$$18^2+24^2=30^2$$
$$3^4\bullet 2^2+2^6\bullet 3^2=3^2\bullet 2^2\bullet 5^2$$
$$3^2(3^2\bullet 2^2+2^6)=3^2\bullet 2^2\bullet 5^2$$
$$\underbrace{3^2}_{k}(\underbrace{3^2\bullet 2^2+2^6}_{L})=\underbrace{3^2}_{M}\bullet\underbrace{2^2\bullet 5^2}_{P}$$

From either Approach 1 or Approach 2, one will next prove the equalities

$\boxed{r^2=t^2}$ and $\boxed{s^2=t^2}$, and deduce $\boxed{r=s=t}$.

 "The simplest solution is usually the best solution"---Albert Einstein

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Option 2
Pythagorean Common Prime Factor Conjecture Proof

Given: $A^2 + B^2 = C^2$, A, B, and C are positive integers .

Required: To prove that A, B and C **may** have a common prime factor.

Plan: Let r, s and t be prime factors of A, B and C, respectively, such that $A = Dr$, $B = Es$, and $C = Ft$, where D, E and F are positive integers, Then, the equation $A^2 + B^2 = C^2$ becomes $D^2r^2 + E^2s^2 = F^2t^2$. The proof would be complete after showing that $r = s = t$. Two conjectured equalities, $r^2 = t^2$ and $s^2 = t^2$, which would imply that $r = s = t$, will be proved. More formally, $r^2 = t^2$ if and only if $(r^2/t^2) = 1$; and $s^2 = t^2$ if and only if $(s^2/t^2) = 1$.

Proof

Step 1: The conjectured equality, $r^2 = t^2$ would be true if and only if $(r^2/t^2) = 1$. The above **biconditional** statement $r^2 = t^2$ if and only if $(r^2/t^2) = 1$. would be split up into two **conditional** statements as follows: 1. If $r^2 = t^2$, then $(r^2/t^2) = 1$ and 2. If $(r^2/t^2) = 1$, then $r^2 = t^2$. For the first statement, one will, assume that $r^2 = t^2$, and show that $(r^2/t^2) = 1$. For the second statement, one will assume that $(r^2/t^2) = 1$, and show that $r^2 = t^2$. After showing that both statements are true, one would have proved that $r^2 = t^2$ if and only if $(r^2/t^2) = 1$

Begin: $D^2r^2 + E^2s^2 = F^2t^2$ (1)

$$\frac{D^2r^2 + E^2s^2}{F^2t^2} = 1 \qquad (2)$$

(Dividing both sides by F^2t^2) (2)

A t^{-2} factor is needed on the right side of equation (1) Therefore, in equation 1, let $t^2 = t^2t^{2-2}$ to obtain $D^2r^2 + E^2s^2 = t^2t^{2-2}F^2$

$D^2r^2 + E^2s^2 = r^2t^{2-2}F^2$

(Replacing t^2 by r^2 ..The hypothesis of the first conditional statement is $r^2 = t^2$)

$D^2r^2 + E^2s^2 = r^2t^{-2}F^2t^2$ (Splitting t^{2-2})

$D^2r^2 + E^2s^2 = \frac{r^2}{t^2}F^2t^2$ (Positive exponents only)

Step 1 continued on next page

Step 2: The second conjectured equality, $s^2 = t^2$,would be true if and only if $(s^2/t^2) = 1$. This biconditional statement will similarly be proved as in Step 1, above. One will split up this biconditional statement into two **conditional** statements, as follows: 1. If $s^2 = t^2$, then $(s^2/t^2) = 1$, and 2. If $(s^2/t^2) = 1$, then $s^2 = t^2$. For the first statement, one will, assume that $s^2 = t^2$, and show that $(s^2/t^2) = 1$. For the second statement,,, one will assume that $(s^2/t^2) = 1$, and show that $s^2 = t^2$. After showing that both conditional statements are true, one would have proved that $s^2 = t^2$ if and only if $(s^2/t^2) = 1$.

Begin: $D^2r^2 + E^2s^2 = F^2t^2$ (1)

A t^{-2} factor is needed on the right side of equation (1).Therefore, in equation 1, let $t^2 = t^2t^{2-2}$ to obtain $D^2r^2 + E^2s^2 = t^2t^{2-2}F^2$

$D^2r^2 + E^2s^2 = s^2t^{2-2}F^2$

(Replacing t^2 by s^2. (The hypothesis of the first conditional statement in Step 2 is $s^2 = t^2$))

$D^2r^2 + E^2s^2 = s^2t^{-2}F^2t^2$ (Splitting t^{2-2})

$D^2r^2 + E^2s^2 = \frac{s^2}{t^2}F^2t^2$ (Positive exponents) only)

$\frac{D^2r^2 + E^2s^2}{F^2t^2} = \frac{s^2}{t^2}$ (Solving for $\frac{s^2}{t^2}$)

Step 2 continued on next page

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Step 1 continued

$$\frac{D^2r^2 + E^2s^2}{F^2t^2} = \frac{r^2}{t^2} \text{ (Solving for } \frac{r^2}{t^2}\text{)}$$

$$1 = \frac{r^2}{t^2} \quad \text{(From (2), } \frac{D^2r^2 + E^2s^2}{F^2t^2} = 1\text{)}$$

Therefore, if $r^2 = t^2$, $\frac{r^2}{t^2} = 1$; and one has shown that the first conditional statement is true. Now, one will show that the second conditional statement is also true,

Since $\frac{r^2}{t^2} = 1$,

(Hypothesis of the second conditional statement)

$$\frac{r^2}{t^2} = \frac{D^2r^2 + E^2s^2}{F^2t^2}$$

$\left(\frac{D^2r^2 + E^2s^2}{F^2t^2} = 1\right.$, from (2) above)

$r^2F^2t^2 = t^2(D^2r^2 + E^2s^2)$ (cross-multiplying)

$r^2 = t^2$ (Divide left side by F^2t^2 and right side by $D^2r^2 + E^2s^2$, since $D^2r^2 + E^2s^2 = F^2t^2$)

Therefore, if $\frac{r^2}{t^2} = 1$, $r^2 = t^2$, and one has shown that the second conditional statement is true, Since the two **conditional** statements, above, have been proved, the **biconditional** statement, $r^2 = t^2$ if and only if $(r^2/t^2) = 1$. has been proved.

Continuing, if $r^2 = t^2$, $r = t$.

($\log r^2 = \log t^2$; $2\log r = 2\log t$; $\log r = \log t$; $r = t$)

Step 2 continued

$$1 = \frac{s^2}{t^2} \quad \text{(From (2) Step 1, } \frac{D^2r^2 + E^2s^2}{F^2t^2} = 1\text{)}$$

Therefore, if $s^2 = t^2$, $(s^2/t^2) = 1$; and one has shown that the first conditional statement of Step 2 is true. Now, one will show that the second conditional statement of Step 2 is also true, Since $\frac{s^2}{t^2} = 1$,

(Hypothesis of the second statement in Step 2))

$$\frac{s^2}{t^2} = \frac{D^2r^2 + E^2s^2}{F^2t^2} \quad \left(\frac{D^2r^2 + E^2s^2}{F^2t^2} = 1 \text{ from Step 1}\right)$$

$s^2F^2t^2 = t^2(D^2r^2 + E^2s^2)$ (cross-multiplying)

$s^2 = t^2$ (Divide left side by F^2t^2 and right side by $D^2r^2 + E^2s^2$, since $D^2r^2 + E^2s^2 = F^2t^2$)

Therefore, if $(s^2/t^2) = 1$, $s^2 = t^2$, and the second conditional statement of Step 2 is true.. Since the two **conditional** statements in Step 2, above, have been proved, the **biconditional** statement, $s^2 = t^2$ if and only if $(s^2/t^2) = 1$. has been proved.

Continuing, if $s^2 = t^2$, $s = t$.

($\log s^2 = \log t^2$; $2\log s = 2\log t$; $\log s = \log t$; $s = t$)

Step 3: It has been shown in Step 1 that $r = t$, and in Step 2 that $s = t$; therefore, $r = s = t$.

Since $A = Dr$, $B = Es$, $C = Ft$ and $r = s = t$, A, B and C may have a common prime factor, and the proof is complete.

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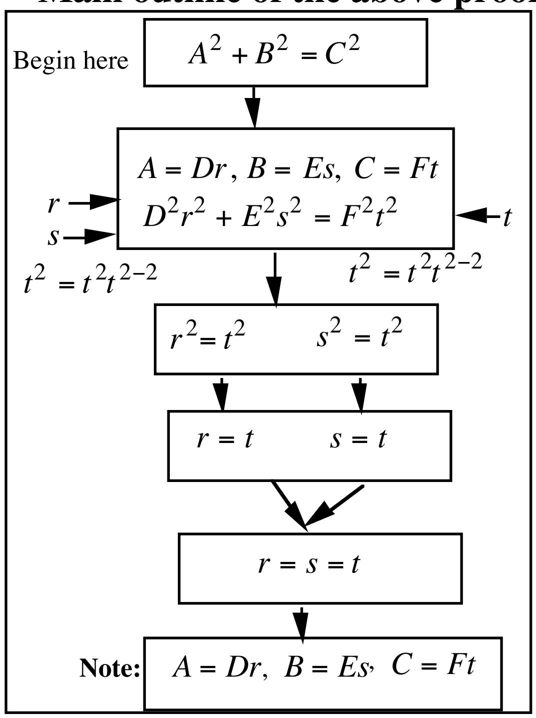
Discussion

The above proof is beautiful mathematics because of the symmetric structure of the proof, One can observe that Step 2 could be viewed as a duplication of Step 1 with r^2 replaced by s^2. The beauty continues when $r^2 = t^2$ and $s^2 = t^2$ imply that $r = t$ and $s = t$, respectively, resulting in the conclusion, $r = s = t$,

Main outline of the above proof



Option 3

Conclusion

The author has surely proved the Pythagorean common prime factor conjecture that if $A^2 + B^2 = C^2$, where A, B, and C are positive integers, then A, B and C may have a common prime factor. The proof was based on the two equalities, $r^2 = t^2$ and $s^2 = t^2$. which were conjectured and proved. These equalities were conjectured using common sense as well as the factorization properties of the factored numerical Pythagorean equations. From these. equalities, it was concluded that $r = s = t$, (where r, s and t are prime factors of A, B and C, respectively), establishing the truthfulness of the Pythagorean common prime factor conjecture that A, B and C may have a common prime factor, The approach used in this paper is exactly the same as the approach used in proving the Beal conjecture (viXra:2012.0120 & viXra:2104.0098). Perhaps, if there had been a Pythagorean common prime factor conjecture and its proof 24 years ago, Beal conjecture would have been proved 23 years ago.

High school students can learn and prove this conjecture as a bonus question on a final class exam.

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