PROOF OF THREE CONJECTURES: BEAL'S CONJECTURE, RIEMANN HYPOTHESIS AND *ABC*
VERSION 1.0 - JUNE 2021

Abdelmajid Ben Hadj Salem, Ingénieur Général

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- The Beal's conjecture.
- The Riemann Hypothesis.
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We give in detail all the proofs.

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Nous donnons les détails des différentes démonstrations.

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Figure 1. Photo of the Author (2011)

To the memory of my Parents, to my wife Wahida, my daughter Sinda and my son Mohamed Mazen

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Résumé. — Cette monographie présente les preuves de 3 conjectures importantes dans le domaine de la théorie des nombres à savoir:
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CONTENTS

CHAPTER 1

A COMPLETE PROOF OF BEAL'S CONJECTURE

Abstract. — In 1997, Andrew Beal announced the following conjecture: *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If $A^m + B^n = C^l$ then $A, B,$ and C have a common factor.* We begin to construct the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ with p, q integers depending of A^m, B^n and C^l. We resolve $x^3 - px + q = 0$ and we obtain the three roots x_1, x_2, x_3 as functions of p, q and a parameter θ. Since $A^m, B^n, -C^l$ are the only roots of $x^3 - px + q = 0$, we discuss the conditions that x_1, x_2, x_3 are integers and have or not a common factor. Three numerical examples are given.

Résumé. — En 1997, Andrew Beal avait annoncé la conjecture suivante: *Soient $A, B, C, m, n,$ et l des entiers positifs avec $m, n, l > 2$. Si $A^m + B^n = C^l$ alors $A, B,$ et C ont un facteur commun.*
Nous commençons par construire le polynôme $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$ avec p, q des entiers qui dépendent de A^m, B^n et C^l. Nous résolvons $x^3 - px + q = 0$ et nous obtenons les trois racines x_1, x_2, x_3 comme fonctions de p, q et d'un paramètre θ. Comme $A^m, B^n, -C^l$ sont les seules racines de $x^3 - px + q = 0$, nous discutons les conditions pourque x_1, x_2, x_3 soient des entiers. Trois exemples numériques sont présentés.

1.1. Introduction

In 1997, Andrew Beal [**4**] announced the following conjecture :

Conjecture 1.1. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.1}$$

then $A, B,$ and C have a common factor.

In this paper, we give a complete proof of the Beal Conjecture. Our idea is to construct a polynomial $P(x)$ of three order having as roots A^m, B^n and $-C^l$ with the condition (1.1). The paper is organized as follows. In Section 1, we begin with the trivial case where $A^m = B^n$. In Section 2, we consider the polynomial $P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - px + q$. We express the three roots of $P(x) = x^3 - px + q = 0$ in function of two parameters ρ, θ that depend of A^m, B^n, C^l. The Sections 3,4 and 5 are the main parts of the paper. We write that $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3}$. As A^{2m} is an integer, it follows that $cos^2 \dfrac{\theta}{3}$ must be written as $\dfrac{a}{b}$ where a, b are two positive coprime integers. We discuss the conditions of divisibility of p, a, b so that the expression of A^{2m} is an integer. Depending of each individual case, we obtain that A, B, C have or not a common factor. We present three numerical examples in section 6 and we give conclusions in the last section.

1.1.1. Trivial Case. — We consider the trivial case when $A^m = B^n$. The equation (1.1) becomes:

$$2A^m = C^l \tag{1.2}$$

then $2|C^l \Longrightarrow 2|C \Longrightarrow \exists\, c \in \mathbb{N}^* /\ C = 2c$, it follows $2A^m = 2^l c^l \Longrightarrow A^m = 2^{l-1} c^l$. As $l > 2$, then $2|A^m \Longrightarrow 2|A \Longrightarrow 2|B^n \Longrightarrow 2|B$. The conjecture (3.1) is verified.

We suppose in the following that $A^m > B^n$.

1.2. Preliminaries

Let $m, n, l \in \mathbb{N}^* \ > 2$ and $A, B, C \in \mathbb{N}^*$ such:

$$A^m + B^n = C^l \tag{1.3}$$

We call:

$$P(x) = (x - A^m)(x - B^n)(x + C^l) = x^3 - x^2(A^m + B^n - C^l)$$
$$+x[A^mB^n - C^l(A^m + B^n)] + C^lA^mB^n \tag{1.4}$$

Using the equation (1.3), $P(x)$ can be written as:

$$\boxed{P(x) = x^3 + x[A^mB^n - (A^m + B^n)^2] + A^mB^n(A^m + B^n)} \tag{1.5}$$

We introduce the notations:

$$p = (A^m + B^n)^2 - A^mB^n \tag{1.6}$$

$$q = A^mB^n(A^m + B^n) \tag{1.7}$$

As $A^m \neq B^n$, we have :

$$p > (A^m - B^n)^2 > 0 \tag{1.8}$$

Equation (1.5) becomes:

$$P(x) = x^3 - px + q \tag{1.9}$$

Using the equation (1.4), $P(x) = 0$ has three different real roots : A^m, B^n and $-C^l$.

Now, let us resolve the equation:

$$P(x) = x^3 - px + q = 0 \tag{1.10}$$

To resolve (1.10) let:

$$x = u + v \tag{1.11}$$

Then $P(x) = 0$ gives:

(1.12)
$$P(x) = P(u+v) = (u+v)^3 - p(u+v) + q = 0 \Longrightarrow u^3 + v^3 + (u+v)(3uv - p) + q = 0$$

To determine u and v, we obtain the conditions:

$$u^3 + v^3 = -q \tag{1.13}$$

$$uv = p/3 > 0 \tag{1.14}$$

Then u^3 and v^3 are solutions of the second order equation:

$$X^2 + qX + p^3/27 = 0 \tag{1.15}$$

Its discriminant Δ is written as :

$$\Delta = q^2 - 4p^3/27 = \frac{27q^2 - 4p^3}{27} = \frac{\bar{\Delta}}{27} \tag{1.16}$$

Let:

$$\bar{\Delta} = 27q^2 - 4p^3 = 27(A^mB^n(A^m+B^n))^2 - 4[(A^m+B^n)^2 - A^mB^n]^3$$
$$= 27A^{2m}B^{2n}(A^m+B^n)^2 - 4[(A^m+B^n)^2 - A^mB^n]^3 \quad (1.17)$$

Noting :

$$\alpha = A^mB^n > 0 \quad (1.18)$$

$$\beta = (A^m+B^n)^2 \quad (1.19)$$

we can write (1.17) as:

$$\bar{\Delta} = 27\alpha^2\beta - 4(\beta-\alpha)^3 \quad (1.20)$$

As $\alpha \neq 0$, we can also rewrite (1.20) as :

$$\bar{\Delta} = \alpha^3\left(27\frac{\beta}{\alpha} - 4\left(\frac{\beta}{\alpha} - 1\right)^3\right) \quad (1.21)$$

We call t the parameter :

$$t = \frac{\beta}{\alpha} \quad (1.22)$$

$\bar{\Delta}$ becomes :

$$\bar{\Delta} = \alpha^3(27t - 4(t-1)^3) \quad (1.23)$$

Let us calling :

$$y = y(t) = 27t - 4(t-1)^3 \quad (1.24)$$

Since $\alpha > 0$, the sign of $\bar{\Delta}$ is also the sign of $y(t)$. Let us study the sign of y. We obtain $y'(t)$:

$$y'(t) = y' = 3(1+2t)(5-2t) \quad (1.25)$$

$y' = 0 \Longrightarrow t_1 = -1/2$ and $t_2 = 5/2$, then the table of variations of y is given below:
The table of the variations of the function y shows that $y < 0$ for $t > 4$. In our case, we are interested for $t > 0$. For $t = 4$ we obtain $y(4) = 0$ and for $t \in]0, 4[\Longrightarrow y > 0$. As we have $t = \frac{\beta}{\alpha} > 4$ because as $A^m \neq B^n$:

$$(A^m - B^n)^2 > 0 \Longrightarrow \beta = (A^m+B^n)^2 > 4\alpha = 4A^mB^n \quad (1.26)$$

Then $y < 0 \Longrightarrow \bar{\Delta} < 0 \Longrightarrow \Delta < 0$. Then, the equation (1.15) does not have real solutions u^3 and v^3. Let us find the solutions u and v with $x = u + v$ is a positive or a negative real and $u.v = p/3$.

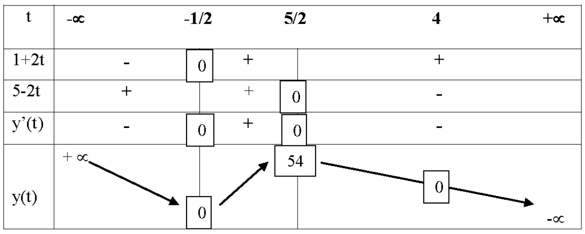

FIGURE 1. The table of variations

1.2.1. Expressions of the roots. —

Proof. — The solutions of (1.15) are:

$$X_1 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{1.27}$$

$$X_2 = \overline{X_1} = \frac{-q - i\sqrt{-\Delta}}{2} \tag{1.28}$$

We may resolve:

$$u^3 = \frac{-q + i\sqrt{-\Delta}}{2} \tag{1.29}$$

$$v^3 = \frac{-q - i\sqrt{-\Delta}}{2} \tag{1.30}$$

Writing X_1 in the form:

$$X_1 = \rho e^{i\theta} \tag{1.31}$$

with:

$$\rho = \frac{\sqrt{q^2 - \Delta}}{2} = \frac{p\sqrt{p}}{3\sqrt{3}} \tag{1.32}$$

$$\text{and } \ sin\theta = \frac{\sqrt{-\Delta}}{2\rho} > 0 \tag{1.33}$$

$$cos\theta = -\frac{q}{2\rho} < 0 \tag{1.34}$$

Then $\theta\,[2\pi] \in]+\dfrac{\pi}{2}, +\pi[$, let:

$$\boxed{\frac{\pi}{2} < \theta < +\pi \Rightarrow \frac{\pi}{6} < \frac{\theta}{3} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < cos\frac{\theta}{3} < \frac{\sqrt{3}}{2}} \tag{1.35}$$

and:

$$\boxed{\frac{1}{4} < cos^2\frac{\theta}{3} < \frac{3}{4}} \tag{1.36}$$

hence the expression of X_2:

$$X_2 = \rho e^{-i\theta} \tag{1.37}$$

Let:

$$u = re^{i\psi} \tag{1.38}$$

$$\text{and } j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \tag{1.39}$$

$$j^2 = e^{i\frac{4\pi}{3}} = -\frac{1 + i\sqrt{3}}{2} = \bar{j} \tag{1.40}$$

j is a complex cubic root of the unity $\iff j^3 = 1$. Then, the solutions u and v are:

$$u_1 = re^{i\psi_1} = \sqrt[3]{\rho}e^{i\frac{\theta}{3}} \tag{1.41}$$

$$u_2 = re^{i\psi_2} = \sqrt[3]{\rho}je^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+2\pi}{3}} \tag{1.42}$$

$$u_3 = re^{i\psi_3} = \sqrt[3]{\rho}j^2e^{i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{+i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{\theta+4\pi}{3}} \tag{1.43}$$

and similarly:

$$v_1 = re^{-i\psi_1} = \sqrt[3]{\rho}e^{-i\frac{\theta}{3}} \tag{1.44}$$

$$v_2 = re^{-i\psi_2} = \sqrt[3]{\rho}j^2e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi}{3}}e^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{4\pi-\theta}{3}} \tag{1.45}$$

$$v_3 = re^{-i\psi_3} = \sqrt[3]{\rho}je^{-i\frac{\theta}{3}} = \sqrt[3]{\rho}e^{i\frac{2\pi-\theta}{3}} \tag{1.46}$$

We may now choose u_k and v_h so that $u_k + v_h$ will be real. In this case, we have necessary :

$$v_1 = \overline{u_1} \tag{1.47}$$

$$v_2 = \overline{u_2} \tag{1.48}$$

$$v_3 = \overline{u_3} \tag{1.49}$$

We obtain as real solutions of the equation (1.12):

$$x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} cos\frac{\theta}{3} > 0 \tag{1.50}$$

$$x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} cos\tfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) < 0 \tag{1.51}$$

$$x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} cos\tfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right) > 0 \tag{1.52}$$

We compare the expressions of x_1 and x_3, we obtain:

$$2\sqrt[3]{p} cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt[3]{p}\left(-cos\tfrac{\theta}{3} + \sqrt{3}sin\tfrac{\theta}{3}\right)$$
$$3cos\tfrac{\theta}{3} \overbrace{>}^{?} \sqrt{3}sin\tfrac{\theta}{3} \tag{1.53}$$

As $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$, then $sin\frac{\theta}{3}$ and $cos\frac{\theta}{3}$ are > 0. Taking the square of the two members of the last equation, we get:

$$\frac{1}{4} < cos^2\frac{\theta}{3} \tag{1.54}$$

which is true since $\frac{\theta}{3} \in]+\frac{\pi}{6}, +\frac{\pi}{3}[$ then $x_1 > x_3$. As A^m, B^n and $-C^l$ are the only real solutions of (1.10), we consider, as A^m is supposed great than B^n, the expressions:

(1.55)

$$\begin{cases} A^m = x_1 = u_1 + v_1 = 2\sqrt[3]{\rho} cos\dfrac{\theta}{3} \\ B^n = x_3 = u_3 + v_3 = 2\sqrt[3]{\rho} cos\dfrac{\theta+4\pi}{3} = \sqrt[3]{\rho}\left(-cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \\ -C^l = x_2 = u_2 + v_2 = 2\sqrt[3]{\rho} cos\dfrac{\theta+2\pi}{3} = -\sqrt[3]{\rho}\left(cos\dfrac{\theta}{3} + \sqrt{3}sin\dfrac{\theta}{3}\right) \end{cases}$$

□

1.3. Preamble of the Proof of the Main Theorem

Theorem 1.2. — *Let $A, B, C, m, n,$ and l be positive integers with $m, n, l > 2$. If:*

$$A^m + B^n = C^l \tag{1.56}$$

then $A, B,$ and C have a common factor.

Proof. — $A^m = 2\sqrt[3]{\rho}cos\dfrac{\theta}{3}$ is an integer $\Rightarrow A^{2m} = 4\sqrt[3]{\rho^2}cos^2\dfrac{\theta}{3}$ is also an integer. But :

$$\sqrt[3]{\rho^2} = \frac{p}{3} \tag{1.57}$$

Then:

$$A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3} = p.\frac{4}{3}.cos^2\frac{\theta}{3} \tag{1.58}$$

As A^{2m} is an integer and p is an integer, then $cos^2\dfrac{\theta}{3}$ must be written under the form:

$$\boxed{cos^2\frac{\theta}{3} = \frac{1}{b} \quad or \quad cos^2\frac{\theta}{3} = \frac{a}{b}} \tag{1.59}$$

with $b \in \mathbb{N}^*$; for the last condition $a \in \mathbb{N}^*$ and a, b coprime.

Notations: In the following of the paper, the scalars $a, b, ..., z,\ \alpha, \beta, ...,$ $A, B, C, ...$ and $\Delta, \Phi, ...$ represent positive integers except the parameters θ, ρ, or others cited in the text, are reals.

1.3.1. Case $cos^2\dfrac{\theta}{3} = \dfrac{1}{b}$. — We obtain:

$$A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p}{3.b} \tag{1.60}$$

As $\dfrac{1}{4} < cos^2\dfrac{\theta}{3} < \dfrac{3}{4} \Rightarrow \dfrac{1}{4} < \dfrac{1}{b} < \dfrac{3}{4} \Rightarrow b < 4 < 3b \Rightarrow b = 1, 2, 3.$

1.3.1.1. $b = 1$. — $b = 1 \Rightarrow 4 < 3$ which is impossible.

1.3.1.2. $b = 2$. — $b = 2 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{2} = \frac{2.p}{3} \Rightarrow 3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, we obtain:

$$A^{2m} = (A^m)^2 = \frac{2p}{3} = 2.p' \Longrightarrow 2|p' \Longrightarrow p' = 2^\alpha p_1^2$$
$$with \quad 2 \nmid p_1, \quad \alpha + 1 = 2\beta$$

$$A^m = 2^\beta p_1 \tag{1.61}$$

$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = p' = 2^\alpha p_1^2 \tag{1.62}$$

From the equation (1.61), it follows that $2|A^m \Longrightarrow A = 2^i A_1$, $i \geq 1$ and $2 \nmid A_1$. Then, we have $\beta = i.m = im$. The equation (1.62) implies that $2|(B^n C^l) \Longrightarrow 2|B^n$ or $2|C^l$.

1.3.1.2.1. *Case* $2|B^n$. — : If $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^j B_1$ with $2 \nmid B_1$. The expression of $B^n C^l$ becomes:

$$B_1^n C^l = 2^{2im-1-jn} p_1^2$$

- If $2im - 1 - jn \geq 1$, $2|C^l \Longrightarrow 2|C$ according to $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im - 1 - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

1.3.1.2.2. *Case* $2|C^l$. — : If $2|C^l$: with the same method used above, we obtain the identical results.

1.3.1.3. $b = 3$. — $b = 3 \Rightarrow A^{2m} = p.\frac{4}{3}.\frac{1}{3} = \frac{4p}{9} \Rightarrow 9|p \Rightarrow p = 9p'$ with $p' \neq 1$, as $9 \ll p$ then $A^{2m} = 4p'$. If p' is prime, it is impossible. We suppose that p' is not a prime, as $m \geq 3$, it follows that $2|p'$, then $2|A^m$. But $B^n C^l = 5p'$ and $2|(B^n C^l)$. Using the same method for the case $b = 2$, we obtain the identical results.

1.3.2. Case $a > 1$, $cos^2\frac{\theta}{3} = \frac{a}{b}$. — We have:

$$cos^2\frac{\theta}{3} = \frac{a}{b}\,; \quad A^{2m} = p.\frac{4}{3}.cos^2\frac{\theta}{3} = \frac{4.p.a}{3.b} \tag{1.63}$$

where a, b verify one of the two conditions:

$$\boxed{\{3|a \quad and \quad b|4p\}} \text{ or } \boxed{\{3|p \quad and \quad b|4p\}} \tag{1.64}$$

and using the equation (1.36), we obtain a third condition:

$$\boxed{b < 4a < 3b} \tag{1.65}$$

For these conditions, $A^{2m} = 4\sqrt[3]{\rho^2}cos^2\frac{\theta}{3} = 4\frac{p}{3}.cos^2\frac{\theta}{3}$ is an integer.

Let us study the conditions given by the equation (1.64) in the following two sections.

1.4. Hypothesis : $\{3|a \quad and \quad b|4p\}$

We obtain :

$$3|a \Longrightarrow \exists a' \in \mathbb{N}^* \ / \ a = 3a' \tag{1.66}$$

1.4.1. Case $b = 2$ and $3|a$: — A^{2m} is written as:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = \frac{4p}{3}.\frac{a}{2} = \frac{2.p.a}{3} \tag{1.67}$$

Using the equation (1.66), A^{2m} becomes :

$$A^{2m} = \frac{2.p.3a'}{3} = 2.p.a' \tag{1.68}$$

but $cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2} > 1$ which is impossible, then $b \neq 2$.

1.4.2. Case $b = 4$ and $3|a$: — A^{2m} is written :

$$A^{2m} = \frac{4.p}{3}cos^2\frac{\theta}{3} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.p}{3}.\frac{a}{4} = \frac{p.a}{3} = \frac{p.3a'}{3} = p.a' \tag{1.69}$$

$$\text{and} \quad cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3.a'}{4} < \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \Longrightarrow a' < 1 \tag{1.70}$$

which is impossible. Then the case $b = 4$ is impossible.

1.4.3. Case $b = p$ and $3|a$: — We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{p} \tag{1.71}$$

and:

$$A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{3a'}{p} = 4a' = (A^m)^2 \tag{1.72}$$

$$\exists a" \ / \ a' = a"^2 \tag{1.73}$$

$$and \quad B^nC^l = p - A^{2m} = b - 4a' = b - 4a"^2 \tag{1.74}$$

The calculation of A^mB^n gives :

$$A^mB^n = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} - 2a'$$

$$(1.75) \qquad or \quad A^mB^n + 2a' = p.\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$$

The left member of (1.75) is an integer and p also, then $2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3}$ is written under the form :

$$(1.76) \qquad 2\frac{\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two coprime integers and $k_2|p \Longrightarrow p = b = k_2.k_3,\ k_3 \in \mathbb{N}^*$.

** A-1- We suppose that $k_3 \neq 1$, we obtain :

$$(1.77) \qquad A^m(A^m + 2B^n) = k_1.k_3$$

Let μ be a prime integer with $\mu|k_3$, then $\mu|b$ and $\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

** A-1-1- If $\mu|A^m \Longrightarrow \mu|A$ and $\mu|A^{2m}$, but $A^{2m} = 4a' \Longrightarrow \mu|4a' \Longrightarrow (\mu = 2$, but $2|a')$ or $(\mu|a')$. Then $\mu|a$ it follows the contradiction with a, b coprime.

** A-1-2- If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$ then $\mu \neq 2$ and $\mu \nmid B^n$. We write $\mu|(A^m + 2B^n)$ as:

$$(1.78) \qquad A^m + 2B^n = \mu.t'$$

It follows :

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p:

$$(1.79) \qquad p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $p = b = k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists\mu'$ and $b = \mu\mu'$, so we can write:

$$(1.80) \qquad \mu'\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

From the last equation, we obtain $\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n$ or $\mu|(B^n - A^m)$.

** A-1-2-1- If $\mu|B^n$ which is in contradiction with $\mu \nmid B^n$.

** A-1-2-2- If $\mu|(B^n - A^m)$ and using that $\mu|(A^m + 2B^n)$, we arrive to :

$$\mu|3B^n \begin{cases} \mu|B^n \\ or \\ \mu = 3 \end{cases} \tag{1.81}$$

** A-1-2-2-1- If $\mu|B^n \Longrightarrow \mu|B$, it is the contradiction with $\mu \nmid B$ cited above.

** A-1-2-2-2- If $\mu = 3$, then $3|b$, but $3|a$ then the contradiction with a, b coprime.

** A-2- We assume now $k_3 = 1$, then :

$$A^{2m} + 2A^m B^n = k_1 \tag{1.82}$$

$$b = k_2 \tag{1.83}$$

$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{1.84}$$

Taking the square of the last equation, we obtain:

$$\frac{4}{3} sin^2\frac{2\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2\frac{\theta}{3} cos^2\frac{\theta}{3} = \frac{k_1^2}{b^2}$$

$$\frac{16}{3} sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{b^2}$$

Finally:

$$4^2 a'(p - a) = k_1^2 \tag{1.85}$$

but $a' = a"^2$, then $p - a$ is a square. Let:

$$\lambda^2 = p - a = b - a = b - 3a"^2 \Longrightarrow \lambda^2 + 3a"^2 = b \tag{1.86}$$

The equation (1.85) becomes:

$$4^2 a"^2 \lambda^2 = k_1^2 \Longrightarrow k_1 = 4a"\lambda \tag{1.87}$$

taking the positive root, but $k_1 = A^m(A^m + 2B^n) = 2a"(A^m + 2B^n)$, then :

$$A^m + 2B^n = 2\lambda \Longrightarrow \lambda = a" + B^n \tag{1.88}$$

** A-2-1- As $A^m = 2a" \Longrightarrow 2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i A_1$, with $i \geq 1$ and $2 \nmid A_1$, then $A^m = 2a" = 2^{im} A_1^m \Longrightarrow a" = 2^{im-1} A_1^m$, but $im \geq 3 \Longrightarrow 4|a"$. As $p = b = A^{2m} + A^m B^n + B^{2n} = \lambda = 2^{im-1} A_1^m + B^n$. Taking its square, then :

$$\lambda^2 = 2^{2im-2} A_1^{2m} + 2^{im} A_1^m B^n + B^{2n}$$

As $im \geq 3$, we can write $\lambda^2 = 4\lambda_1 + B^{2n} \Longrightarrow \lambda^2 \equiv B^{2n}(\bmod 4) \Longrightarrow \lambda^2 \equiv B^{2n} \equiv 0(\bmod 4)$ or $\lambda^2 \equiv B^{2n} \equiv 1(\bmod 4)$.

** A-2-1-1- We suppose that $\lambda^2 \equiv B^{2n} \equiv 0(\bmod 4) \Longrightarrow 4|\lambda^2 \Longrightarrow 2|(b-a)$. But $2|a$ because $a = 3a' = 3a"^2 = 3 \times 2^{2(im-1)} A_1^{2m}$ and $im \geq 3$. Then $2|b$, it follows the contradiction with a, b coprime.

** A-2-1-2- We suppose now that $\lambda^2 \equiv B^{2n} \equiv 1(\bmod 4)$. As $A^m = 2^{im-1} A_1^m$ and $im - 1 \geq 2$, then $A^m \equiv 0(\bmod 4)$. As $B^{2n} \equiv 1(\bmod 4)$, then B^n verifies $B^n \equiv 1(\bmod 4)$ or $B^n \equiv 3(\bmod 4)$ which gives for the two cases $B^nC^l \equiv 1(\bmod 4)$.

We have also $p = b = A^{2m} + A^mB^n + B^{2n} = 4a' + B^n.C^l = 4a"^2 + B^nC^l \Longrightarrow B^nC^l = \lambda^2 - a"^2 = B^n.C^l$, then $\lambda, a" \in \mathbb{N}^*$ are solutions of the Diophantine equation :

$$x^2 - y^2 = N \tag{1.89}$$

with $N = B^nC^l > 0$. Let $Q(N)$ be the number of the solutions of (1.89) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the equation (1.89) (see theorem 27.3 in [**6**]):

- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (u, v), $u, v \in \mathbb{N}^*$ be another pair, solution of the equation (1.89), then $u^2 - v^2 = x^2 - y^2 = N = B^nC^l$, but $\lambda = x$ and $a" = y$ verify the equation (1.88) given by $x - y = B^n$, it follows u, v verify also $u - v = B^n$, that gives $u + v = C^l$, then $u = x = \lambda = a" + B^n$ and $v = a"$. We have given a proof of the uniqueness of the solutions of the equation (1.89) with the condition $x - y = B^n$. As $N = B^nC^l \equiv 1(\bmod 4) \Longrightarrow Q(N) = [\tau(N)/2] > 1$. But $Q(N) = 1$, then the contradiction.

Hence, the case $k_3 = 1$ is impossible.

Let us verify the condition (1.65) given by $b < 4a < 3b$. In our case, the condition becomes :

$$p < 3A^{2m} < 3p \quad with \quad p = A^{2m} + B^{2n} + A^m B^n \tag{1.90}$$

and $3A^{2m} < 3p \Longrightarrow A^{2m} < p$ that is verified. If :

$$p < 3A^{2m} \Longrightarrow 2A^{2m} - A^m B^n - B^{2n} \overbrace{>}^{?} 0$$

Studying the sign of the polynomial $Q(Y) = 2Y^2 - B^n Y - B^{2n}$ and taking $Y = A^m > B^n$, the condition $2A^{2m} - A^m B^n - B^{2n} > 0$ is verified, then the condition $b < 4a < 3b$ is true.

In the following of the paper, we verify easily that the condition $b < 4a < 3b$ implies to verify that $A^m > B^n$ which is true.

1.4.4. Case $b|p \Rightarrow \ p = b.p', p' > 1$, $b \neq 2$, $b \neq 4$ and $3|a$: —

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.b.p'.3.a'}{3.b} = 4.p'a' \tag{1.91}$$

We calculate $B^n C^l$:

$$B^n C^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) \tag{1.92}$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$, we obtain:

(1.93)

$$B^n C^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = p'(b - 4a')$$

As $p = b.p'$, and $p' > 1$, so we have :

$$B^n C^l = p'(b - 4a') \tag{1.94}$$

$$\text{and} \quad A^{2m} = 4.p'.a' \tag{1.95}$$

** B-1- We suppose that p' is prime, then $A^{2m} = 4ap' = (A^m)^2 \Longrightarrow p'|a$. But $B^n C^l = p'(b - 4a') \Longrightarrow p'|B^n$ or $p'|C^l$.

** B-1-1- If $p'|B^n \Longrightarrow p'|B \Longrightarrow B = p'B_1$ with $B_1 \in \mathbb{N}^*$. Hence : $p'^{n-1}B_1^n C^l = b - 4a'$. But $n > 2 \Rightarrow (n-1) > 1$ and $p'|a'$, then $p'|b \Longrightarrow a$ and b are not coprime, then the contradiction.

** B-1-2- If $p'|C^l \Longrightarrow p'|C$. The same method used above, we obtain the same results.

** B-2- We consider that p' is not a prime integer.

** B-2-1- p', a are supposed coprime: $A^{2m} = 4ap' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2|a'$ or $2|p_1$.

** B-2-1-1- $2|a'$, then $2|a' \Longrightarrow 2 \nmid p_1$. But $p' = p_1^2$.

** B-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** B-2-1-1-2- We suppose that p_1 is not prime, we can write it as $p_1 = \omega^m \Longrightarrow p' = \omega^{2m}$, then: $B^nC^l = \omega^{2m}(b - 4a')$.

** B-2-1-1-2-1- If ω is prime, it is different of 2, then $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** B-2-1-1-2-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(b - 4a')$.

** B-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = b - 4a'$. As $C^l = A^m + B^n \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, and $\omega|(b - 4a')$. But $\omega \neq 2$ and ω is coprime with a' then coprime with a, then $\omega \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-1-1-2- If $2m - nj \geq 1$, in this case with the same method, we obtain $\omega|C^l \Longrightarrow \omega|C$ and $\omega|(b-4a')$ and $\omega \nmid a$ and $\omega \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = b - 4a'$. As $\omega|C$ using $C^l = A^m + B^n$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l = b - 4a'$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega|B_1^nC_1^l$, it follows the contradiction that $\omega \nmid B_1$ or $\omega \nmid C_1$. Then if $n.j - 2m + h.l > 0$ and $\omega|(b - 4a')$ with ω, a, b coprime and the conjecture (3.1) is verified.

** B-2-1-1-2-1-2- We obtain the same results if $\omega|C^l$.

** B-2-1-1-2-2- Now, $p' = \omega^{2m}$ and ω not prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1|A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(b-4a') \Longrightarrow \omega_1|(B^nC^l) \Longrightarrow \omega_1|B^n$ or $\omega_1|C^l$.

** B-2-1-1-2-2-1- If $\omega_1|B^n \Longrightarrow \omega_1|B \Longrightarrow B = \omega_1^jB_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2mf-nj}\Omega^{2m}(b-4a')$:

** B-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(b-4a')$. As $C^l = A^m + B^n \Longrightarrow \omega_1|C^l \Longrightarrow \omega_1|C \Longrightarrow \omega_1|(b-4a')$. But $\omega_1 \neq 2$ and ω_1 is coprime with a', then coprime with a, we deduce $\omega_1 \nmid b$. Then the conjecture (3.1) is verified.

** B-2-1-1-2-2-1-2- If $2f.m-n.j \geq 1$, we have $\omega_1|C^l \Longrightarrow \omega_1|C \Longrightarrow \omega_1|(b-4a')$ and $\omega_1 \nmid a$ and $\omega_1 \nmid b$. The conjecture (3.1) is verified.

** B-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(b-4a')$. As $\omega_1|C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(b-4a')$. If $n.j-2m.f+h.l < 0 \Longrightarrow \omega_1|B_1^nC_1^l$, it follows the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j-2m.f+h.l > 0$ and $\omega_1|(b-4a')$ with ω_1, a, b coprime and the conjecture (3.1) is verified.

** B-2-1-1-2-2-2- We obtain the same results if $\omega_1|C^l$.

** B-2-1-2- If $2|p_1$, then $2|p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$. But $p' = p_1^2$.

** B-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2|a'$, then the contradiction with a, b coprime.

** B-2-1-2-2- We suppose that p_1 is not prime and $2|p_1$, as $A^m = 2a'p_1$, p_1 is written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p' = 2^{2m-2}\omega^{2m}$. It follows $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a') \Longrightarrow 2|B^n$ or $2|C^l$.

** B-2-1-2-2-1- If $2|B^n \Longrightarrow 2|B$, as $2|A$, then $2|C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(b-4a')$, it follows if $2|(b-4a') \Longrightarrow 2|b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** B-2-1-2-2-2- If $2|C^l$, using the same method as above, we obtain the identical results.

** B-2-2- p', a are supposed not coprime. Let ω be a prime integer so that $\omega|a$ and $\omega|p'$.

** B-2-2-1- We suppose firstly $\omega = 3$. As $A^{2m} = 4ap' \Longrightarrow 3|A$, but $3|p' \Longrightarrow 3|p$, as $p = A^{2m} + B^{2n} + A^m B^n \Longrightarrow 3|B^{2n} \Longrightarrow 3|B$, then $3|C^l \Longrightarrow 3|C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ and 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im} A_1^{2m} + 3^{2nj} B_1^{2n} + 3^{im+jn} A_1^m B_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3)|a$ and $(\omega = 3)|p'$ that gives $a = 3^\alpha a_1 = 3a' \Longrightarrow a' = 3^{\alpha-1} a_1$, $3 \nmid a_1$ and $p' = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4a'p' = 3^{2im} A_1^{2m} = 4 \times 3^{\alpha-1+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu - 1 = 2im$. As $p = bp' = b.3^\mu p_1 = 3^\mu.b.p_1$. The exponent of the term 3 of p is k, the exponent of the term 3 of the left member of the last equation is μ. If $3|b$ it is a contradiction with a, b coprime. Then, we suppose that $3 \nmid b$, and the equality of the exponents: $min(2im, 2jn, im + jn) = \mu$, recall that $\alpha + \mu - 1 = 2im$. But $B^n C^l = p'(b - 4a')$ that gives $3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1)$. We have also $A^m + B^n = C^l$ gives $3^{im} A_1^m + 3^{jn} B_1^n = 3^{hl} C_1^l$. Let $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. Then, we obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu \tag{1.96}$$

$$\alpha + \mu - 1 = 2im \tag{1.97}$$

$$\epsilon = hl = min(im, jn) \tag{1.98}$$

$$3^{(nj+hl)} B_1^n C_1^l = 3^\mu p_1 (b - 4 \times 3^{(\alpha-1)} a_1) \tag{1.99}$$

** B-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1 = 3a'$ and $3 \nmid a_1$, the equation (1.97) becomes:

$$\mu = 2im$$

and the first equation (1.96) is written as:

$$k = min(2im, 2jn, im + jn) = 2im$$

- If $k = 2im$, then $2im \leq 2jn \Longrightarrow im \leq jn \Longrightarrow hl = im$, and (1.99) gives $\mu = 2im = nj + hl = im + nj \Longrightarrow im = jn = hl$. Hence $3|A, 3|B$ and $3|C$ and the conjecture (3.1) is verified.
- If $k = 2jn \Longrightarrow 2jn = 2im \Longrightarrow im = jn = hl$. Hence $3|A, 3|B$ and $3|C$ and the conjecture (3.1) is verified.

- If $k = im + jn = 2im \Longrightarrow im = jn \Longrightarrow \epsilon = hl = im = jn$ case that is seen above and we deduce that $3|A, 3|B$ and $3|C$, and the conjecture (3.1) is verified.

** B-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$ and $a' = 3^{\alpha-1}a_1$.

- If $k = 2im \Longrightarrow 2im = \mu$, but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = 2jn = \mu \Longrightarrow 2jn = 2im + 1 - \alpha$. We obtain $2jn < 2im \Longrightarrow jn < im \Longrightarrow 2jn < im + jn$, $k = 2jn$ is just the minimum of $(2im, 2jn, im + jn)$. We obtain $jn = hl < im$ and the equation (1.99) becomes:

$$B_1^n C_1^l = p_1(b - 4 \times 3^{(\alpha-1)} a_1)$$

The conjecture (3.1) is verified.

- If $k = im + jn \leq 2im \Longrightarrow jn \leq im$ and $k = im + jn \leq 2jn \Longrightarrow im \leq jn \Longrightarrow im = jn \Longrightarrow k = im + jn = 2im = \mu$ but $\mu = 2im + 1 - \alpha$ that is impossible.
- If $k = im + jn < 2im \Longrightarrow jn < im$ and $2jn < im + jn = k$ that is a contradiction with $k = min(2im, 2jn, im + jn)$.

** B-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $p' = \omega^\mu p_1$ with $\omega \nmid p_1$. As $A^{2m} = 4ap' = 4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega|A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = p'(b - 4a') = \omega^\mu p_1(b - 4a') \Longrightarrow \omega|B^n C^l \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** B-2-2-2-1- $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega|C^l \Longrightarrow \omega|C$. As $p = bp' = \omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$ with $k = min(2im, 2jn, im + jn)$. Then :

- If $\mu = k$, then $\omega \nmid b$ and the conjecture (3.1) is verified.
- If $k > \mu$, then $\omega|b$, but $\omega|a$ we deduce the contradiction with a, b coprime.
- If $k < \mu$, it follows from :

$$\omega^\mu b p_1 = \omega^k(\omega^{2im-k}A_1^{2m} + \omega^{2jn-k}B_1^{2n} + \omega^{im+jn-k}A_1^m B_1^n)$$

that $\omega|A_1$ or $\omega|B_1$ that is a contradiction with the hypothesis.

** B-2-2-2-2- If $\omega|C^l \Longrightarrow \omega|C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega|(C^l - A^m) \Longrightarrow \omega|B$. Then, we obtain the same results as B-2-2-2-1- above.

1.4.5. Case $b = 2p$ and $3|a$: — We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{2p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{2p} = 2a' = (A^m)^2 \Longrightarrow 2|a' \Longrightarrow 2|a$$

Then $2|a$ and $2|b$ that is a contradiction with a, b coprime.

1.4.6. Case $b = 4p$ and $3|a$: — We have :

$$cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{3a'}{4p} \Longrightarrow A^{2m} = \frac{4p.a}{3b} = \frac{4p}{3}.\frac{3a'}{4p} = a' = (A^m)^2 = a"^2$$
$$\text{with} \quad A^m = a"$$

Let us calculate A^mB^n, we obtain:

$$A^mB^n = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{2p}{3}cos^2\frac{\theta}{3} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3} - \frac{a'}{2} \Longrightarrow$$
$$A^mB^n + \frac{A^{2m}}{2} = \frac{p\sqrt{3}}{3}.sin\frac{2\theta}{3}$$

Let:

$$A^{2m} + 2A^mB^n = \frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3} \tag{1.100}$$

The left member of (1.100) is an integer and p is an integer, then $\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3}$ will be written as :

$$\frac{2\sqrt{3}}{3}sin\frac{2\theta}{3} = \frac{k_1}{k_2}$$

where k_1, k_2 are two integers coprime and $k_2|p \Longrightarrow p = k_2.k_3$.

** C-1- Firstly, we suppose that $k_3 \neq 1$. Then :

$$A^{2m} + 2A^mB^n = k_3.k_1$$

Let μ be a prime integer and $\mu|k_3$, then $\mu|A^m(A^m + 2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

** C-1-1- If $\mu|(A^m = a") \Longrightarrow \mu|(a"^2 = a') \Longrightarrow \mu|(3a' = a)$. As $\mu|k_3 \Longrightarrow \mu|p \Longrightarrow \mu|(4p = b)$, then the contradiction with a, b coprime.

** C-1-2- If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then:

$$\mu \neq 2 \quad and \quad \mu \nmid B^n \tag{1.101}$$

$\mu|(A^m + 2B^n)$, we write:

$$A^m + 2B^n = \mu.t'$$

Then:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^mB^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$
$$\Longrightarrow p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$$

As $b = 4p = 4k_2.k_3$ and $\mu|k_3$ then $\mu|b \Longrightarrow \exists \mu'$ so that $b = \mu.\mu'$, we obtain:

$$\mu'.\mu = \mu(4\mu t'^2 - 8t'B^n) + 4B^n(B^n - A^m)$$

The last equation implies $\mu|4B^n(B^n - A^m)$, but $\mu \neq 2$ then $\mu|B^n$ or $\mu|(B^n - A^m)$.

** C-1-1-1- If $\mu|B^n \Longrightarrow$ then the contradiction with (1.101).

** C-1-1-2- If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we have :

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n \\ or \\ \mu = 3 \end{cases}$$

** C-1-1-2-1- If $\mu|B^n$ then the contradiction with (1.101).

** C-1-1-2-2- If $\mu = 3$, then $3|b$, but $3|a$ then the contradiction with a, b coprime.

** C-2- We assume now that $k_3 = 1$, then:

$$A^{2m} + 2A^mB^n = k_1 \tag{1.102}$$
$$p = k_2$$
$$\frac{2\sqrt{3}}{3} sin\frac{2\theta}{3} = \frac{k_1}{p}$$

We take the square of the last equation, we obtain :

$$\frac{4}{3} sin^2\frac{2\theta}{3} = \frac{k_1^2}{p^2}$$
$$\frac{16}{3} sin^2\frac{\theta}{3} cos^2\frac{\theta}{3} = \frac{k_1^2}{p^2}$$
$$\frac{16}{3} sin^2\frac{\theta}{3}.\frac{3a'}{b} = \frac{k_1^2}{p^2}$$

Finally:

$$a'(4p - 3a') = k_1^2 \tag{1.103}$$

but $a' = a"^2$, then $4p - 3a'$ is a square. Let :

$$\lambda^2 = 4p - 3a' = 4p - a = b - a$$

The equation (1.103) becomes :

$$a"^2\lambda^2 = k_1^2 \Longrightarrow k_1 = a"\lambda \tag{1.104}$$

taking the positive root. Using (1.102), we have:

$$k_1 = A^m(A^m + 2B^n) = a"(A^m + 2B^n)$$

Then :

$$A^m + 2B^n = \lambda$$

Now, we consider that $b - a = \lambda^2 \Longrightarrow \lambda^2 + 3a"^2 = b$, then the couple $(\lambda, a")$ is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{1.105}$$

with $X = \lambda$ and $Y = a"$. But using one theorem on the solutions of the equation given by (1.105), b is written under the form (see theorem 37.4 in [**1**]):

$$b = 2^{2s} \times 3^t.p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime integers so that $p_i \equiv 1(\text{mod}\, 6)$, the q_j are also prime integers so that $q_j \equiv 5(\text{mod}\, 6)$. Then, as $b = 4p$:
 - If $t \geq 1 \Longrightarrow 3|b$, but $3|a$, then the contradiction with a, b coprime.

** C-2-2-1- Hence, we suppose that p is written under the form:

$$p = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1(\text{mod}6)$ and $q_j \equiv 5(\text{mod}6)$. Finally, we obtain that $p \equiv 1(\text{mod}\, 6)$. We will verify if this condition does not give contradictions.

We will present the table of the value modulo 6 of $p = A^{2m} + A^mB^n + B^{2n}$ in function of the values of $A^m, B^n(\text{mod}\, 6)$. We obtain the table below:

A^m, B^n	0	1	2	3	4	5
0	0	**1**	4	3	4	**1**
1	**1**	3	**1**	**1**	3	**1**
2	4	**1**	0	**1**	4	3
3	3	**1**	**1**	3	**1**	**1**
4	4	3	4	**1**	0	**1**
5	**1**	**1**	3	**1**	**1**	3

TABLE 1. Table of $p \pmod 6$

** C-2-2-1-1- Case $A^m \equiv 0 (\mod 6) \Longrightarrow 2|(A^m = a") \Longrightarrow 2|(a' = a"^2) \Longrightarrow 2|a$, but $2|b$, then the contradiction with a, b coprime. All the cases of the first line of the table 1 are to reject.

** C-2-2-1-2- Case $A^m \equiv 1(\mod 6)$ and $B^n \equiv 0(\mod 6)$, then $2|B^n \Longrightarrow B^n = 2B'$ and p is written as $p = (A^m + B')^2 + 3B'^2$ with $(p, 3) = 1$, if not $3|p$, then $3|b$, but $3|a$, then the contradiction with a, b coprime. Hence, the pair $(A^m + B', B')$ is solution of the Diophantine equation:

$$x^2 + 3y^2 = p \tag{1.106}$$

The solution $x = A^m + B', y = B'$ is unique because $x - y$ verify $x - y = A^m$. If (u, v) another pair solution of (1.106), with $u, v \in \mathbb{N}^*$, then we obtain:

$$u^2 + 3v^2 = p$$
$$u - v = A^m$$

Then $u = v + A^m$ and we obtain the equation of second degree $4v^2 + 2vA^m - 2B'(A^m+2B') = 0$ that gives as positive root $v_1 = B' = y$, then $u = A^m+B' = x$. It follows that p in (1.106) has an unique representation under the form X^2+3Y^2 with $X, 3Y$ coprime. As p is an odd integer number, we applique one of Euler's theorems on convenient numbers "numerus idoneus" (see [**2**],[**3**]) : *Let m be an odd number relatively prime to n which is properly represented by x^2+ny^2. If the equation $m = x^2+ny^2$ has only one solution with $x, y > 0$, then m is a prime number.* Then p is prime and $4p$ has an unique representation (we put $U = 2u$, $V = 2v$, with $U^2 + 3V^2 = 4p$ and $U - V = 2A^m$). But

$b = 4p \Longrightarrow \lambda^2 + 3a"^2 = (2(A^m + B'))^2 + 3(2B')^2$, the representation of $4p$ is unique gives:

$$\lambda = 2(A^m + B') = 2a" + B^n = 2a" + B^n$$
$$and \quad a" = 2B' = B^n = A^m$$

But $A^m > B^n$, then the contradiction.

** C-2-2-1-3- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-4- Case $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$, then $3|B^n \Longrightarrow B^n = 3B'$. We can write $b = 4p = (2A^m + 3B')^2 + 3(3B')^2 = \lambda^2 + 3a"^2$. The unique representation of b as $x^2 + 3y^2 = \lambda^2 + 3a"^2 \Longrightarrow a" = A^m = 3B' = B^n$, then the contradiction with $A^m > B^n$.

** C-2-2-1-5- Case $A^m \equiv 1(\mathrm{mod} 6)$ and $B^n \equiv 5(\mathrm{mod} 6)$, then $C^l \equiv 0(\bmod 6) \Longrightarrow 2|C^l$, see C-2-2-1-2-.

** C-2-2-1-6- Case $A^m \equiv 2(\mathrm{mod} 6) \Longrightarrow 2|a" \Longrightarrow 2|a$, but $2|b$, then the contradiction with a, b coprime.

** C-2-2-1-7- Case $A^m \equiv 3(\mathrm{mod} 6)$ and $B^n \equiv 1(\mathrm{mod} 6)$, then $C^l \equiv 4(\bmod 6) \Longrightarrow 2|C^l \Longrightarrow C^l = 2C'$, we can write that $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-8- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-9- Case $A^m \equiv 3(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-10- Case $A^m \equiv 3(\mathrm{mod} 6)$ and $B^n \equiv 5(\mathrm{mod} 6)$, then $C^l \equiv 2(\bmod 6) \Longrightarrow 2|C^l$, see C-2-2-1-2-.

** C-2-2-1-11- Case $A^m \equiv 4(\mathrm{mod} 6) \Longrightarrow 2|a" \Longrightarrow 2|a$, but $2|b$, then the contradiction with a, b coprime.

** C-2-2-1-12- Case $A^m \equiv 5(\mathrm{mod}\, 6)$ and $B^n \equiv 0(\mathrm{mod}\, 6)$, then B^n is even, see C-2-2-1-2-.

** C-2-2-1-13- Case $A^m \equiv 5(\mathrm{mod} 6)$ and $B^n \equiv 1(\mathrm{mod} 6)$, then $C^l \equiv 0(\mathrm{mod}\, 6) \Longrightarrow 2|C^l$, see C-2-2-1-2-.

** C-2-2-1-14- Case $A^m \equiv 5(\mathrm{mod} 6)$ and $B^n \equiv 3(\mathrm{mod} 6)$, then $C^l \equiv 2(\mathrm{mod}\, 6) \Longrightarrow 2|C^l \Longrightarrow C^l = 2C'$, p is written as $p = (C' - B^n)^2 + 3C'^2$, see C-2-2-1-2-.

** C-2-2-1-15- Case $A^m \equiv 5(\mathrm{mod}\, 6)$ and $B^n \equiv 4(\mathrm{mod}\, 6)$, then B^n is even, see C-2-2-1-2-.

We have achieved the study all the cases of the table 1 giving contradictions.

Then the case $k_3 = 1$ is impossible.

1.4.7. Case $3|a$ **and** $b = 2p'$ $b \neq 2$ **with** $p'|p$ **:** — $3|a \Longrightarrow a = 3a'$, $b = 2p'$ with $p = k.p'$, then:

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{6p'} = 2.k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - 2a')$$

As $p = b.p'$, and $p' > 1$, then we have:

$$B^nC^l = k(p' - 2a') \tag{1.107}$$

$$\text{and} \quad A^{2m} = 2k.a' \tag{1.108}$$

** D-1- We suppose that k is prime.

** D-1-1- If $k = 2$, then we have $p = 2p' = b \Longrightarrow 2|b$, but $A^{2m} = 4a' = (A^m)^2 \Longrightarrow A^m = 2a"$ with $a' = a"^2$, then $2|a" \Longrightarrow 2|(a = 3a"^2)$, it follows the contradiction with a, b coprime.

** D-1-2- We suppose $k \neq 2$. From $A^{2m} = 2k.a' = (A^m)^2 \Longrightarrow k|a'$ and $2|a' \Longrightarrow a' = 2.k.a"^2 \Longrightarrow A^m = 2.k.a"$. Then $k|A^m \Longrightarrow k|A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{im}A_1^m = 2ka" \Longrightarrow 2a" = k^{im-1}A_1^m$. From $B^nC^l = k(p' - 2a') \Longrightarrow k|(B^nC^l) \Longrightarrow k|B^n$ or $k|C^l$.

** D-1-2-1- We suppose that $k|B^n \Longrightarrow k|B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. It follows $k^{nj-1}B_1^nC^l = p' - 2a' = p' - 4ka"^2$. As $n \geq 3 \Longrightarrow nj - 1 \geq 2$, then $k|p'$ but $k \neq 2 \Longrightarrow k|(2p' = b)$, but $k|a' \Longrightarrow k|(3a' = a)$. It follows the contradiction with a, b coprime.

** D-1-2-2- If $k|C^l$ we obtain the identical results.

** D-2- We suppose that k is not prime. Let ω be an integer prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.107-1.108) become:

$$B^nC^l = \omega^s.k_1(p' - 2a')$$
$$\text{and} \quad A^{2m} = 2\omega^s.k_1.a'$$

** D-2-1- We suppose that $\omega = 2$, then we have the equations:

$$A^{2m} = 2^{s+1}.k_1.a' \tag{1.109}$$
$$B^nC^l = 2^s.k_1(p' - 2a') \tag{1.110}$$

** D-2-1-1- Case: $2|a' \Longrightarrow 2|a$, but $2|b$, then the contradiction with a, b coprime.

** D-2-1-2- Case: $2 \nmid a'$. As $2 \nmid k_1$, the equation (1.109) gives $2|A^{2m} \Longrightarrow A = 2^iA_1$, with $i \geq 1$ and $2 \nmid A_1$. It follows that $2im = s + 1$.

** D-2-1-2-1- We suppose that $2 \nmid (p' - 2a') \Longrightarrow 2 \nmid p'$. From the equation (1.110), we obtain that $2|B^nC^l \Longrightarrow 2|B^n$ or $2|C^l$.

** D-2-1-2-1-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$ and $j \geq 1$, then $B_1^nC^l = 2^{s-jn}k_1(p' - 2a')$:

- If $s - jn \geq 1$, then $2|C^l \Longrightarrow 2|C$, and no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$, and the conjecture (3.1) is verified.

- If $s-jn\leq 0$, from $B_1^nC^l=2^{s-jn}k_1(p'-2a')\Longrightarrow 2\nmid C^l$, then the contradiction with $C^l=2^{im}A_1^m+2^{jn}B_1^n\Longrightarrow 2|C^l$.

** D-2-1-2-1-2- Using the same method of the proof above, we obtain the identical results if $2|C^l$.

** D-2-1-2-2- We suppose now that $2|(p'-2a')\Longrightarrow p'-2a'=2^\mu.\Omega$, with $\mu\geq 1$ and $2\nmid\Omega$. We recall that $2\nmid a'$. The equation (1.110) is written as:

$$B^nC^l=2^{s+\mu}.k_1.\Omega$$

This last equation implies that $2|(B^nC^l)\Longrightarrow 2|B^n$ or $2|C^l$.

** D-2-1-2-2-1- We suppose that $2|B^n\Longrightarrow 2|B\Longrightarrow B=2^jB_1$ with $j\geq 1$ and $2\nmid B_1$. Then $B_1^nC^l=2^{s+\mu-jn}.k_1.\Omega$:

- If $s+\mu-jn\geq 1$, then $2|C^l\Longrightarrow 2|C$, no contradiction with $C^l=2^{im}A_1^m+2^{jn}B_1^n$, and the conjecture (3.1) is verified.

- If $s+\mu-jn\leq 0$, from $B_1^nC^l=2^{s+\mu-jn}k_1.\Omega\Longrightarrow 2\nmid C^l$, then contradiction with $C^l=2^{im}A_1^m+2^{jn}B_1^n\Longrightarrow 2|C^l$.

** D-2-1-2-2-2- We obtain the identical results if $2|C^l$.

** D-2-2- We suppose that $\omega\neq 2$. We have then the equations:

$$A^{2m}=2\omega^s.k_1.a' \tag{1.111}$$

$$B^nC^l=\omega^s.k_1.(p'-2a') \tag{1.112}$$

As $\omega\neq 2$, from the equation (1.111), we have $2|(k_1.a')$. If $2|a'\Longrightarrow 2|a$, but $2|b$, then the contradiction with a,b coprime.

** D-2-2-1- Case: $2\nmid a'$ and $2|k_1\Longrightarrow k_1=2^\mu.\Omega$ with $\mu\geq 1$ and $2\nmid\Omega$. From the equation (1.111), we have $2|A^{2m}\Longrightarrow 2|A\Longrightarrow A=2^iA_1$ with $i\geq 1$ and $2\nmid A_1$, then $2im=1+\mu$. The equation (1.112) becomes:

$$B^nC^l=\omega^s.2^\mu.\Omega.(p'-2a') \tag{1.113}$$

From the equation (1.113), we obtain $2|(B^nC^l)\Longrightarrow 2|B^n$ or $2|C^l$.

** D-2-2-1-1- We suppose that $2|B^n\Longrightarrow 2|B\Longrightarrow B=2^jB_1$, with $j\in\mathbb{N}^*$ and $2\nmid B_1$.

** D-2-2-1-1-1- We suppose that $2 \nmid (p' - 2a')$, then we have $B_1^n C^l = \omega^s 2^{\mu - jn}\Omega(p' - 2a')$:

- If $\mu - jn \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $\mu - jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-1-2- We suppose that $2|(p' - 2a') \Longrightarrow p' - 2a' = 2^\alpha.P$, with $\alpha \in \mathbb{N}^*$ and $2 \nmid P$. It follows that $B_1^n C^l = \omega^s 2^{\mu+\alpha-jn}\Omega.P$:

- If $\mu + \alpha - jn \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $\mu + \alpha - jn \leq 0 \Longrightarrow 2 \nmid C^l$ then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** D-2-2-1-2- We suppose now that $2|C^n \Longrightarrow 2|C$. Using the same method described above, we obtain the identical results.

1.4.8. Case $3|a$ and $b = 4p'$ $b \neq 2$ with $p'|p$: — $3|a \Longrightarrow a = 3a'$, $b = 4p'$ with $p = k.p'$, $k \neq 1$ if not $b = 4p$ this case has been studied (see paragraph 1.4.6), then we have :

$$A^{2m} = \frac{4.p}{3}.\frac{a}{b} = \frac{4.k.p'.3.a'}{12p'} = k.a'$$

We calculate B^nC^l:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3} - cos^2\frac{\theta}{3}\right) = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right)$$

but $\sqrt[3]{\rho^2} = \dfrac{p}{3}$, then using $cos^2\dfrac{\theta}{3} = \dfrac{3.a'}{b}$:

$$B^nC^l = \sqrt[3]{\rho^2}\left(3 - 4cos^2\frac{\theta}{3}\right) = \frac{p}{3}\left(3 - 4\frac{3.a'}{b}\right) = p.\left(1 - \frac{4.a'}{b}\right) = k(p' - a')$$

As $p = b.p'$, and $p' > 1$, we have :

$$B^nC^l = k(p' - a') \tag{1.114}$$

$$\text{and} \quad A^{2m} = k.a' \tag{1.115}$$

** E-1- We suppose that k is prime. From $A^{2m} = k.a' = (A^m)^2 \Longrightarrow k|a'$ and $a' = k.a"^2 \Longrightarrow A^m = k.a"$. Then $k|A^m \Longrightarrow k|A \Longrightarrow A = k^i.A_1$ with $i \geq 1$ and $k \nmid A_1$. $k^{mi}A_1^m = ka" \Longrightarrow a" = k^{mi-1}A_1^m$. From $B^nC^l = k(p' - a') \Longrightarrow k|(B^nC^l) \Longrightarrow k|B^n$ or $k|C^l$.

** E-1-1- We suppose that $k|B^n \Longrightarrow k|B \Longrightarrow B = k^j.B_1$ with $j \geq 1$ and $k \nmid B_1$. Then $k^{n.j-1}B_1^nC^l = p' - a'$. As $n.j - 1 \geq 2 \Longrightarrow k|(p' - a')$. But $k|a' \Longrightarrow k|a$, then $k|p' \Longrightarrow k|(4p' = b)$ and we arrive to the contradiction that a, b are coprime.

** E-1-2- We suppose that $k|C^l$, using the same method with the above hypothesis $k|B^n$, we obtain the identical results.

** E-2- We suppose that k is not prime.

** E-2-1- We take $k = 4 \Longrightarrow p = 4p' = b$, it is the case 1.4.3 studied above.

** E-2-2- We suppose that $k \geq 6$ not prime. Let ω be a prime so that $k = \omega^s.k_1$, with $s \geq 1$, $\omega \nmid k_1$. The equations (1.114-1.115) become:

$$B^nC^l = \omega^s.k_1(p' - a') \tag{1.116}$$

$$\text{and} \quad A^{2m} = \omega^s.k_1.a' \tag{1.117}$$

** E-2-2-1- We suppose that $\omega = 2$.

** E-2-2-1-1- If $2|a' \Longrightarrow 2|(3a' = a)$, but $2|(4p' = b)$, then the contradiction with a, b coprime.

** E-2-2-1-2- We consider that $2 \nmid a'$. From the equation (1.117), it follows that $2|A^{2m} \Longrightarrow 2|A \Longrightarrow A = 2^iA_1$ with $2 \nmid A_1$ and:

$$B^nC^l = 2^sk_1(p' - a')$$

** E-2-2-1-2-1- We suppose that $2 \nmid (p' - a')$, from the above expression, we have $2|(B^nC^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** E-2-2-1-2-1-1- If $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$ with $2 \nmid B_1$. Then $B_1^nC^l = 2^{2im-jn}k_1(p' - a')$:

- If $2im - jn \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im - jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2|C^l$.

** E-2-2-1-2-1-2- If $2|C^l \Longrightarrow 2|C$, using the same method described above, we obtain the identical results.

** E-2-2-1-2-2- We suppose that $2|(p'-a')$. As $2 \nmid a' \Longrightarrow 2 \nmid p'$, $2|(p'-a') \Longrightarrow p'-a' = 2^\alpha.P$ with $\alpha \geq 1$ and $2 \nmid P$. The equation (1.116) is written as :

$$B^nC^l = 2^{s+\alpha}k_1.P = 2^{2im+\alpha}k_1.P \tag{1.118}$$

then $2|(B^nC^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** E-2-2-1-2-2-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$, with $2 \nmid B_1$. The equation (1.118) becomes $B_1^nC^l = 2^{2im+\alpha-jn}k_1P$:

- If $2im+\alpha-jn \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im+\alpha-jn \leq 0 \Longrightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2|C^l$.

** E-2-2-1-2-2-2- We suppose that $2|C^l \Longrightarrow 2|C$. Using the same method described above, we obtain the identical results.

** E-2-2-2- We suppose that $\omega \neq 2$. We recall the equations:

$$A^{2m} = \omega^s.k_1.a' \tag{1.119}$$

$$B^nC^l = \omega^s.k_1(p'-a') \tag{1.120}$$

** E-2-2-2-1- We suppose that ω, a' are coprime, then $\omega \nmid a'$. From the equation (1.119), we have $\omega|A^{2m} \Longrightarrow \omega|A \Longrightarrow A = \omega^iA_1$ with $\omega \nmid A_1$ and $s = 2im$.

** E-2-2-2-1-1- We suppose that $\omega \nmid (p'-a')$. From the equation (1.120) above, we have $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** E-2-2-2-1-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$. Then $B_1^nC^l = 2^{2im-jn}k_1(p'-a')$:

- If $2im-jn \geq 1 \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im-jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \Longrightarrow \omega|C^l$.

** E-2-2-2-1-1-2- If $\omega|C^l \Longrightarrow \omega|C$, using the same method described above, we obtain the identical results.

** E-2-2-2-1-2- We suppose that $\omega|(p'-a') \Longrightarrow \omega \nmid p'$ if not $\omega|a'$, $\omega|(p'-a') \Longrightarrow p'-a' = \omega^\alpha.P$ with $\alpha \geq 1$ and $\omega \nmid P$. The equation (1.120) becomes :

$$B^nC^l = \omega^{s+\alpha}k_1.P = \omega^{2im+\alpha}k_1.P \tag{1.121}$$

then $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** E-2-2-2-1-2-1- We suppose that $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$, with $\omega \nmid B_1$. The equation (1.121) is written as $B_1^nC^l = 2^{2im+\alpha-jn}k_1P$:

- If $2im+\alpha-jn \geq 1 \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im+\alpha-jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \Longrightarrow \omega|C^l$.

** E-2-2-2-1-2-2- We suppose that $\omega|C^l \Longrightarrow \omega|C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2- We suppose that ω, a' are not coprime, then $a' = \omega^\beta.a$" with $\omega \nmid a$". The equation (1.119) becomes:

$$A^{2m} = \omega^sk_1a' = \omega^{s+\beta}k_1.a\text{''}$$

We have $\omega|A^{2m} \Longrightarrow \omega|A \Longrightarrow A = \omega^iA_1$ with $\omega \nmid A_1$ and $s+\beta = 2im$.

** E-2-2-2-2-1- We suppose that $\omega \nmid (p'-a') \Longrightarrow \omega \nmid p' \Longrightarrow \omega \nmid (b = 4p')$. From the equation (1.120), we obtain $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** E-2-2-2-2-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$. Then $B_1^nC^l = 2^{s-jn}k_1(p'-a')$:

- If $s-jn \geq 1 \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $s-jn \leq 0 \Longrightarrow \omega \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n \Longrightarrow \omega|C^l$.

** E-2-2-2-2-1-2- If $\omega|C^l \Longrightarrow \omega|C$, using the same method described above, we obtain the identical results.

** E-2-2-2-2-2- We suppose that $\omega|(p'-a'=p'-\omega^\beta.a") \Longrightarrow \omega|p' \Longrightarrow \omega|(4p'=b)$, but $\omega|a' \Longrightarrow \omega|a$. Then the contradiction with a,b coprime.

The study of the cases of 1.4.8 is achieved.

1.4.9. Case $3|a$ and $b|4p$:. — $a=3a'$ and $4p=k_1b$. As $A^{2m}=\dfrac{4p}{3}cos^2\dfrac{\theta}{3}=\dfrac{4p}{3}\dfrac{3a'}{b}=k_1a'$ and B^nC^l:

$$B^nC^l=\sqrt[3]{\rho^2}\left(3sin^2\frac{\theta}{3}-cos^2\frac{\theta}{3}\right)=\frac{p}{3}\left(3-4cos^2\frac{\theta}{3}\right)=\frac{p}{3}\left(3-4\frac{3a'}{b}\right)=\frac{k_1}{4}(b-4a')$$

As B^nC^l is an integer, we must obtain $4|k_1$, or $4|(b-4a')$ or $(2|k_1$ and $2|(b-4a'))$.
** F-1- If $k_1=1 \Rightarrow b=4p$: it is the case 1.4.6.

** F-2- If $k_1=4 \Rightarrow p=b$: it is the case 1.4.3.

** F-3- If $k_1=2$ and $2|(b-4a')$: in this case, we have $A^{2m}=2a' \Longrightarrow 2|a' \Longrightarrow 2|a$. $2|(b-4a') \Longrightarrow 2|b$ then the contradiction with a,b coprime.

** F-4- If $2|k_1$ and $2|(b-4a')$: $2|(b-4a') \Longrightarrow b-4a'=2^\alpha\lambda$, α and $\lambda\in\mathbb{N}^*\geq 1$ with $2\nmid\lambda$; $2|k_1 \Longrightarrow k_1=2^tk'_1$ with $t\geq 1\in\mathbb{N}^*$ with $2\nmid k'_1$ and we have:

$$A^{2m}=2^tk'_1a' \tag{1.122}$$

$$B^nC^l=2^{t+\alpha-2}k'_1\lambda \tag{1.123}$$

From the equation (1.122), we have $2|A^{2m} \Longrightarrow 2|A \Longrightarrow A=2^iA_1$, $i\geq 1$ and $2\nmid A_1$.
** F-4-1- We suppose that $t=\alpha=1$, then the equations (1.122-1.123) become :

$$A^{2m}=2k'_1a' \tag{1.124}$$

$$B^nC^l=k'_1\lambda \tag{1.125}$$

From the equation (1.124) it follows that $2|a' \Longrightarrow 2|(a=3a')$. But $b=4a'+2\lambda \Longrightarrow 2|b$, then the contradiction with a,b coprime.

** F-4-2- We suppose that $t+\alpha-2\geq 1$ and we have the expressions:

$$A^{2m}=2^tk'_1a' \tag{1.126}$$

$$B^nC^l=2^{t+\alpha-2}k'_1\lambda \tag{1.127}$$

** F-4-2-1- We suppose that $2|a' \Longrightarrow 2|a$, but $b = 2^\alpha\lambda + 4a' \Longrightarrow 2|b$, then the contradiction with a, b coprime.

** F-4-2-2- We suppose that $2 \nmid a'$. From (1.126), we have $2|A^{2m} \Longrightarrow 2|A \Longrightarrow A = 2^i A_1$ and $B^nC^l = 2^{t+\alpha-2}k_1'\lambda \Longrightarrow 2|B^nC^l \Longrightarrow 2|B^n$ or $2|C^l$.

** F-4-2-2-1- We suppose that $2|B^n$. We have $2|B \Longrightarrow B = 2^jB_1$, $j \geq 1$ and $2 \nmid B_1$. The equation (1.127) becomes $B_1^nC^l = 2^{t+\alpha-2-jn}k_1'\lambda$:

- If $t+\alpha-2-jn > 0 \Longrightarrow 2|C^l \Longrightarrow 2|C$, no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $t+\alpha-2-jn < 0 \Longrightarrow 2|k_1'\lambda$, but $2 \nmid k_1'$ and $2 \nmid \lambda$. Then this case is impossible.
- If $t+\alpha-2-jn = 0 \Longrightarrow B_1^nC^l = k_1'\lambda \Longrightarrow 2 \nmid C^l$ then it is a contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** F-4-2-2-2- We suppose that $2|C^l$. We use the same method described above, we obtain the identical results.

** F-5- We suppose that $4|k_1$ with $k_1 > 4 \Rightarrow k_1 = 4k_2'$, we have :

$$A^{2m} = 4k_2'a' \tag{1.128}$$

$$B^nC^l = k_2'(b - 4a') \tag{1.129}$$

** F-5-1- We suppose that k_2' is prime, from (1.128), we have $k_2'|a'$. From (1.129), $k_2'|(B^nC^l) \Longrightarrow k_2'|B^n$ or $k_2'|C^l$.

** F-5-1-1- We suppose that $k_2'|B^n \Longrightarrow k_2'|B \Longrightarrow B = k_2'^{\beta}.B_1$ with $\beta \geq 1$ and $k_2' \nmid B_1$. It follows that we have $k_2'^{n\beta-1}B_1^nC^l = b - 4a' \Longrightarrow k_2'|b$ then the contradiction with a, b coprime.

** F-5-1-2- We obtain identical results if we suppose that $k_2'|C^l$.

** F-5-2- We suppose that k_2' is not prime.

** F-5-2-1- We suppose that k_2' and a' are coprime. From (1.128), k_2' can be written under the form $k_2' = q_1^{2j}.q_2^2$ and $q_1 \nmid q_2$ and q_1 prime. We have $A^{2m} = 4q_1^{2j}.q_2^2a' \Longrightarrow q_1|A$ and $B^nC^l = q_1^{2j}.q_2^2(b - 4a') \Longrightarrow q_1|B^n$ or $q_1|C^l$.

** F-5-2-1-1- We suppose that $q_1|B^n \Longrightarrow q_1|B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^nC^l = q_1^{2j-fn}q_2^2(b-4a')$:
- If $2j - f.n \geq 1 \Longrightarrow q_1|C^l \Longrightarrow q_1|C$ but $C^l = A^m + B^n$ gives also $q_1|C$ and the conjecture (3.1) is verified.
- If $2j - f.n = 0$, we have $B_1^nC^l = q_2^2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1|C$, then $q_1|(b-4a')$. As q_1 and a' are coprime, then $q_1 \nmid b$, and the conjecture (3.1) is verified.
- If $2j - f.n < 0 \Longrightarrow q_1|(b-4a') \Longrightarrow q_1 \nmid b$ because a' is coprime with q_1, and $C^l = A^m + B^n$ gives $q_1|C$, and the conjecture (3.1) is verified.

** F-5-2-1-2- We obtain identical results if we suppose that $q_1|C^l$.

** F-5-2-2- We suppose that k_2', a' are not coprime. Let q_1 be a prime so that $q_1|k_2'$ and $q_1|a'$. We write k_2' under the form $q_1^j.q_2$ with $j \geq 1$, $q_1 \nmid q_2$. From $A^{2m} = 4k_2'a' \Longrightarrow q_1|A^{2m} \Longrightarrow q_1|A$. Then from $B^nC^l = q_1^jq_2(b-4a')$, it follows that $q_1|(B^nC^l) \Longrightarrow q_1|B^n$ or $q_1|C^l$.

** F-5-2-2-1- We suppose that $q_1|B^n \Longrightarrow q_1|B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then, we have $q_1^{n\beta}B_1^nC^l = q_1^jq_2(b-4a') \Longrightarrow B_1^nC^l = q_1^{j-n\beta}q_2(b-4a')$.
- If $j - n\beta \geq 1$, then $q_1|C^l \Longrightarrow q_1|C$, but $C^l = A^m + B^n$ gives $q_1|C$, then the conjecture (3.1) is verified.
- If $j - n\beta = 0$, we obtain $B_1^nC^l = q_2(b-4a')$, but $C^l = A^m + B^n$ gives $q_1|C$, then $q_1|(b-4a') \Longrightarrow q_1|b$ because $q_1|a' \Longrightarrow q_1|a$, then the contradiction with a, b coprime.
- If $j - n\beta < 0 \Longrightarrow q_1|(b-4a') \Longrightarrow q_1|b$, because $q_1|a' \Longrightarrow q_1|a$, then the contradiction with a, b coprime.

** F-5-2-2-2- We obtain identical results if we suppose that $q_1|C^l$.

** F-6- If $4 \nmid (b-4a')$ and $4 \nmid k_1$ it is impossible. We suppose that $4|(b-4a') \Rightarrow 4|b$, and $b - 4a' = 4^t.g$, $t \geq 1$ with $4 \nmid g$, then we have :

$$A^{2m} = k_1a'$$
$$B^nC^l = k_1.4^{t-1}.g$$

** F-6-1- We suppose that k_1 is prime. From $A^{2m} = k_1a'$ we deduce easily that $k_1|a'$. From $B^nC^l = k_1.4^{t-1}.g$ we obtain that $k_1|(B^nC^l) \Longrightarrow k_1|B^n$ or $k_1|C^l$.

** F-6-1-1- We suppose that $k_1|B^n \Longrightarrow k_1|B \Longrightarrow B = k_1^j.B_1$ with $j > 0$ and $k_1 \nmid B_1$, then $k_1^{n.j}B_1^nC^l = k_1.4^{t-1}.g \Longrightarrow k_1^{n.j-1}B_1^nC^l = 4^{t-1}.g$. But $n \geq 3$ and $j \geq 1$, then $n.j - 1 \geq 2$. We deduce as $k_1 \neq 2$ that $k_1|g \Longrightarrow k_1|(b - 4a')$, but $k_1|a' \Longrightarrow k_1|b$, then the contradiction with a, b coprime.

** F-6-1-2- We obtain identical results if we suppose that $k_1|C^l$.

** F-6-2- We suppose that k_1 is not prime $\neq 4$, ($k_1 = 4$ see case F-2, above) with $4 \nmid k_1$.

** F-6-2-1- If $k_1 = 2k'$ with k' odd > 1. Then $A^{2m} = 2k'a' \Longrightarrow 2|a' \Longrightarrow 2|a$, as $4|b$ it follows the contradiction with a, b coprime.

** F-6-2-2- We suppose that k_1 is odd with k_1 and a' coprime. We write k_1 under the form $k_1 = q_1^j.q_2$ with $q_1 \nmid q_2$, q_1 prime and $j \geq 1$. $B^nC^l = q_1^j.q_2 4^{t-1}g \Longrightarrow q_1|B^n$ or $q_1|C^l$.

** F-6-2-2-1- We suppose that $q_1|B^n \Longrightarrow q_1|B \Longrightarrow B = q_1^f.B_1$ with $q_1 \nmid B_1$. We obtain $B_1^nC^l = q_1^{j-f.n}q_2 4^{t-1}g$.
- If $j - f.n \geq 1 \Longrightarrow q_1|C^l \Longrightarrow q_1|C$, but $C^l = A^m + B^n$ gives also $q_1|C$ and the conjecture (3.1) is verified.
- If $j - f.n = 0$, we have $B_1^nC^l = q_2 4^{t-1}g$, but $C^l = A^m + B^n$ gives $q_1|C$, then $q_1|(b - 4a')$. As q_1 and a' are coprime then $q_1 \nmid b$ and the conjecture (3.1) is verified.
- If $j - f.n < 0 \Longrightarrow q_1|(b - 4a') \Longrightarrow q_1 \nmid b$ because q_1, a' are primes. $C^l = A^m + B^n$ gives $q_1|C$ and the conjecture (3.1) is verified.

** F-6-2-2-2- We obtain identical results if we suppose that $q_1|C^l$.

** F-6-2-3- We suppose that k_1 and a' are not coprime. Let q_1 be a prime so that $q_1|k_1$ and $q_1|a'$. We write k_1 under the form $q_1^j.q_2$ with $q_1 \nmid q_2$. From $A^{2m} = k_1a' \Longrightarrow q_1|A^{2m} \Longrightarrow q_1|A$. From $B^nC^l = q_1^jq_2(b - 4a')$, it follows that $q_1|(B^nC^l) \Longrightarrow q_1|B^n$ or $q_1|C^l$.

** F-6-2-3-1- We suppose that $q_1|B^n \Longrightarrow q_1|B \Longrightarrow B = q_1^\beta.B_1$ with $\beta \geq 1$ and $q_1 \nmid B_1$. Then we have $q_1^{n\beta}B_1^nC^l = q_1^jq_2(b - 4a') \Longrightarrow B_1^nC^l = q_1^{j-n\beta}q_2(b - 4a')$:
- If $j - n\beta \geq 1$, then $q_1|C^l \Longrightarrow q_1|C$, but $C^l = A^m + B^n$ gives $q_1|C$, and the conjecture (3.1) is verified.

- If $j - n\beta = 0$, we obtain $B_1^n C^l = q_2(b - 4a')$, but $q_1|A$ and $q_1|B$ then $q_1|C$ and we obtain $q_1|(b - 4a') \Longrightarrow q_1|b$ because $q_1|a' \Longrightarrow q_1|a$, then the contradiction with a, b coprime.

- If $j - n\beta < 0 \Longrightarrow q_1|(b - 4a') \Longrightarrow q_1|b$, then the contradiction with a, b coprime.

** F-6-2-3-2- We obtain identical results as above if we suppose that $q_1|C^l$.

1.5. Hypothèse: $\{3|p \quad and \quad b|4p\}$

1.5.1. Case $b = 2$ and $3|p$: — $3|p \Rightarrow p = 3p'$ with $p' \neq 1$ because $3 \ll p$, and $b = 2$, we obtain:

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3b} = \frac{4.p'.a}{2} = 2.p'.a$$

As:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{2} < \frac{3}{4} \Rightarrow 1 < 2a < 3 \Rightarrow a = 1 \Longrightarrow cos^2\frac{\theta}{3} = \frac{1}{2}$$

but this case was studied (see case 1.3.1.2).

1.5.2. Case $b = 4$ and $3|p$: — we have $3|p \Longrightarrow p = 3p'$ with $p' \in \mathbb{N}^*$, it follows :

$$A^{2m} = \frac{4p.a}{3b} = \frac{4.3p'.a}{3 \times 4} = p'.a$$

and:

$$\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{4} < \frac{3}{4} \Rightarrow 1 < a < 3 \Rightarrow a = 2$$

as a, b are coprime, then the case $b = 4$ and $3|p$ is impossible.

1.5.3. Case: $b \neq 2, b \neq 4, b \neq 3$, $b|p$ and $3|p$: — As $3|p$, then $p = 3p'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{b} = \frac{4p'a}{b}$$

We consider the case: $b|p' \Longrightarrow p' = bp"$ and $p" \neq 1$ (If $p" = 1$, then $p = 3b$, see paragraph 1.5.8 Case $k' = 1$). Finally, we obtain:

$$A^{2m} = \frac{4bp"a}{b} = 4ap"\,; \quad B^nC^l = p".(3b - 4a)$$

** G-1- We suppose that $p"$ est prime, then $A^{2m} = 4ap" = (A^m)^2 \Longrightarrow p"|a$. But $B^nC^l = p"(3b - 4a) \Longrightarrow p"|B^n$ or $p"|C^l$.

** G-1-1- If $p"|B^n \Longrightarrow p"|B \Longrightarrow B = p"B_1$ with $B_1 \in \mathbb{N}^*$. Then $p"^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $p"|a$, then $p"|3b \Longrightarrow p" = 3$ or $p"|b$.

** G-1-1-1- If $p" = 3 \Longrightarrow 3|a$, with a that we write as $a = 3a'^2$, but $A^m = 6a' \Longrightarrow 3|A^m \Longrightarrow 3|A \Longrightarrow A = 3A_1$, then $3^{m-1}A_1^m = 2a' \Longrightarrow 3|a' \Longrightarrow a' = 3a"$. As $p"^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n > 2 \Longrightarrow n - 2 \geq 1$, then $3|b$ and the contradiction with a, b coprime.

** G-1-1-2- We suppose that $p"|b$, as $p"|a$, then the contradiction with a, b coprime.

** G-1-2- If we suppose $p"|C^l$, we obtain identical results (contradictions).

** G-2- We consider now that $p"$ is not prime.

** G-2-1- $p", a$ coprime: $A^{2m} = 4ap" \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $p" = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$, then $2|a'$ or $2|p_1$.

** G-2-1-1- We suppose that $2|a'$, then $2|a' \Longrightarrow 2 \nmid p_1$, but $p" = p_1^2$.

** G-2-1-1-1- If p_1 est prime, it is impossible with $A^m = 2a'.p_1$.

** G-2-1-1-2- We suppose that p_1 is not prime so we can write $p_1 = \omega^m \Longrightarrow p" = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b - 4a)$.

** G-2-1-1-2-1- If ω est prime, $\omega \neq 2$, then $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** G-2-1-1-2-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

** G-2-1-1-2-1-1-1- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, and $\omega|(3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then ω, a are coprime, it follows $\omega \nmid (3b)$, then $\omega \neq 3$ and $\omega \nmid b$, the conjecture (3.1) is verified.

** G-2-1-1-2-1-1-2- If $2m - nj \geq 1$, using the method as above, we obtain $\omega|C^l \Longrightarrow \omega|C$ and $\omega|(3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$, then the conjecture (3.1) is verified.

** G-2-1-1-2-1-1-3- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = 3b - 4a$. From $A^m + B^n = C^l \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, then $C = \omega^h.C_1$, with $\omega \nmid C_1$, we obtain $\omega^{n.j-2m+h.l}B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega|B_1^nC_1^l$ then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. It follows $n.j - 2m + h.l > 0$ and $\omega|(3b - 4a)$ with ω, a, b coprime and the conjecture is verified.

** G-2-1-1-2-1-2- Using the same method above, we obtain identical results if $\omega|C^l$.

** G-2-1-1-2-2- We suppose that $p" = \omega^{2m}$ and ω is not prime. We write $\omega = \omega_1^f.\Omega$ with ω_1 prime $\nmid \Omega$, $f \geq 1$, and $\omega_1|A$. Then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b - 4a) \Longrightarrow \omega_1|(B^nC^l) \Longrightarrow \omega_1|B^n$ or $\omega_1|C^l$.

** G-2-1-1-2-2-1- If $\omega_1|B^n \Longrightarrow \omega_1|B \Longrightarrow B = \omega_1^jB_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.m-nj}\Omega^{2m}(3b - 4a)$:

** G-2-1-1-2-2-1-1- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1|C^l \Longrightarrow \omega_1|C$, and $\omega_1|(3b - 4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, it follows $\omega_1 \nmid (3b)$, then $\omega_1 \neq 3$ and $\omega_1 \nmid b$, and the conjecture (3.1) is verified.

** G-2-1-1-2-2-1-2- If $2f.m - n.j \geq 1$, we have $\omega_1|C^l \Longrightarrow \omega_1|C$ and $\omega_1|(3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, it follows that the conjecture (3.1) is verified.

** G-2-1-1-2-2-1-3- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(3b - 4a)$. As $\omega_1|C$ using $C^l = A^m + B^n$, then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(3b - 4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1|B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1|(3b - 4a)$ with ω_1, a, b coprime and the conjecture (3.1) is verified.

** G-2-1-1-2-2-2- Using the same method above, we obtain identical results if $\omega_1|C^l$.

** G-2-1-2- We suppose that $2|p_1$: then $2|p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $p" = p_1^2$.

** G-2-1-2-1- We suppose that $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2|a'$, then the contradiction with a, b coprime.

** G-2-1-2-2- We suppose that p_1 is not prime and $2|p_1$. As $A^m = 2a'p_1$, p_1 can written as $p_1 = 2^{m-1}\omega^m \Longrightarrow p" = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2|B^n$ or $2|C^l$.

** G-2-1-2-2-1- We suppose that $2|B^n \Longrightarrow 2|B$. As $2|A$, then $2|C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2|(3b-4a) \Longrightarrow 2|b$ but as $2 \nmid a$ there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** G-2-1-2-2-2- We suppose that $2|C^l$, using the same method above, we obtain identical results.

** G-2-2- We suppose that $p", a$ are not coprime: let ω be an integer prime so that $\omega|a$ and $\omega|p"$.

** G-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ap" \Longrightarrow 3|A$, or $3|p$, As $p = A^{2m} + B^{2n} + A^mB^n \Longrightarrow 3|B^{2n} \Longrightarrow 3|B$, then $3|C^l \Longrightarrow 3|C$. We write $A = 3^iA_1$, $B = 3^jB_1$, $C = 3^hC_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^mB_1^n = 3^k.g$ with $k = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3)|a$ and $(\omega = 3)|p"$ that gives $a = 3^\alpha a_1$, $3 \nmid a_1$ and $p" = 3^\mu p_1$, $3 \nmid p_1$ with $A^{2m} = 4ap" = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_1 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.p" = 3b.3^\mu p_1 = 3^{\mu+1}.b.p_1$, the exponent of the factor 3 of p is k, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the term b, with $\beta \geq 0$, let $min(2im, 2jn, im + jn) = \mu + 1 + \beta$ and we recall that $\alpha + \mu = 2im$. But $B^nC^l = p"(3b-4a)$, we obtain $3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_1(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l \Longrightarrow 3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call $\epsilon = min(im, jn)$, we have $\epsilon = hl = min(im, jn)$. We obtain the conditions:

$$k = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{1.130}$$

$$\alpha + \mu = 2im \tag{1.131}$$

$$\epsilon = hl = min(im, jn)$$

$$3^{(nj+hl)}B_1^nC_1^l = 3^{\mu+1}p_1(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$$

** G-2-2-1-1- $\alpha=1 \Longrightarrow a=3a_1$ and $3\nmid a_1$, the equation (1.131) becomes:

$$1+\mu=2im$$

and the first equation (1.130) is written as:

$$k=min(2im,2jn,im+jn)=2im+\beta$$

- If $k=2im \Longrightarrow \beta=0$ then $3\nmid b$. We obtain $2im\leq 2jn \Longrightarrow im\leq jn$, and $2im\leq im+jn \Longrightarrow im\leq jn$. The third equation gives $hl=im$ and the last equation gives $nj+hl=\mu+1=2im \Longrightarrow im=nj$, then $im=nj=hl$ and $B_1^nC_1^l=p_1(b-4a_1)$. As a,b are coprime, the conjecture (3.1) is verified.

- If $k=2jn$ or $k=im+jn$, we obtain $\beta=0$, $im=jn=hl$ and $B_1^nC_1^l=p_1(b-4a_1)$. As a,b are coprime, the conjecture (3.1) is verified.

** G-2-2-1-2- $\alpha>1 \Longrightarrow \alpha\geq 2$.

- If $k=2im \Longrightarrow 2im=\mu+1+\beta$, but $\mu=2im-\alpha$ that gives $\alpha=1+\beta\geq 2 \Longrightarrow \beta\neq 0 \Longrightarrow 3|b$, but $3|a$ then the contradiction with a,b coprime.

- If $k=2jn=\mu+1+\beta\leq 2im \Longrightarrow \mu+1+\beta\leq \mu+\alpha \Longrightarrow 1+\beta\leq\alpha \Longrightarrow \beta\geq 1$. If $\beta\geq 1 \Longrightarrow 3|b$ but $3|a$, then the contradiction with a,b coprime.

- If $k=im+jn \Longrightarrow im+jn\leq 2im \Longrightarrow jn\leq im$, and $im+jn\leq 2jn \Longrightarrow im\leq jn$, then $im=jn$. As $k=im+jn=2im=1+\mu+\beta$ and $\alpha+\mu=2im$, we obtain $\alpha=1+\beta\geq 2 \Longrightarrow \beta\geq 1 \Longrightarrow 3|b$, then the contradiction with a,b coprime.

** G-2-2-2- We suppose that $\omega\neq 3$. We write $a=\omega^\alpha a_1$ with $\omega\nmid a_1$ and $p"=\omega^\mu p_1$ with $\omega\nmid p_1$. As $A^{2m}=4ap"=4\omega^{\alpha+\mu}.a_1.p_1 \Longrightarrow \omega|A \Longrightarrow A=\omega^iA_1$, $\omega\nmid A_1$. But $B^nC^l=p"(3b-4a)=\omega^\mu p_1(3b-4a) \Longrightarrow \omega|B^nC^l \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** G-2-2-2-1- We suppose that $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B=\omega^jB_1$ and $\omega\nmid B_1$. From $A^m+B^n=C^l \Longrightarrow \omega|C^l \Longrightarrow \omega|C$. As $p=bp'=3bp"=3\omega^\mu bp_1=\omega^k(\omega^{2im-k}A_1^{2m}+\omega^{2jn-k}B_1^{2n}+\omega^{im+jn-k}A_1^mB_1^n)$ with $k=min(2im,2jn,im+jn)$. Then:

- If $k=\mu$, then $\omega\nmid b$ and the conjecture (3.1) is verified.

- If $k>\mu$, then $\omega|b$, but $\omega|a$ then the contradiction with a,b coprime.

- If $k<\mu$, it follows from:

$$3\omega^\mu bp_1=\omega^k(\omega^{2im-k}A_1^{2m}+\omega^{2jn-k}B_1^{2n}+\omega^{im+jn-k}A_1^mB_1^n)$$

that $\omega|A_1$ or $\omega|B_1$ then the contradiction with $\omega\nmid A_1$ or $\omega\nmid B_1$.

** G-2-2-2-2- If $\omega|C^l \Longrightarrow \omega|C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega|(C^l - A^m) \Longrightarrow \omega|B$. Then, using the same method as for the case G-2-2-2-1-, we obtain identical results.

1.5.4. Case $b = 3$ and $3|p$: — As $3|p \Longrightarrow p = 3p'$, We write :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{4 \times 3p'}{3}\frac{a}{3} = \frac{4p'a}{3}$$

As A^{2m} is an integer and a, b are coprime and $cos^2\frac{\theta}{3} < 1$ (see equation (1.35)), then we have necessary $3|p' \Longrightarrow p' = 3p"$ with $p" \neq 1$, if not $p = 3p' = 3 \times 3p" = 9$, but $9 \ll (p = A^{2m} + B^{2n} + A^m B^n)$, the hypothesis $p" = 1$ is impossible, then $p" > 1$, and we obtain:

$$A^{2m} = \frac{4p'a}{3} = \frac{4 \times 3p"a}{3} = 4p"a\,; \quad B^n C^l = p".(9 - 4a)$$

As $\frac{1}{4} < cos^2\frac{\theta}{3} = \frac{a}{b} = \frac{a}{3} < \frac{3}{4} \Longrightarrow 3 < 4a < 9 \Longrightarrow$ as $a > 1, a = 2$ and we obtain:

$$A^{2m} = 4p"a = 8p"\,; \quad B^n C^l = \frac{3p"(9 - 4a)}{3} = p" \tag{1.132}$$

The two last equations above imply that $p"$ is not a prime. We can write $p"$ as :$p" = \prod_{i\in I} p_i^{\alpha_i}$ where p_i are distinct primes, α_i elements of $\mathbb{N}$ and $i \in I$ a finite set of indices. We can write also $p" = p_1^{\alpha_1}.q_1$ with $p_1 \nmid q_1$. From (1.132), we have $p_1|A$ and $p_1|B^nC^l \Longrightarrow p_1|B^n$ or $p_1|C^l$.

** H-1- We suppose that $p_1|B^n \Longrightarrow B = p_1^{\beta_1}.B_1$ with $p_1 \nmid B_1$ and $\beta_1 \geq 1$. Then, we obtain $B_1^n C^l = p_1^{\alpha_1 - n\beta_1}.q_1$ with the following cases:

- If $\alpha_1 - n\beta_1 \geq 1 \Longrightarrow p_1|C^l \Longrightarrow p_1|C$, in accord with $p_1|(C^l = A^m + B^n)$, it follows that the conjecture (3.1) is verified.
- If $\alpha_1 - n\beta_1 = 0 \Longrightarrow B_1^n C^l = q_1 \Longrightarrow p_1 \nmid C^l$, it is a contradiction with $p_1|(A^m - B^n) \Longrightarrow p_1|C^l$. Then this case is impossible.
- If $\alpha_1 - n\beta_1 < 0$, we obtain $p_1^{n\beta_1 - \alpha_1} B_1^n C^l = q_1 \Longrightarrow p_1|q_1$, it is a contradiction with $p_1 \nmid q_1$. Then this case is impossible.

** H-2- We suppose that $p_1|C^l$, using the same method as for the case $p_1|B^n$, we obtain identical results.

1.5.5. Case $3|p$ and $b = p$: — We have $cos^2\dfrac{\theta}{3} = \dfrac{a}{b} = \dfrac{a}{p}$ et:

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{p} = \frac{4a}{3}$$

As A^{2m} is an integer, it implies that $3|a$, but $3|p \Longrightarrow 3|b$. As a and b are coprime, then the contradiction and the case $3|p$ and $b = p$ is impossible.

1.5.6. Case $3|p$ and $b = 4p$: — $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 4p = 12p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{a}{3} \Longrightarrow 3|a$$

as A^{2m} is an integer. But $3|p \Longrightarrow 3|\left[(4p) = b\right]$, then the contradiction with a, b coprime and the case $b = 4p$ is impossible.

1.5.7. Case $3|p$ and $b = 2p$: — $3|p \Longrightarrow p = 3p'$, $p' \neq 1$ because $3 \ll p$, then $b = 2p = 6p'$.

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}\frac{a}{b} = \frac{2a}{3} \Longrightarrow 3|a$$

as A^{2m} is an integer. But $3|p \Longrightarrow 3|(2p) \Longrightarrow 3|b$, then the contradiction with a, b coprime and the case $b = 2p$ is impossible.

1.5.8. Case $3|p$ and $b \neq 3$ a divisor of p : — We have $b = p' \neq 3$, and p is written as $p = kp'$ with $3|k \Longrightarrow k = 3k'$ and :

$$A^{2m} = \frac{4p}{3}cos^2\frac{\theta}{3} = \frac{4p}{3}.\frac{a}{b} = 4ak'$$

$$B^nC^l = \frac{p}{3}.\left(3 - 4cos^2\frac{\theta}{3}\right) = k'(3p' - 4a) = k'(3b - 4a)$$

** I-1- $k' \neq 1$:

** I-1-1- We suppose that k' est prime, then $A^{2m} = 4ak' = (A^m)^2 \Longrightarrow k'|a$. But $B^nC^l = k'(3b - 4a) \Longrightarrow k'|B^n$ or $k'|C^l$.

** I-1-1-1- If $k'|B^n \Longrightarrow k'|B \Longrightarrow B = k'B_1$ with $B_1 \in \mathbb{N}^*$. Then $k'^{n-1}B_1^nC^l = 3b - 4a$. As $n > 2$, then $(n-1) > 1$ and $k'|a$, then $k'|3b \Longrightarrow k' = 3$ or $k'|b$.

** I-1-1-1-1- If $k' = 3 \Longrightarrow 3|a$, with a that we can write it under the form $a = 3a'^2$. But $A^m = 6a' \Longrightarrow 3|A^m \Longrightarrow 3|A \Longrightarrow A = 3A_1$ with $A_1 \in \mathbb{N}^*$. Then $3^{m-1}A_1^m = 2a' \Longrightarrow 3|a' \Longrightarrow a' = 3a"$. But $k'^{n-1}B_1^nC^l = 3^{n-1}B_1^nC^l = 3b - 4a \Longrightarrow 3^{n-2}B_1^nC^l = b - 36a"^2$. As $n \geq 3 \Longrightarrow n-2 \geq 1$, then $3|b$. Hence the contradiction with a, b coprime.

** I-1-1-1-2- We suppose that $k'|b$, but $k'|a$, then the contradiction with a, b coprime.

** I-1-1-2- We suppose that $k'|C^l$, using the same method as for the case $k'|B^n$, we obtain identical results.

** I-1-2- We consider that k' is not a prime.

** I-1-2-1- We suppose that k', a coprime: $A^{2m} = 4ak' \Longrightarrow A^m = 2a'.p_1$ with $a = a'^2$ and $k' = p_1^2$, then a', p_1 are also coprime. As $A^m = 2a'.p_1$ then $2|a'$ or $2|p_1$.

** I-1-2-1-1- We suppose that $2|a'$, then $2|a' \Longrightarrow 2 \nmid p_1$, but $k' = p_1^2$.

** I-1-2-1-1-1- If p_1 is prime, it is impossible with $A^m = 2a'.p_1$.

** I-1-2-1-1-2- We suppose that p_1 is not prime and it can be written as $p_1 = \omega^m \Longrightarrow k' = \omega^{2m}$. Then $B^nC^l = \omega^{2m}(3b - 4a)$.

** I-1-2-1-1-2-1- If ω is prime $\neq 2$, then $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** I-1-2-1-1-2-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$ with $\omega \nmid B_1$, then $B_1^n.C^l = \omega^{2m-nj}(3b - 4a)$.

- If $2m - n.j = 0$, we obtain $B_1^n.C^l = 3b - 4a$, as $C^l = A^m + B^n \Longrightarrow \omega|C^l \Longrightarrow \omega|C$, and $\omega|(3b - 4a)$. But $\omega \neq 2$ and ω, a' are coprime, then $\omega \nmid (3b) \Longrightarrow \omega \neq 3$ and $\omega \nmid b$. Hence, the conjecture (3.1) is verified.

- If $2m - nj \geq 1$, using the same method, we have $\omega|C^l \Longrightarrow \omega|C$ and $\omega|(3b - 4a)$ and $\omega \nmid a$ and $\omega \neq 3$ and $\omega \nmid b$. Then, the conjecture (3.1) is verified.

- If $2m - nj < 0 \Longrightarrow \omega^{n.j-2m}B_1^n.C^l = 3b - 4a$. As $C^l = A^m + B^n \Longrightarrow \omega|C$ then $C = \omega^h.C_1 \Longrightarrow \omega^{n.j-2m+h.l}B_1^n.C_1^l = 3b - 4a$. If $n.j - 2m + h.l < 0 \Longrightarrow \omega|B_1^nC_1^l$, then the contradiction with $\omega \nmid B_1$ or $\omega \nmid C_1$. If

$n.j - 2m + h.l > 0 \Longrightarrow \omega|(3b - 4a)$ with ω, a, b coprime, it implies that the conjecture (3.1) is verified.

** I-1-2-1-1-2-1-2- We suppose that $\omega|C^l$, using the same method as for the case $\omega|B^n$, we obtain identical results.

** I-1-2-1-1-2-2- Now $k' = \omega^{2m}$ and ω not a prime, we write $\omega = \omega_1^f.\Omega$ with ω_1 a prime $\nmid \Omega$ and $f \geq 1$ an integer, and $\omega_1|A$, then $B^nC^l = \omega_1^{2f.m}\Omega^{2m}(3b-4a) \Longrightarrow \omega_1|(B^nC^l) \Longrightarrow \omega_1|B^n$ or $\omega_1|C^l$.

** I-1-2-1-1-2-2-1- If $\omega_1|B^n \Longrightarrow \omega_1|B \Longrightarrow B = \omega_1^jB_1$ with $\omega_1 \nmid B_1$, then $B_1^n.C^l = \omega_1^{2.fm-nj}\Omega^{2m}(3b-4a)$.

- If $2f.m - n.j = 0$, we obtain $B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1|C^l \Longrightarrow \omega_1|C$, and $\omega_1|(3b-4a)$. But $\omega_1 \neq 2$ and ω_1, a' are coprime, then ω, a are coprime, then $\omega_1 \nmid (3b) \Longrightarrow \omega_1 \neq 3$ and $\omega_1 \nmid b$. Hence, the conjecture (3.1) is verified.
- If $2f.m - n.j \geq 1$, we have $\omega_1|C^l \Longrightarrow \omega_1|C$ and $\omega_1|(3b-4a)$ and $\omega_1 \nmid a$ and $\omega_1 \neq 3$ and $\omega_1 \nmid b$, then the conjecture (3.1) is verified.
- If $2f.m - n.j < 0 \Longrightarrow \omega_1^{n.j-2m.f}B_1^n.C^l = \Omega^{2m}(3b-4a)$. As $C^l = A^m + B^n \Longrightarrow \omega_1|C$ using , then $C = \omega_1^h.C_1 \Longrightarrow \omega^{n.j-2m.f+h.l}B_1^n.C_1^l = \Omega^{2m}(3b-4a)$. If $n.j - 2m.f + h.l < 0 \Longrightarrow \omega_1|B_1^nC_1^l$, then the contradiction with $\omega_1 \nmid B_1$ and $\omega_1 \nmid C_1$. Then if $n.j - 2m.f + h.l > 0$ and $\omega_1|(3b-4a)$ with ω_1, a, b coprime, then the conjecture (3.1) is verified.

** I-1-2-1-1-2-2-2- As in the case $\omega_1|B^n$, we obtain identical results if $\omega_1|C^l$.

** I-1-2-1-2- If $2|p_1$: then $2|p_1 \Longrightarrow 2 \nmid a' \Longrightarrow 2 \nmid a$, but $k' = p_1^2$.

** I-1-2-1-2-1- If $p_1 = 2$, we obtain $A^m = 4a' \Longrightarrow 2|a'$, then the contradiction with $2 \nmid a'$. Case to reject.

** I-1-2-1-2-2- We suppose that p_1 is not prime and $2|p_1$. As $A^m = 2a'p_1$, p_1 is written under the form $p_1 = 2^{m-1}\omega^m \Longrightarrow p_1^2 = 2^{2m-2}\omega^{2m}$. Then $B^nC^l = k'(3b-4a) = 2^{2m-2}\omega^{2m}(3b-4a) \Longrightarrow 2|B^n$ or $2|C^l$.

** I-1-2-1-2-2-1- If $2|B^n \Longrightarrow 2|B$, as $2|A \Longrightarrow 2|C$. From $B^nC^l = 2^{2m-2}\omega^{2m}(3b-4a)$ it follows that if $2|(3b-4a) \Longrightarrow 2|b$ but as $2 \nmid a$, there is no contradiction with a, b coprime and the conjecture (3.1) is verified.

** I-1-2-1-2-2-2- We obtain identical results as above if $2|C^l$.

** I-1-2-2- We suppose that k', a are not coprime: let ω be a prime integer so that $\omega|a$ and $\omega|p_1^2$.

** I-1-2-2-1- We suppose that $\omega = 3$. As $A^{2m} = 4ak' \Longrightarrow 3|A$, but $3|p$. As $p = A^{2m} + B^{2n} + A^m B^n \Longrightarrow 3|B^{2n} \Longrightarrow 3|B$, then $3|C^l \Longrightarrow 3|C$. We write $A = 3^i A_1$, $B = 3^j B_1$, $C = 3^h C_1$ with 3 coprime with A_1, B_1 and C_1 and $p = 3^{2im}A_1^{2m} + 3^{2nj}B_1^{2n} + 3^{im+jn}A_1^m B_1^n = 3^s.g$ with $s = min(2im, 2jn, im+jn)$ and $3 \nmid g$. We have also $(\omega = 3)|a$ and $(\omega = 3)|k'$ that give $a = 3^\alpha a_1$, $3 \nmid a_1$ and $k' = 3^\mu p_2$, $3 \nmid p_2$ with $A^{2m} = 4ak' = 3^{2im}A_1^{2m} = 4 \times 3^{\alpha+\mu}.a_1.p_2 \Longrightarrow \alpha + \mu = 2im$. As $p = 3p' = 3b.k' = 3b.3^\mu p_2 = 3^{\mu+1}.b.p_2$. The exponent of the factor 3 of p is s, the exponent of the factor 3 of the left member of the last equation is $\mu + 1$ added of the exponent β of 3 of the factor b, with $\beta \geq 0$, let $min(2im, 2jn, im + jn) = \mu + 1 + \beta$, we recall that $\alpha + \mu = 2im$. But $B^n C^l = k'(4b - 3a)$ that gives $3^{(nj+hl)}B_1^n C_1^l = 3^{\mu+1}p_2(b - 4 \times 3^{(\alpha-1)}a_1) = 3^{\mu+1}p_2(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1)$, $3 \nmid b_1$. We have also $A^m + B^n = C^l$ that gives $3^{im}A_1^m + 3^{jn}B_1^n = 3^{hl}C_1^l$. We call $\epsilon = min(im, jn)$, we obtain $\epsilon = hl = min(im, jn)$. We have then the conditions:

$$s = min(2im, 2jn, im + jn) = \mu + 1 + \beta \tag{1.133}$$

$$\alpha + \mu = 2im \tag{1.134}$$

$$\epsilon = hl = min(im, jn) \tag{1.135}$$

$$3^{(nj+hl)}B_1^n C_1^l = 3^{\mu+1}p_2(3^\beta b_1 - 4 \times 3^{(\alpha-1)}a_1) \tag{1.136}$$

** I-1-2-2-1-1- $\alpha = 1 \Longrightarrow a = 3a_1$ and $3 \nmid a_1$, the equation (1.134) becomes:

$$1 + \mu = 2im$$

and the first equation (1.133) is written as :

$$s = min(2im, 2jn, im + jn) = 2im + \beta$$

- If $s = 2im \Longrightarrow \beta = 0 \Longrightarrow 3 \nmid b$. We obtain $2im \leq 2jn \Longrightarrow im \leq jn$, and $2im \leq im + jn \Longrightarrow im \leq jn$. The third equation (1.135) gives $hl = im$. The last equation (1.136) gives $nj + hl = \mu + 1 = 2im \Longrightarrow im = jn$, then $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. As a, b are coprime, the conjecture (3.1) is verified.

- If $s = 2jn$ or $s = im + jn$, we obtain $\beta = 0$, $im = jn = hl$ and $B_1^n C_1^l = p_2(b - 4a_1)$. Then as a, b are coprime, the conjecture (3.1) est is verified.

** I-1-2-2-1-2- $\alpha > 1 \Longrightarrow \alpha \geq 2$.

- If $s = 2im \Longrightarrow 2im = \mu + 1 + \beta$, but $\mu = 2im - \alpha$ it gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \neq 0 \Longrightarrow 3|b$, but $3|a$ then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s = 2jn = \mu+1+\beta \leq 2im \Longrightarrow \mu+1+\beta \leq \mu+\alpha \Longrightarrow 1+\beta \leq \alpha \Longrightarrow \beta = 1$. If $\beta = 1 \Longrightarrow 3|b$ but $3|a$, then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s = im + jn \Longrightarrow im + jn \leq 2im \Longrightarrow jn \leq im$, and $im + jn \leq 2jn \Longrightarrow im \leq jn$, then $im = jn$. As $s = im + jn = 2im = 1 + \mu + \beta$ and $\alpha + \mu = 2im$ it gives $\alpha = 1 + \beta \geq 2 \Longrightarrow \beta \geq 1 \Longrightarrow 3|b$, then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

** I-1-2-2-2- We suppose that $\omega \neq 3$. We write $a = \omega^\alpha a_1$ with $\omega \nmid a_1$ and $k' = \omega^\mu p_2$ with $\omega \nmid p_2$. As $A^{2m} = 4ak' = 4\omega^{\alpha+\mu}.a_1.p_2 \Longrightarrow \omega|A \Longrightarrow A = \omega^i A_1$, $\omega \nmid A_1$. But $B^n C^l = k'(3b - 4a) = \omega^\mu p_2(3b - 4a) \Longrightarrow \omega|B^n C^l \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** I-1-2-2-2-1- $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B^n = \omega^j B_1$ and $\omega \nmid B_1$. From $A^m + B^n = C^l \Longrightarrow \omega|C^l \Longrightarrow \omega|C$. As $p = bp' = 3bk' = 3\omega^\mu bp_2 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^m B_1^n)$ with $s = min(2im, 2jn, im + jn)$. Then:

- If $s = \mu$, then $\omega \nmid b$ and the conjecture (3.1) is verified.

- If $s > \mu$, then $\omega|b$, but $\omega|a$ then the contradiction with a, b coprime and the conjecture (3.1) is not verified.

- If $s < \mu$, it follows from:

$$3\omega^\mu bp_1 = \omega^s(\omega^{2im-s}A_1^{2m} + \omega^{2jn-s}B_1^{2n} + \omega^{im+jn-s}A_1^m B_1^n)$$

that $\omega|A_1$ or $\omega|B_1$ that is the contradiction with the hypothesis and the conjecture (3.1) is not verified.

** I-1-2-2-2-2- If $\omega|C^l \Longrightarrow \omega|C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$. From $A^m + B^n = C^l \Longrightarrow \omega|(C^l - A^m) \Longrightarrow \omega|B$. Then we obtain identical results as the case above I-1-2-2-2-1-.

** I-2- We suppose $k'=1$: then $k'=1 \Longrightarrow p=3b$, then we have $A^{2m}=4a=(2a')^2 \Longrightarrow A^m=2a'$, then $a=a'^2$ is even and :

$$A^mB^n = 2\sqrt[3]{\rho}cos\frac{\theta}{3}.\sqrt[3]{\rho}\left(\sqrt{3}sin\frac{\theta}{3}-cos\frac{\theta}{3}\right)=\frac{p\sqrt{3}}{3}sin\frac{2\theta}{3}-2a$$

and we have also:

$$A^{2m}+2A^mB^n=\frac{2p\sqrt{3}}{3}sin\frac{2\theta}{3}=2b\sqrt{3}sin\frac{2\theta}{3} \tag{1.137}$$

The left member of the equation (1.137) is a naturel number and also b, then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written under the form :

$$2\sqrt{3}sin\frac{2\theta}{3}=\frac{k_1}{k_2}$$

where k_1, k_2 are two natural numbers coprime and $k_2|b \Longrightarrow b=k_2.k_3$.

** I-2-1- $k'=1$ and $k_3 \neq 1$: then $A^{2m}+2A^mB^n=k_3.k_1$. Let μ be a prime so that $\mu|k_3$. If $\mu=2 \Rightarrow 2|b$, but $2|a$, it is a contradiction with a, b coprime. We suppose that $\mu \neq 2$ and $\mu|k_3$, then $\mu|A^m(A^m+2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m+2B^n)$.

** I-2-1-1- $\mu|A^m$: If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|4a \Longrightarrow \mu|a$. As $\mu|k_3 \Longrightarrow \mu|b$, the contradiction with a, b coprime.

** I-2-1-2- $\mu|(A^m+2B^n)$: If $\mu|(A^m+2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m+2B^n)$, we can write $A^m+2B^n=\mu.t'$. It follows:

$$A^m+B^n=\mu t'-B^n \Longrightarrow A^{2m}+B^{2n}+2A^mB^n=\mu^2t'^2-2t'\mu B^n+B^{2n}$$

Using the expression of p, we obtain:

$$p=t'^2\mu^2-2t'B^n\mu+B^n(B^n-A^m)$$

As $p=3b=3k_2.k_3$ and $\mu|k_3$ then $\mu|p \Longrightarrow \ p=\mu.\mu'$, then we obtain:

$$\mu'.\mu=\mu(\mu t'^2-2t'B^n)+B^n(B^n-A^m)$$

and $\mu|B^n(B^n-A^m) \Longrightarrow \mu|B^n$ or $\mu|(B^n-A^m)$.

** I-2-1-2-1- $\mu|B^n$: If $\mu|B^n \Longrightarrow \mu|B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2- $\mu|(B^n - A^m)$: If $\mu|(B^n - A^m)$ and using that $\mu|(A^m + 2B^n)$, we obtain :

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n \Longrightarrow \mu|B \\ or \\ \mu = 3 \end{cases}$$

** I-2-1-2-2-1- $\mu|B^n$: If $\mu|B^n \Longrightarrow \mu|B$, that is the contradiction with I-2-1-2- above.

** I-2-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3|k_3 \Longrightarrow k_3 = 3k'_3$, and we have $b = k_2k_3 = 3k_2k'_3$, it follows $p = 3b = 9k_2k'_3$, then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$ then:

$$9k_2k'_3 - 3A^mB^n = (A^m - B^n)^2$$

that we write as:

$$3(3k_2k'_3 - A^mB^n) = (A^m - B^n)^2 \tag{1.138}$$

then:

$$3|(3k_2k'_3 - A^mB^n) \Longrightarrow\ 3|A^mB^n \Longrightarrow 3|A^m \ or \ 3|B^n$$

** I-2-1-2-2-2-1- $3|A^m$: If $3|A^m \Longrightarrow 3|A$ and we have also $3|A^{2m}$, but $A^{2m} = 4a \Longrightarrow 3|4a \Longrightarrow 3|a$. As $b = 3k_2k'_3$ then $3|b$, but a, b are coprime, then the contradiction and $3 \nmid A$.

** I-2-1-2-2-2-2- $3|B^m$: If $3|B^n \Longrightarrow 3|B$, but the equation (1.138) implies $3|(A^m - B^n)^2 \Longrightarrow 3|(A^m - B^n) \Longrightarrow 3|A^m \Longrightarrow 3|A$. The last case above has given that $3 \nmid A$. Then case $3|B^m$ is to reject.

Finally the hypothesis $k_3 \neq 1$ is impossible.

** I-2-2- Now, we suppose that $k_3 = 1 \Longrightarrow b = k_2$ and $p = 3b = 3k_2$, then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_1}{b} \tag{1.139}$$

with k_1, b coprime. We write (1.139) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_1}{b}$$

Taking the square of the two members and replacing $cos^2\dfrac{\theta}{3}$ by $\dfrac{a}{b}$, we obtain:

$$3 \times 4^2.a(b - a) = k_1^2 \Longrightarrow k_1^2 = 3 \times 4^2.a'^2(b - a)$$

it implies that :

$$b - a = 3\alpha^2 \Longrightarrow b = a'^2 + 3\alpha^2 \Longrightarrow k_1 = 12a'\alpha$$

As:

$$k_1 = 12a'\alpha = A^m(A^m + 2B^n) \Longrightarrow 3\alpha = a' + B^n$$

We consider now that $3|(b-a)$ with $b = a'^2 + 3\alpha^2$. The case $\alpha = 1$ gives $a' + B^n = 3$ that is impossible. We suppose $\alpha > 1$, the pair (a', α) is a solution of the Diophantine equation:

$$X^2 + 3Y^2 = b \tag{1.140}$$

with $X = a'$ and $Y = \alpha$. But using a theorem on the solutions of the equation given by (1.140), b is written as (see theorem in [**6**]):

$$b = 2^{2s} \times 3^t.p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

where p_i are prime numbers verifying $p_i \equiv 1(\bmod 6)$, the q_j are also prime numbers so that $q_j \equiv 5(\bmod 6)$, then :

- If $s \geq 1 \Longrightarrow 2|b$, as $2|a$, then the contradiction with a, b coprime.
- If $t \geq 1 \Longrightarrow 3|b$, but $3|(b-a) \Longrightarrow 3|a$, then the contradiction with a, b coprime.

** I-2-2-1- We suppose that b is written as :

$$b = p_1^{t_1} \cdots p_g^{t_g} q_1^{2s_1} \cdots q_r^{2s_r}$$

with $p_i \equiv 1(\text{mod}6)$ and $q_j \equiv 5(\text{mod}6)$. Finally, we obtain that $b \equiv 1(\bmod 6)$. We will verify then this condition.

** I-2-2-1-1- We present the table giving the value of $A^m + B^n = C^l$ modulo 6 in function of the value of $A^m, B^n(\bmod 6)$. We obtain the table below after retiring the lines (respectively the colones) of $A^m \equiv 0(\text{mod}6)$ and $A^m \equiv 3(\bmod 6)$ (respectively of $B^n \equiv 0(\bmod 6)$ and $B^n \equiv 3(\bmod 6)$), they present cases with contradictions:

A^m, B^n	1	2	4	5
1	2	3	5	0
2	3	4	0	1
4	5	0	2	3
5	0	1	3	4

TABLE 2. Table of $C^l(\bmod 6)$

** I-2-2-1-1-1- For the case $C^l \equiv 0(\bmod 6)$ and $C^l \equiv 3(\bmod 6)$, we deduce that $3|C^l \Longrightarrow 3|C \Longrightarrow C = 3^h C_1$, with $h \geq 1$ and $3 \nmid C_1$. It follows that $p - B^nC^l = 3b - 3^{lh}C_1^lB^n = A^{2m} \Longrightarrow 3|(A^{2m} = 4a) \Longrightarrow 3|a \Longrightarrow 3|b$, then the contradiction with a, b coprime.

** I-2-2-1-1-2- For the case $C^l \equiv 0(\text{mod}6)$, $C^l \equiv 2(\text{mod}6)$ and $C^l \equiv 4(\bmod 6)$, we deduce that $2|C^l \Longrightarrow 2|C \Longrightarrow C = 2^hC_1$, with $h \geq 1$ and $2 \nmid C_1$. It follows that $p = 3b = A^{2m} + B^nC^l = 4a + 2^{lh}C_1^lB^n \Longrightarrow 2|3b \Longrightarrow 2|b$, then the contradiction with a, b coprime.

** I-2-2-1-1-3- We consider the cases $A^m \equiv 1(\bmod 6)$ and $B^n \equiv 4(\bmod 6)$ (respectively $B^n \equiv 2(\bmod 6)$): then $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows from $3b = A^{2m} + B^nC^l = 4a + 2^{jn}B_1^nC^l$, then $2|b$, then the contradiction with a, b coprime.

** I-2-2-1-1-4- We consider the case $A^m \equiv 5(\bmod 6)$ and $B^n \equiv 2(\bmod 6)$: then $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$ with $j \geq 1$ and $2 \nmid B_1$. It follows that $3b = A^{2m} + B^nC^l = 4a + 2^{jn}B_1^nC^l$, then $2|b$, then the contradiction with a, b coprime.

** I-2-2-1-1-5- We consider the case $A^m \equiv 2(\bmod 6)$ and $B^n \equiv 5(\bmod 6)$: as $A^m \equiv 2(\bmod 6) \Longrightarrow A^m \equiv 2(\bmod 3)$, then A^m is not a square and also for B^n. Hence, we can write A^m and B^n as:

$$A^m = a_0.\mathscr{A}^2$$
$$B^n = b_0\mathscr{B}^2$$

where a_0 (respectively b_0) regroups the product of the prime numbers of A^m with exponent 1 (respectively of B^n) with not necessary $(a_0, \mathscr{A}) = 1$

and $(b_0, \mathscr{B}) = 1$. We have also $p = 3b = A^{2m} + A^mB^n + B^{2n} = (A^m - B^n)^2 + 3A^mB^n \Longrightarrow 3|(b - A^mB^n) \Longrightarrow A^mB^n \equiv b(\text{mod}3)$ but $b = a + 3\alpha^2 \Longrightarrow b \equiv a \equiv a'^2(\text{mod}3)$, then $A^mB^n \equiv a'^2(\text{mod}3)$. But $A^m \equiv 2(\bmod 6) \Longrightarrow 2a' \equiv 2(\bmod 6) \Longrightarrow 4a'^2 \equiv 4(\bmod 6) \Longrightarrow a'^2 \equiv 1(\bmod 3)$. It follows that A^mB^n is a square, let $A^mB^n = \mathscr{N}^2 = \mathscr{A}^2.\mathscr{B}^2.a_0.b_0$. We call $\mathscr{N}_1^2 = a_0.b_0$. Let p_1 be a prime number so that $p_1|a_0 \Longrightarrow a_0 = p_1.a_1$ with $p_1 \nmid a_1$. $p_1|\mathscr{N}_1^2 \Longrightarrow p_1|\mathscr{N}_1 \Longrightarrow \mathscr{N}_1 = p_1^t\mathscr{N}_1'$ with $t \geq 1$ and $p_1 \nmid \mathscr{N}_1'$, then $p_1^{2t-1}\mathscr{N}_1'^2 = a_1.b_0$. As $2t \geq 2 \Longrightarrow 2t - 1 \geq 1 \Longrightarrow p_1|a_1.b_0$ but $(p_1, a_1) = 1$, then $p_1|b_0 \Longrightarrow p_1|B^n \Longrightarrow p_1|B$. But $p_1|(A^m = 2a')$, and $p_1 \neq 2$ because $p_1|B^n$ and B^n is odd, then the contradiction. Hence, $p_1|a' \Longrightarrow p_1|a$. If $p_1 = 3$, from $3|(b - a) \Longrightarrow 3|b$ then the contradiction with a, b coprime. Then $p_1 > 3$ a prime that divides A^m and B^n, then $p_1|(p = 3b) \Longrightarrow p_1|b$, it follows the contradiction with a, b coprime, knowing that $p = 3b \equiv 3(\text{mod}6)$ and we choose the case $b \equiv 1(\bmod 6)$ of our interest.

** I-2-2-1-1-6- We consider the last case of the table above $A^m \equiv 4(\bmod 6)$ and $B^n \equiv 1(\bmod 6)$. We return to the equation (1.140) that b verifies :

$$b = X^2 + 3Y^2 \tag{1.141}$$
$$with \quad X = a'; \quad Y = \alpha$$
$$and \quad 3\alpha = a' + B^n$$

Suppose that it exists another solution of (1.141):

$$b = X^2 + 3Y^3 = u^2 + 3v^2 \Longrightarrow 2u \neq A^m,\ 3v \neq a' + B^n$$

But $B^n = \dfrac{6\alpha - A^m}{2} = 3\alpha - a'$ and b verify also :$3b = p = A^{2m} + A^mB^n + B^{2n}$, it is impossible that u, v verify:

$$6v = 2u + 2B^n$$
$$3b = 4u^2 + 2uB^n + B^{2n}$$

If we consider that : $6v - 2u = 6\alpha - 2a' \Longrightarrow u = 3v - 3\alpha + a'$, then $b = u^2 + 3v^2 = (3v - 3\alpha + a')^2 + 3v^2$, it gives:

$$2v^2 - B^nv + \alpha^2 - a'\alpha = 0$$
$$2v^2 - B^nv - \frac{(a' + B^n)(A^m - B^n)}{9} = 0$$

The resolution of the last equation gives with taking the positive root (because $A^m > B^n$), $v_1 = \alpha$, then $u = a'$. It follows that b in (1.141) has an unique representation under the form $X^2 + 3Y^2$ with $X, 3Y$ coprime. As b is odd, we applique one of Euler's theorems on the convenient numbers "numerus idoneus" as cited above (Case C-2-2-1-2). It follows that b is prime.

We have also $p = 3b = A^{2m} + A^mB^n + B^{2n} = 4a'^2 + B^n.C^l \Longrightarrow 9\alpha^2 - a'^2 = B^n.C^l$, then $3\alpha, a' \in \mathbb{N}^*$ are solutions of the Diophantine equation:

$$x^2 - y^2 = N \tag{1.142}$$

with $N = B^nC^l > 0$. Let $Q(N)$ be the number of the solutions of (1.142) and $\tau(N)$ the number of ways to write the factors of N, then we announce the following result concerning the number of the solutions of (1.142) (see theorem 27.3 in [**6**]):
- If $N \equiv 2(\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\bmod 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\bmod 4)$, then $Q(N) = [\tau(N/4)/2]$.

We recall that $A^m \equiv 0(\bmod 4)$. Concerning B^n, for $B^n \equiv 0(\bmod 4)$ or $B^n \equiv 2(\bmod 4)$, we find that $2|B^n \Longrightarrow 2|\alpha \Longrightarrow 2|b$, then the contradiction with a, b coprime. For the last case $B^n \equiv 3(\bmod 4) \Longrightarrow C^l \equiv 3(\bmod 4) \Longrightarrow N = B^nC^l \equiv 1(\bmod 4) \Longrightarrow Q(N) = [\tau(N)/2] > 1$. But $Q(N) = 1$, because the unknowns of (1.142) are also the unknowns of (1.141) and we have an unique solution of the two Diophantine equations, then the contradiction.

It follows that the condition $3|(b - a)$ is a contradiction.

The study of the case 1.5.8 is achieved.

1.5.9. Case $3|p$ and $b|4p$: — The following cases have been soon studied:
* $3|p$, $b = 2 \Longrightarrow b|4p$: case 1.5.1
* $3|p$, $b = 4 \Longrightarrow b|4p$: case 1.5.2
* $3|p \Longrightarrow p = 3p'$, $b|p' \Longrightarrow p' = bp"$, $p" \neq 1$: case 1.5.3
* $3|p$, $b = 3 \Longrightarrow b|4p$: case 1.5.4
* $3|p \Longrightarrow p = 3p'$, $b = p' \Longrightarrow b|4p$: case 1.5.8

** J-1- Particular case: $b = 12$. In fact $3|p \Longrightarrow p = 3p'$ and $4p = 12p'$. Taking $b = 12$, we have $b|4p$. But $b < 4a < 3b$, that gives $12 < 4a < 36 \Longrightarrow 3 < a < 9$. As $2|b$ and $3|b$, the possible values of a are 5 and 7.

** J-1-1- $a = 5$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3}.\frac{a}{b} = \frac{5bp'}{3b} = \frac{5p'}{3} \Longrightarrow 3|p' \Longrightarrow p' = 3p"$ with $p" \in \mathbb{N}^*$, then $p = 9p"$, we obtain the expressions:

$$A^{2m} = 5p" \tag{1.143}$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 4p" \tag{1.144}$$

As $n, l \geq 3$, we deduce from the equation (1.144) that $2|p" \Longrightarrow p" = 2^\alpha p_1$ with $\alpha \geq 1$ and $2 \nmid p_1$. Then (1.143) becomes: $A^{2m} = 5p" = 5 \times 2^\alpha p_1 \Longrightarrow 2|A \Longrightarrow A = 2^iA_1$, $i \geq 1$ and $2 \nmid A_1$. We have also $B^nC^l = 2^{\alpha+2}p_1 \Longrightarrow 2|B^n$ or $2|C^l$.

** J-1-1-1- We suppose that $2|B^n \Longrightarrow B = 2^jB_1$, $j \geq 1$ and $2 \nmid B_1$. We obtain $B_1^nC^l = 2^{\alpha+2-jn}p_1$:

- If $\alpha + 2 - jn > 0 \Longrightarrow 2|C^l$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2|C^l$ and the conjecture (3.1) is verified.
- If $\alpha + 2 - jn = 0 \Longrightarrow B_1^nC^l = p_1$. From $C^{=}2^{im}A_1^m + 2^{jn}B_1^n \Longrightarrow 2|C^l$ that implies that $2|p_1$, then the contradiction with $2 \nmid p_1$.
- If $\alpha + 2 - jn < 0 \Longrightarrow 2^{jn-\alpha-2}B_1^nC^l = p_1$, it implies that $2|p_1$, then the contradiction as above.

** J-1-1-2- We suppose that $2|C^l$, using the same method above, we obtain the identical results.

** J-1-2- We suppose that $a = 7$ and $b = 12 \Longrightarrow 4p = 12p' = bp'$. But $A^{2m} = \frac{4p}{3}.\frac{a}{b} = \frac{12p'}{3}.\frac{7}{12} = \frac{7p'}{3} \Longrightarrow 3|p' \Longrightarrow p = 9p"$, we obtain:

$$A^{2m} = 7p"$$

$$B^nC^l = \frac{p}{3}\left(3 - 4cos^2\frac{\theta}{3}\right) = 2p"$$

The last equation implies that $2|B^nC^l$. Using the same method as for the case J-1-1- above, we obtain the identical results.

We study now the general case. As $3|p \Rightarrow p = 3p'$ and $b|4p \Rightarrow \exists k_1 \in \mathbb{N}^*$ and $4p = 12p' = k_1b$.

** J-2- $k_1 = 1$: If $k_1 = 1$ then $b = 12p'$, ($p' \neq 1$, if not $p = 3 \ll A^{2m} + B^{2n} + A^m B^n$). But $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{12p'}{3}\frac{a}{b} = \frac{4p'.a}{12p'} = \frac{a}{3} \Rightarrow 3|a$ because A^{2m} is a natural number, then the contradiction with a, b coprime.

** J-3- $k_1 = 3$: If $k_1 = 3$, then $b = 4p'$ and $A^{2m} = \frac{4p}{3}.cos^2\frac{\theta}{3} = \frac{k_1.a}{3} = a = (A^m)^2 = a'^2 \Longrightarrow A^m = a'$. The term $A^m B^n$ gives $A^m B^n = \frac{p\sqrt{3}}{3} sin\frac{2\theta}{3} - \frac{a}{2}$, then:

$$A^{2m} + 2A^m B^n = \frac{2p\sqrt{3}}{3} sin\frac{2\theta}{3} = 2p'\sqrt{3} sin\frac{2\theta}{3} \tag{1.145}$$

The left member of (1.145) is a natural number and also p', then $2\sqrt{3}sin\frac{2\theta}{3}$ can be written under the form:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{k_3}$$

where k_2, k_3 are two natural numbers and are coprime and $k_3|p' \Longrightarrow p' = k_3.k_4$.

** J-3-1- $k_4 \neq 1$: We suppose that $k_4 \neq 1$, then:

$$A^{2m} + 2A^m B^n = k_2.k_4 \tag{1.146}$$

Let μ be a prime natural number so that $\mu|k_4$, then $\mu|A^m(A^m+2B^n) \Longrightarrow \mu|A^m$ or $\mu|(A^m + 2B^n)$.

** J-3-1-1- $\mu|A^m$: If $\mu|A^m \Longrightarrow \mu|A^{2m} \Longrightarrow \mu|a$. As $\mu|k_4 \Longrightarrow \mu|p' \Rightarrow \mu|(4p' = b)$. But a, b are coprime, then the contradiction.

** J-3-1-2- $\mu|(A^m + 2B^n)$: If $\mu|(A^m + 2B^n) \Longrightarrow \mu \nmid A^m$ and $\mu \nmid 2B^n$, then $\mu \neq 2$ and $\mu \nmid B^n$. $\mu|(A^m + 2B^n)$, we can write $A^m + 2B^n = \mu.t'$. It follows:

$$A^m + B^n = \mu t' - B^n \Longrightarrow A^{2m} + B^{2n} + 2A^m B^n = \mu^2 t'^2 - 2t'\mu B^n + B^{2n}$$

Using the expression of p, we obtain $p = t'^2\mu^2 - 2t'B^n\mu + B^n(B^n - A^m)$. As $p = 3p'$ and $\mu|p' \Rightarrow \mu|(3p') \Rightarrow \mu|p$, we can write : $\exists \mu'$ and $p = \mu\mu'$, then we arrive to:

$$\mu'.\mu = \mu(\mu t'^2 - 2t'B^n) + B^n(B^n - A^m)$$

and $\mu|B^n(B^n - A^m) \Longrightarrow \mu|B^n$ or $\mu|(B^n - A^m)$.

** J-3-1-2-1- $\mu|B^n$: If $\mu|B^n \Longrightarrow \mu|B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2- $\mu|(B^n - A^m)$: If $\mu|(B^n - A^m)$ and using $\mu|(A^m + 2B^n)$, we obtain :

$$\mu|3B^n \Longrightarrow \begin{cases} \mu|B^n \\ or \\ \mu = 3 \end{cases}$$

** J-3-1-2-2-1- $\mu|B^n$: If $\mu|B^n \Longrightarrow \mu|B$, it is in contradiction with J-3-1-2-.

** J-3-1-2-2-2- $\mu = 3$: If $\mu = 3 \Longrightarrow 3|k_4 \Longrightarrow k_4 = 3k'_4$, and we have $p' = k_3k_4 = 3k_3k'_4$, it follows that $p = 3p' = 9k_3k'_4$, then $9|p$, but $p = (A^m - B^n)^2 + 3A^mB^n$, then we obtain:

$$9k_3k'_4 - 3A^mB^n = (A^m - B^n)^2$$

that we write : $3(3k_3k'_4 - A^mB^n) = (A^m - B^n)^2$, then : $3|(3k_3k'_4 - A^mB^n) \Longrightarrow 3|A^mB^n \Longrightarrow 3|A^m$ or $3|B^n$.

** J-3-1-2-2-2-1- $3|A^m$: If $3|A^m \Longrightarrow 3|A^{2m} \Rightarrow 3|a$, but $3|p' \Rightarrow 3|(4p') \Rightarrow 3|b$, then the contradiction with a, b coprime and $3 \nmid A$.

** J-3-1-2-2-2-2- $3|B^n$: If $3|B^n$ but $A^m = \mu t' - 2B^n = 3t' - 2B^n \Longrightarrow 3|A^m$, it is in contradiction with $3 \nmid A$.

Then the hypothesis $k_4 \neq 1$ is impossible.

** J-3-2- $k_4 = 1$: We suppose now that $k_4 = 1 \Longrightarrow p' = k_3k_4 = k_3$. Then we have:

$$2\sqrt{3}sin\frac{2\theta}{3} = \frac{k_2}{p'} \tag{1.147}$$

with k_2, p' coprime, we write (1.147) as :

$$4\sqrt{3}sin\frac{\theta}{3}cos\frac{\theta}{3} = \frac{k_2}{p'}$$

Taking the square of the two members and replacing $cos^2\frac{\theta}{3}$ by $\frac{a}{b}$ and $b = 4p'$, we obtain:

$$3.a(b - a) = k_2^2$$

As $A^{2m} = a = a'^2$, it implies that :

$$3|(b - a), \quad and \quad b - a = b - a'^2 = 3\alpha^2$$

As $k_2 = A^m(A^m + 2B^n)$ following the equation (1.146) and that $3|k_2 \Longrightarrow 3|A^m(A^m + 2B^n) \Longrightarrow 3|A^m$ or $3|(A^m + 2B^n)$.

** J-3-2-1- $3|A^m$: If $3|A^m \Longrightarrow 3|A^{2m} \Longrightarrow 3|a$, but $3|(b-a) \Longrightarrow 3|b$, then the contradiction with a, b coprime.

** J-3-2-2- $3|(A^m + 2B^n) \Longrightarrow 3 \nmid A^m$ and $3 \nmid B^n$. As $k_2^2 = 9a\alpha^2 = 9a'^2\alpha^2 \Longrightarrow k_2 = 3a'\alpha = A^m(A^m + 2B^n)$, then :

$$3\alpha = A^m + 2B^n \tag{1.148}$$

As b can be written under the form $b = a'^2 + 3\alpha^2$, then the pair (a', α) is a solution of the Diophantine equation:

$$x^2 + 3y^2 = b \tag{1.149}$$

As $b = 4p'$, then :

** J-3-2-2-1- If x, y are even, then $2|a' \Longrightarrow 2|a$, it is a contradiction with a, b coprime.

** J-3-2-2-2- If x, y are odd, then a', α are odd, it implies $A^m = a' \equiv 1(\bmod 4)$ or $A^m \equiv 3(\bmod 4)$. If u, v verify (1.149), then $b = u^2 + 3v^2$, with $u \neq a'$ and $v \neq \alpha$, then u, v do not verify (1.148): $3v \neq u + 2B^n$, if not, $u = 3v - 2B^n \Longrightarrow b = (3v - 2B^n)^2 + 3v^2 = a'^2 + 3\alpha$, the resolution of the obtained equation of second degree in v gives the positive root $v_1 = \alpha$, then $u = 3\alpha - 2B^n = a'$, then the uniqueness of the representation of b by the equation (1.149).

** J-3-2-2-2-1- We suppose that $A^m \equiv 1(\bmod 4)$ and $B^n \equiv 0(\bmod 4)$, then B^n is even and $B^n = 2B'$. The expression of p becomes:

$$p = a'^2 + 2a'B' + 4B'^2 = (a' + B')^2 + 3B'^2 = 3p' \Longrightarrow 3|(a' + B') \Longrightarrow a' + B' = 3B"$$
$$p' = B'^2 + 3B"^2 \Longrightarrow b = 4p' = (2B')^2 + 3(2B")^2 = a'^2 + 3\alpha^2$$

that gives $2B' = B^n = a' = A^m$, then the contradiction with $A^m > B^n$.

** J-3-2-2-2-2- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 1 (\bmod 4)$, then C^l is even and $C^l = 2C'$. The expression of p becomes:

$$p = C^{2l} - C^l B^n + B^{2n} = 4C'^2 - 2C'B^n + B^{2n} = (C' - B^n)^2 + 3C'^2 = 3p'$$
$$\Longrightarrow 3|(C' - B^n) \Longrightarrow C' - B^n = 3C"$$
$$p' = C'^2 + 3C"^2 \Longrightarrow b = 4p' = (2C')^2 + 3(2C")^2 = a'^2 + 3\alpha^2$$

We obtain $2C' = C^l = a' = A^m$, then the contradiction.

** J-3-2-2-2-3- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 2 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-4- We suppose that $A^m \equiv 1 (\bmod 4)$ and $B^n \equiv 3 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-5- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 0 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-6- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 1 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

** J-3-2-2-2-7- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 2 (\bmod 4)$, then B^n is even, see J-3-2-2-2-1-.

** J-3-2-2-2-8- We suppose that $A^m \equiv 3 (\bmod 4)$ and $B^n \equiv 3 (\bmod 4)$, then C^l is even, see J-3-2-2-2-2-.

We have achieved the study of the case J-3-2-2- given contradictions.

** J-4- We suppose that $k_1 \neq 3$ and $3|k_1 \Longrightarrow k_1 = 3k'_1$ with $k'_1 \neq 1$, then $4p = 12p' = k_1 b = 3k'_1 b \Rightarrow 4p' = k'_1 b$. A^{2m} can be written as $A^{2m} = \dfrac{4p}{3} cos^2 \dfrac{\theta}{3} = \dfrac{3k'_1 b}{3} \dfrac{a}{b} = k'_1 a$ and $B^n C^l = \dfrac{p}{3}\left(3 - 4cos^2 \dfrac{\theta}{3}\right) = \dfrac{k'_1}{4}(3b - 4a)$. As $B^n C^l$ is a natural number, we must have $4|(3b - 4a)$ or $4|k'_1$ or $[2|k'_1$ and $2|(3b - 4a)]$.

** J-4-1- We suppose that $4|(3b - 4a)$.

** J-4-1-1- We suppose that $3b - 4a = 4 \Longrightarrow 4|b \Longrightarrow 2|b$. Then, we have:

$$A^{2m} = k'_1 a$$
$$B^n C^l = k'_1$$

** J-4-1-1-1- If k'_1 is prime, from $B^n C^l = k'_1$, it is impossible.

** J-4-1-1-2- We suppose that $k'_1 > 1$ is not prime. Let ω be a prime natural number so that $\omega|k'_1$.

** J-4-1-1-2-1- We suppose that $k'_1 = \omega^s$, with $s \geq 6$. Then we have :

$$A^{2m} = \omega^s.a \tag{1.150}$$
$$B^n C^l = \omega^s \tag{1.151}$$

** J-4-1-1-2-1-1- We suppose that $\omega = 2$. If a, k'_1 are not coprime , then $2|a$, as $2|b$, it is the contradiction with a, b coprime.

** J-4-1-1-2-1-2- We suppose $\omega = 2$ and a, k'_1 are coprime, then $2 \nmid a$. From (1.151), we deduce that $B = C = 2$ and $n + l = s$, and $A^{2m} = 2^s.a$, but $A^m = 2^l - 2^n \Longrightarrow A^{2m} = (2^l - 2^n)^2 = 2^{2l} + 2^{2n} - 2(2^{l+n}) = 2^{2l} + 2^{2n} - 2 \times 2^s = 2^s.a \Longrightarrow 2^{2l} + 2^{2n} = 2^s(a + 2)$. If $l = n$, we obtain $a = 0$ then the contradiction. If $l \neq n$, as $A^m = 2^l - 2^n > 0 \Longrightarrow n < l \Longrightarrow 2n < s$, then $2^{2n}(1+2^{2l-2n}-2^{s+1-2n}) = 2^n 2^l.a$. We call $l = n+n_1 \Longrightarrow 1+2^{2l-2n}-2^{s+1-2n} = 2^{n_1}.a$, but the left term is odd and the right member is even, then the contradiction. Then the case $\omega = 2$ is impossible.

** J-4-1-1-2-1-3- We suppose that $k'_1 = \omega^s$ with $\omega \neq 2$:

** J-4-1-1-2-1-3-1- Suppose that a, k'_1 are not coprime, then $\omega|a \Longrightarrow a = \omega^t.a_1$ and $t \nmid a_1$. Then, we have:

$$A^{2m} = \omega^{s+t}.a_1 \tag{1.152}$$
$$B^n C^l = \omega^s \tag{1.153}$$

From (1.153), we deduce that $B^n = \omega^n$, $C^n = \omega^l$, $s = n+l$ and $A^m = \omega^l - \omega^n > 0 \Longrightarrow l > n$. We have also $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2 = \omega^{2l} + \omega^{2n} - 2 \times \omega^s$. As $\omega \neq 2 \Longrightarrow \omega$ is odd, then $A^{2m} = \omega^{s+t}.a_1 = (\omega^l - \omega^n)^2$ is even, then $2|a_1 \Longrightarrow 2|a$, it is in contradiction with a, b coprime, then this case is impossible.

** J-4-1-1-2-1-3-2- Suppose that a, k'_1 are coprime, with :

$$A^{2m} = \omega^s.a \tag{1.154}$$

$$B^nC^l = \omega^s \tag{1.155}$$

From (1.155), we deduce that $B^n = \omega^n$, $C^l = \omega^l$ and $s = n + l$. As $\omega \neq 2 \Longrightarrow \omega$ is odd and $A^{2m} = \omega^s.a = (\omega^l - \omega^n)^2$ is even, then $2|a$. It follows the contradiction with a, b coprime and this case is impossible.

** J-4-1-1-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$. We have :

$$A^{2m} = \omega^s.k_2.a$$

$$B^nC^l = \omega^s.k_2$$

** J-4-1-1-2-2-1- If k_2 is prime, from the last equation above, $\omega = k_2$, it is in contradiction with $\omega \nmid k_2$. Then this case is impossible.

** J-4-1-1-2-2-2- We suppose that $k'_1 = \omega^s.k_2$, with $s \geq 6$, $\omega \nmid k_2$ and k_2 not a prime. Then, we have:

$$A^{2m} = \omega^s.k_2.a$$

$$B^nC^l = \omega^s.k_2 \tag{1.156}$$

** J-4-1-1-2-2-2-1- We suppose that ω, a are coprime, then $\omega \nmid a$. As $A^{2m} = \omega^s.k_2.a \Longrightarrow \omega|A \Longrightarrow A = \omega^iA_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s = 2im$. From (1.156), we have $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** J-4-1-1-2-2-2-1-1- We suppose that $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^jB_1$ with $j \geq 1$ and $\omega \nmid B_1$. then :

$$B_1^nC^l = \omega^{2im-jn}k_2$$

- If $2im - jn > 0$, $\omega|C^l \Longrightarrow \omega|C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn = 0 \Longrightarrow B_1^nC^l = k_2$, as $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega|(C^l = A^m + B^n)$.

- If $2im - jn < 0 \Longrightarrow \omega^{jn-2im}B_1^nC^l = k_2 \Longrightarrow \omega|k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-1-2- We suppose that $\omega|C^l$. Using the same method used above, we obtain identical results.

** J-4-1-1-2-2-2-2- We suppose that a, ω are not coprime, then $\omega|a \Longrightarrow a = \omega^t.a_1$ and $\omega \nmid a_1$. So we have :

$$A^{2m} = \omega^{s+t}.k_2.a_1 \tag{1.157}$$

$$B^nC^l = \omega^s.k_2 \tag{1.158}$$

As $A^{2m} = \omega^{s+t}.k_2.a_1 \Longrightarrow \omega|A \Longrightarrow A = \omega^i A_1$ with $i \geq 1$ and $\omega \nmid A_1$, then $s+t = 2im$. From (1.158), we have $\omega|(B^nC^l) \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** J-4-1-1-2-2-2-2-1- We suppose that $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^j B_1$ with $j \geq 1$ and $\omega \nmid B_1$. then:

$$B_1^nC^l = \omega^{2im-t-jn}k_2$$

- If $2im-t-jn > 0$, $\omega|C^l \Longrightarrow \omega|C$, no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im - t - jn = 0 \Longrightarrow B_1^nC^l = k_2$, As $\omega \nmid k_2 \Longrightarrow \omega \nmid C^l$, then the contradiction with $\omega|(C^l = A^m + B^n)$.

- If $2im - t - jn < 0 \Longrightarrow \omega^{jn+t-2im}B_1^nC^l = k_2 \Longrightarrow \omega|k_2$, then the contradiction with $\omega \nmid k_2$.

** J-4-1-1-2-2-2-2-2- We suppose that $\omega|C^l$. Using the same method used above, we obtain identical results.

** J-4-1-2- $3b - 4a \neq 4$ and $4|(3b-4a) \Longrightarrow 3b - 4a = 4^s\Omega$ with $s \geq 1$ and $4 \nmid \Omega$. We obtain:

$$A^{2m} = k_1'a \tag{1.159}$$

$$B^nC^l = 4^{s-1}k_1'\Omega \tag{1.160}$$

** J-4-1-2-1- We suppose that $k_1' = 2$. From (1.159), we deduce that $2|a$. As $4|(3b-4a) \Longrightarrow 2|b$, then the contradiction with a, b coprime and this case is impossible.

** J-4-1-2-2- We suppose that $k_1' = 3$. From (1.159) we deduce that $3^3|A^{2m}$. From (1.160), it follows that $3^3|B^n$ or $3^3|C^l$. In the last two cases, we obtain $3^3|p$. But $4p = 3k_1'b = 9b \Longrightarrow 3|b$, then the contradiction with a, b coprime. Then this case is impossible.

** J-4-1-2-3- We suppose that k_1' is prime ≥ 5:

** J-4-1-2-3-1- Suppose that k'_1 and a are coprime. The equation (1.159) gives $(A^m)^2 = k'_1.a$, that is impossible with $k'_1 \nmid a$. Then this case is impossible.

** J-4-1-2-3-2- Suppose that k'_1 and a are not coprime. Let $k'_1|a \Longrightarrow a = k'^{\alpha}_1 a_1$ with $\alpha \geq 1$ and $k'_1 \nmid a_1$. The equation (1.159) is written as :

$$A^{2m} = k'_1 a = k'^{\alpha+1}_1 a_1$$

The last equation gives $k'_1|A^{2m} \Longrightarrow k'_1|A \Longrightarrow A = k'^i_1.A_1$, with $k'_1 \nmid A_1$. If $2i.m \neq (\alpha+1)$, it is impossible. We suppose that $2i.m = \alpha + 1$, then $k'_1|A^m$. We return to the equation (1.160). If k'_1 and Ω are coprime, it is impossible. We suppose that k'_1 and Ω are not coprime, then $k'_1|\Omega$ and the exponent of k'_1 in Ω is so the equation (1.160) is satisfying. We deduce easily that $k'_1|B^n$. Then $k'^2_1|(p = A^{2m} + B^{2n} + A^m B^n)$, but $4p = 3k'_1 b \Longrightarrow k'_1|b$, then the contradiction with a, b coprime.

** J-4-1-2-4- We suppose that $k'_1 \geq 4$ is not a prime.

** J-4-1-2-4-1- Supposons que $k'_1 = 4$, we have then $A^{2m} = 4a$ and $B^n C^l = 3b - 4a = 3p' - 4a$. This case was studied in the paragraph 1.5.8, case ** I-2-.

** J-4-1-2-4-2- We suppose that $k'_1 > 4$ is not a prime.

** J-4-1-2-4-2-1- We suppose that a, k'_1 are coprime. From the expression $A^{2m} = k'_1.a$, we deduce that $a = a_1^2$ and $k'_1 = k"^2_1$. It gives :

$$A^m = a_1.k"_1$$
$$B^n C^l = 4^{s-1} k"^2_1.\Omega$$

Let ω be a prime so that $\omega|k"_1$ and $k"_1 = \omega^t.k"_2$ with $\omega \nmid k"_2$. The last two equations become :

$$A^m = a_1.\omega^t.k"_2 \tag{1.161}$$

$$B^n C^l = 4^{s-1}\omega^{2t}.k"^2_2.\Omega \tag{1.162}$$

From (1.161), $\omega|A^m \Longrightarrow \omega|A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$ and $im = t$. From (1.162), we obtain $\omega|B^n C^l \Longrightarrow \omega|B^n$ or $\omega|C^l$.

** J-4-1-2-4-2-1-1- If $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^j.B_1$ with $\omega \nmid B_1$. From (1.161), we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k"_2^2.\Omega$.

** J-4-1-2-4-2-1-1-1- If $\omega = 2$ and $2 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s-j.n-2}k"_2^2.\Omega$:

- If $2t + 2s - jn - 2 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t+2s-jn-2 \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-1-2- If $\omega = 2$ and if $2|\Omega \Longrightarrow \Omega = 2.\Omega_1$ because $4 \nmid \Omega$, we have $B_1^nC^l = 2^{2t+2s+1-j.n-2}k"_2^2\Omega_1$:

- If $2t + 2s - jn - 3 \leq 0$ then $2 \nmid C^l$, then the contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
- If $2t+2s-jn-3 \geq 1 \Longrightarrow 2|C^l \Longrightarrow 2|C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-1-3- If $\omega \neq 2$, we have $B_1^nC^l = \omega^{2t-j.n}4^{s-1}.k"_2^2.\Omega$:

-If $2t - jn \leq 0 \Longrightarrow \omega \nmid C^l$ it is in contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$.
-If $2t - jn \geq 1 \Longrightarrow \omega|C^l \Longrightarrow \omega|C$ and the conjecture (3.1) is verified.

** J-4-1-2-4-2-1-2- If $\omega|C^l \Longrightarrow \omega|C \Longrightarrow C = \omega^h.C_1$, with $\omega \nmid C_1$. Using the same method as in the case J-4-1-2-4-2-1-1 above, we obtain identical results.

** J-4-1-2-4-2-2- We suppose that a, k_1' are not coprime. Let ω be a prime so that $\omega|a$ and $\omega|k_1'$. We write:

$$a = \omega^\alpha.a_1$$
$$k_1' = \omega^\mu.k"_1$$

with $a_1, k"_1$ coprime. The expression of A^{2m} becomes $A^{2m} = \omega^{\alpha+\mu}.a_1.k"_1$. The term B^nC^l becomes:

$$B^nC^l = 4^{s-1}.\omega^\mu.k"_1.\Omega \qquad (1.163)$$

** J-4-1-2-4-2-2-1- If $\omega = 2 \Longrightarrow 2|a$, but $2|b$, then the contradiction with a, b coprime, this case is impossible.

** J-4-1-2-4-2-2-2- If $\omega \geq 3$, we have $\omega|a$. If $\omega|b$ then the contradiction with a, b coprime. We suppose that $\omega \nmid b$. From the expression of A^{2m}, we obtain $\omega|A^{2m} \Longrightarrow \omega|A \Longrightarrow A = \omega^i.A_1$ with $\omega \nmid A_1$, $i \geq 1$ and $2i.m = \alpha + \mu$. From (1.163), we deduce that $\omega|B^n$ or $\omega|C^l$.

** J-4-1-2-4-2-2-2-1- We suppose that $\omega|B^n \Longrightarrow \omega|B \Longrightarrow B = \omega^j B_1$ with $\omega \nmid B_1$ and $j \geq 1$. Then, $B_1^n C^l = 4^{s-1}\omega^{\mu-jn}.k"_1.\Omega$:

* $\omega \nmid \Omega$:

- If $\mu - jn \geq 1$, we have $\omega|C^l \Longrightarrow \omega|C$, there is no contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $\mu - jn \leq 0$, then $\omega \nmid C^l$ and it is a contradiction with $C^l = \omega^{im}A_1^m + \omega^{jn}B_1^n$. Then this case is impossible.

* $\omega|\Omega$: we write $\Omega = \omega^\beta.\Omega_1$ with $\beta \geq 1$ and $\omega \nmid \Omega_1$. As $3b - 4a = 4^s.\Omega = 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4a + 4^s.\omega^\beta.\Omega_1 = 4\omega^\alpha.a_1 + 4^s.\omega^\beta.\Omega_1 \Longrightarrow 3b = 4\omega(\omega^{\alpha-1}.a_1 + 4^{s-1}.\omega^{\beta-1}.\Omega_1)$. If $\omega = 3$ and $\beta = 1$, we obtain $b = 4(3^{\alpha-1}a_1 + 4^{s-1}\Omega_1)$ and $B_1^n C^l = 4^{s-1}3^{\mu+1-jn}.k"_1\Omega_1$.

- If $\mu - jn + 1 \geq 1$, then $3|C^l$ and the conjecture (3.1) is verified.

- If $\mu - jn + 1 \leq 0$, then $3 \nmid C^l$ and it is the contradiction with $C^l = 3^{im}A_1^m + 3^{jn}B_1^n$.

Now, if $\beta \geq 2$ and $\alpha = im \geq 3$, we obtain $3b = 4\omega^2(\omega^{\alpha-2}a_1 + 4^{s-1}\omega^{\beta-2}\Omega_1)$. If $\omega = 3$ or not, then $\omega|b$, but $\omega|a$, then the contradiction with a, b coprime.

** J-4-1-2-4-2-2-2-2- We suppose that $\omega|C^l \Longrightarrow \omega|C \Longrightarrow C = \omega^h C_1$ with $\omega \nmid C_1$ and $h \geq 1$. then, $B^n C_1^l = 4^{s-1}\omega^{\mu-hl}.k"_1.\Omega$. Using the same method as above, we obtain identical results.

** J-4-2- We suppose that $4|k'_1$.

** J-4-2-1- $k'_1 = 4 \Longrightarrow 4p = 3k'_1 b = 12b \Longrightarrow p = 3b = 3p'$, this case has been studied (see case I-2- paragraph 1.5.8).

** J-4-2-2- $k'_1 > 4$ with $4|k'_1 \Longrightarrow k'_1 = 4^s k"_1$ and $s \geq 1$, $4 \nmid k"_1$. Then, we obtain:

$$A^{2m} = 4^s k"_1 a = 2^{2s} k"_1 a$$
$$B^n C^l = 4^{s-1} k"_1(3b - 4a) = 2^{2s-2} k"_1(3b - 4a)$$

** J-4-2-2-1- We suppose that $s = 1$ and $k'_1 = 4k"_1$ with $k"_1 > 1$, so $p = 3p'$ and $p' = k"_1 b$, this is the case 1.5.3 already studied.

** J-4-2-2-2- We suppose that $s > 1$, then $k'_1 = 4^s k"_1 \Longrightarrow 4p = 3 \times 4^s k"_1 b$ and we obtain:

$$A^{2m} = 4^s k"_1 a \tag{1.164}$$

$$B^n C^l = 4^{s-1} k"_1 (3b - 4a) \tag{1.165}$$

** J-4-2-2-2-1- We suppose that $2 \nmid (k"_1.a) \Longrightarrow 2 \nmid k"_1$ and $2 \nmid a$. As $(A^m)^2 = (2^s)^2.(k"_1.a)$, we call $d^2 = k"_1.a$, then $A^m = 2^s.d \Longrightarrow 2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i A_1$ with $2 \nmid A_1$ and $i \geq 1$, then: $2^{im} A_1^m = 2^s.d \Longrightarrow s = im$. From the equation (1.165), we have $2|(B^n C^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** J-4-2-2-2-1-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^j.B_1$, with $j \geq 1$ and $2 \nmid B_1$. The equation (1.165) becomes:

$$B_1^n C^l = 2^{2s-jn-2} k"_1 (3b - 4a) = 2^{2im-jn-2} k"_1 (3b - 4a)$$

* We suppose that $2 \nmid (3b - 4a)$:

- If $2im - jn - 2 \geq 1$, then $2|C^l$, there is no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

* We suppose that $2^\mu|(3b - 4a)$, $\mu \geq 1$:

- If $2im + \mu - jn - 2 \geq 1$, then $2|C^l$, no contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$ and the conjecture (3.1) is verified.

- If $2im + \mu - jn - 2 \leq 0$, then $2 \nmid C^l$, then the contradiction with $C^l = 2^{im} A_1^m + 2^{jn} B_1^n$.

** J-4-2-2-2-1-2- We suppose that $2|C^l \Longrightarrow 2|C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2- We suppose that $2|(k"_1.a)$:

** J-4-2-2-2-2-1- We suppose that $k"_1$ and a are coprime:

** J-4-2-2-2-2-1-1- We suppose that $2 \nmid a$ and $2|k"_1 \Longrightarrow k"_1 = 2^{2\mu}.k"^2_2$ and $a = a_1^2$, then the equations (1.164-1.165) become:

$$A^{2m} = 4^s.2^{2\mu} k"^2_2 a_1^2 \Longrightarrow A^m = 2^{s+\mu}.k"_2.a_1 \tag{1.166}$$

$$B^n C^l = 4^{s-1} 2^{2\mu} k"^2_2 (3b - 4a) = 2^{2s+2\mu-2} k"^2_2 (3b - 4a) \tag{1.167}$$

The equation (1.166) gives $2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.167), we have $2|(B^nC^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** J-4-2-2-2-2-1-1-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$, then $B_1^nC^l = 2^{2s+2\mu-jn-2}k"_2^2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:
- If $2im + 2\mu - jn - 2 \geq 1 \Rightarrow 2|C^l$, then there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im + 2\mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha|(3b-4a)$, $\alpha \geq 1$ so that a, b remain coprime:
- If $2im + 2\mu + \alpha - jn - 2 \geq 1 \Rightarrow 2|C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2im + 2\mu + \alpha - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-2-2-2-2-1-1-2- We suppose that $2|C^l \Longrightarrow 2|C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-1-2- We suppose that $2 \nmid k"_1$ and $2|a \Longrightarrow a = 2^{2\mu}.a_1^2$ and $k"_1 = k"_2^2$, then the equations (1.164-1.165) become:

$$A^{2m} = 4^s.2^{2\mu}a_1^2k"_2^2 \Longrightarrow A^m = 2^{s+\mu}.a_1.k"_2. \tag{1.168}$$

$$B^nC^l = 4^{s-1}k"_2^2(3b-4a) = 2^{2s-2}k"_2^2(3b-4a) \tag{1.169}$$

The equation (1.168) gives $2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i.A_1$ with $2 \nmid A_1$, $i \geq 1$ and $im = s + \mu$. From the equation (1.169), we have $2|(B^nC^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** J-4-2-2-2-2-1-2-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^nC^l = 2^{2s-jn-2}k"_2^2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a) \Longrightarrow 2 \nmid b$:
- If $2im-jn-2 \geq 1 \Rightarrow 2|C^l$, then no contradiction with $C^l = 2^{im}A_1^m+2^{jn}B_1^n$ and the conjecture (3.1) is verified.

- If $2im - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha|(3b-4a)$, $\alpha \geq 1$, in this case a, b are not coprime, then the contradiction.

** J-4-2-2-2-2-1-2-2- We suppose that $2|C^l \Longrightarrow 2|C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-2-2-2-2-2- We suppose that $k"_1$ and a are not coprime $2|a$ and $2|k"_1$. Let $a = 2^t.a_1$ and $k"_1 = 2^\mu k"_2$ and $2 \nmid a_1$ and $2 \nmid k"_2$. From (1.164), we have $\mu + t = 2\lambda$ and $a_1.k"_2 = \omega^2$. The equations (1.164-1.165) become:

$$A^{2m} = 4^s k"_1 a = 2^{2s}.2^\mu k"_2.2^t.a_1 = 2^{2s+2\lambda}.\omega^2 \Longrightarrow A^m = 2^{s+\lambda}.\omega \tag{1.170}$$

$$B^n C^l = 4^{s-1}2^\mu k"_2(3b-4a) = 2^{2s+\mu-2}k"_2(3b-4a) \tag{1.171}$$

From (1.170) we have $2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i A_1, i \geq 1$ and $2 \nmid A_1$. From(1.171), $2s + \mu - 2 \geq 1$, we deduce that $2|(B^n C^l) \Longrightarrow 2|B^n$ or $2|C^l$.

** J-4-2-2-2-2-2-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^j.B_1$, $2 \nmid B_1$ and $j \geq 1$. Then we obtain $B_1^n C^l = 2^{2s+\mu-jn-2}k"^2(3b-4a)$:

* We suppose that $2 \nmid (3b-4a)$:

- If $2s + \mu - jn - 2 \geq 1 \Rightarrow 2|C^l$, then no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2s + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C^l$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

* We suppose that $2^\alpha|(3b-4a)$, for one value $\alpha \geq 1$. As $2|a$, then $2^\alpha|(3b-4a) \Longrightarrow 2|(3b-4a) \Longrightarrow 2|(3b) \Longrightarrow 2|b$, then the contradiction with a, b coprime.

** J-4-2-2-2-2-2-2- We suppose that $2|C^l \Longrightarrow 2|C \Longrightarrow C = 2^h.C_1$, with $h \geq 1$ and $2 \nmid C_1$. With the same method used above, we obtain identical results.

** J-4-3- $2|k'_1$ and $2|(3b-4a)$: then we obtain $2|k'_1 \Longrightarrow k'_1 = 2^t.k"_1$ with $t \geq 1$ and $2 \nmid k"_1$, $2|(3b-4a) \Longrightarrow 3b - 4a = 2^\mu.d$ with $\mu \geq 1$ and $2 \nmid d$. We have also

$2|b$. If $2|a$, it is a contradition with a, b coprime.

We suppose, in the following, that $2 \nmid a$. The equations (1.164-1.165) become:

$$A^{2m} = 2^t.k"_1.a = (A^m)^2 \tag{1.172}$$

$$B^nC^l = 2^{t-1}k"_1.2^{\mu-1}d = 2^{t+\mu-2}k"_1.d \tag{1.173}$$

From (1.172), we deduce that the exponent t is even, let $t = 2\lambda$. Then we call $\omega^2 = k"_1.a$, it gives $A^m = 2^\lambda.\omega \Longrightarrow 2|A^m \Longrightarrow 2|A \Longrightarrow A = 2^i.A_1$ with $i \geq 1$ and $2 \nmid A_1$. From (1.173), we have $2\lambda + \mu - 2 \geq 1$, then $2|(B^nC^l) \Longrightarrow 2|B^n$ or $2|C^l$:

** J-4-3-1- We suppose that $2|B^n \Longrightarrow 2|B \Longrightarrow B = 2^jB_1$, with $j \geq 1$ and $2 \nmid B_1$. Then we obtain $B_1^nC^l = 2^{2\lambda+\mu-jn-2}.k"_1.d$.

- If $2\lambda + \mu - jn - 2 \geq 1 \Rightarrow 2|C^l \Longrightarrow 2|C$, there is no contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$ and the conjecture (3.1) is verified.
- If $2s + t + \mu - jn - 2 \leq 0 \Rightarrow 2 \nmid C$, then the contradiction with $C^l = 2^{im}A_1^m + 2^{jn}B_1^n$.

** J-4-3-2- We suppose that $2|C^l \Longrightarrow 2|C$. With the same method used above, we obtain identical results.

□

The Main Theorem is proved.

1.6. Numerical Examples

1.6.1. Example 1: — We consider the example : $6^3 + 3^3 = 3^5$ with $A^m = 6^3$, $B^n = 3^3$ and $C^l = 3^5$. With the notations used in the paper, we obtain:

$$p = 3^6 \times 73, \quad q = 8 \times 3^{11}, \quad \bar{\Delta} = 4 \times 3^{18}(3^7 \times 4^2 - 73^3) < 0$$

$$\rho = \frac{3^8 \times 73\sqrt{73}}{\sqrt{3}}, \quad cos\theta = -\frac{4 \times 3^3 \times \sqrt{3}}{73\sqrt{73}} \tag{1.174}$$

As $A^{2m} = \dfrac{4p}{3}.cos^2\dfrac{\theta}{3} \Longrightarrow cos^2\dfrac{\theta}{3} = \dfrac{3A^{2m}}{4p} = \dfrac{3 \times 2^4}{73} = \dfrac{a}{b} \Longrightarrow a = 3 \times 2^4,\ b = 73$; then we obtain:

$$cos\frac{\theta}{3} = \frac{4\sqrt{3}}{\sqrt{73}}, \quad p = 3^6.b \tag{1.175}$$

We verify easily the equation (1.174) to calculate $cos\theta$ using (1.175). For this example, we can use the two conditions from (1.64) as $3|a$,$b|4p$ and $3|p$. The cases 1.4.4 and 1.5.3 are respectively used. For the case 1.4.4, it is the case B-2-2-1- that was used and the conjecture (3.1) is verified. Concerning the case 1.5.3, it is the case G-2-2-1- that was used and the conjecture (3.1) is verified.

1.6.2. Example 2: — The second example is: $7^4+7^3=14^3$. We take $A^m=7^4\,, B^n=7^3$ and $C^l=14^3$. We obtain $p=57\times 7^6=3\times 19\times 7^6\,,\quad q=8\times 7^{10}\,,\quad \overline{\Delta}=27q^2-4p^3=27\times 4\times 7^{18}(16\times 49-19^3)=-27\times 4\times 7^{18}\times 6075<0\,,\quad \rho=19\times 7^9\times\sqrt{19}\,,\quad cos\theta=-\dfrac{4\times 7}{19\sqrt{19}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{7^2}{4\times 19}=\dfrac{a}{b}\Longrightarrow a=7^2,\ b=4\times 19$, then $cos\dfrac{\theta}{3}=\dfrac{7}{2\sqrt{19}}$ and we have the two principal conditions $3|p$ and $b|(4p)$. The calculation of $cos\theta$ from the expression of $cos\dfrac{\theta}{3}$ is confirmed by the value below:

$$cos\theta=cos3(\theta/3)=4cos^3\frac{\theta}{3}-3cos\frac{\theta}{3}=4\left(\frac{7}{2\sqrt{19}}\right)^3-3\frac{7}{2\sqrt{19}}=-\frac{4\times 7}{19\sqrt{19}}$$

Then, we obtain $3|p\Rightarrow p=3p'$, $b|(4p)$ with $b\neq 2,4$ then $12p'=k_1b=3\times 7^6b$. It concerns the paragraph 1.5.9 of the second hypothesis. As $k_1=3\times 7^6=3k'_1$ with $k'_1=7^6\neq 1$. It is the case J-4-1-2-4-2-2- with the condition $4|(3b-4a)$. So we verify :

$$3b-4a=3\times 4\times 19-4\times 7^2=32\Longrightarrow 4|(3b-4a)$$

with $A^{2m}=7^8=7^6\times 7^2=k'_1.a$ and k'_1 not a prime, with a and k'_1 not coprime with $\omega=7\nmid\Omega(=2)$. We find that the conjecture (3.1) is verified with a common factor equal to 7 (prime and divisor of $k'_1=7^6$).

1.6.3. Example 3: — The third example is: $19^4+38^3=57^3$ with $A^m=19^4$, $B^n=38^3$ and $C^l=57^3$. We obtain $p=19^6\times 577\,,\quad q=8\times 27\times 19^{10}\,,\quad \overline{\Delta}=27q^2-4p^3=4\times 19^{18}(27^3\times 16\times 19^2-577^3)<0\,,\quad \rho=\dfrac{19^9\times 577\sqrt{577}}{3\sqrt{3}}\,,\quad cos\theta=-\dfrac{4\times 3^4\times 19\sqrt{3}}{577\sqrt{577}}$. As $A^{2m}=\dfrac{4p}{3}.cos^2\dfrac{\theta}{3}\Longrightarrow cos^2\dfrac{\theta}{3}=\dfrac{3A^{2m}}{4p}=\dfrac{3\times 19^2}{4\times 577}=\dfrac{a}{b}\Longrightarrow a=3\times 19^2,\ b=4\times 577$, then $cos\dfrac{\theta}{3}=\dfrac{19\sqrt{3}}{2\sqrt{577}}$ and we have the first hypothesis $3|a$ and $b|(4p)$. Here again, the calculation of $cos\theta$ from the expression of $cos\frac{\theta}{3}$ is confirmed by the value below:

$$cos\theta = cos3(\theta/3) = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3} = 4\left(\frac{19\sqrt{3}}{2\sqrt{577}}\right)^3 - 3\frac{19\sqrt{3}}{2\sqrt{577}} = -\frac{4 \times 3^4 \times 19\sqrt{3}}{577\sqrt{577}}$$

Then, we obtain $3|a \Rightarrow a = 3a' = 3 \times 19^2$, $b|(4p)$ with $b \neq 2, 4$ and $b = 4p'$ with $p = kp'$ soit $p' = 577$ and $k = 19^6$. This concerns the paragraph 1.4.8 of the first hypothesis. It is the case E-2-2-2-2-1- with $\omega = 19$, a', ω not coprime and $\omega = 19 \nmid (p' - a') = (577 - 19^2)$ with $s - jn = 6 - 1 \times 3 = 3 \geq 1$, and the conjecture (3.1) is verified.

1.7. Conclusion

The method used to give the proof of the conjecture of Beal has discussed many possibles cases, using elementary number theory and the results of some theorems about Diophantine equations. We have confirmed the method by three numerical examples. In conclusion, we can announce the theorem:

Theorem 1.3. — *Let $A, B, C, m, n,$ and l be positive natural numbers with $m, n, l > 2$. If :*

$$A^m + B^n = C^l \tag{1.176}$$

then $A, B,$ and C have a common factor.

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BIBLIOGRAPHY

[1] E.D. Bolker, *Elementary Number Theory: An Algebraic Approach.* W.A. Benjamin, Inc., New-York, (1970) 195 pages.

[2] D.A. Cox, *Primes of the form :*$x^2 + ny^2$*, Fermat, class field theory and complex multiplication.* A Wiley-Interscience Publication, John Wiley & Sons Inc., New-York, (1989) 363 pages.

[3] G. Frei, *Leonhard Euler's convenient numbers. The Mathematical Intelligencer*, **Vol. 7**, $n°3$, (1985) pp.55-58 and 64.

[4] D.R. Mauldin, *A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem. Notice of AMS*, **Vol 44**, $n°11$, (1977) pp 1436-1437.

[5] B.M. Stewart, *Theory of Numbers*, Second edition. The Macmillan Compagny, New-York, (1964) 390 pages.

CHAPTER 2

TOWARDS A SOLUTION OF THE RIEMANN HYPOTHESIS

Abstract. — In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros)* $s = \sigma + it$ *of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \quad \Re(s) > 1$$

have real part $\sigma = \dfrac{1}{2}$.

We give a proof that $\sigma = \dfrac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

Résumé. — En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux* $s = \sigma + it$ *de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ pour } \quad \Re(s) > 1$$

ont comme parties réelles $\sigma = \dfrac{1}{2}$.

On donne une démonstration que $\sigma = \dfrac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

2.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 2.1. — . *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \frac{1}{2}$ except at most for a finite number of zeros.

2.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 2.2. — . *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line. We have also the theorem (see page 16, [**3**]):

Theorem 2.3. — . *For all* $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]). Then, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

The Riemann Hypothesis is formulated as:

Conjecture 2.4. — . *(The Riemann Hypothesis,*[**2**]*) All nontrivial zeros of* $\zeta(s)$ *lie on the critical line* $\Re(s) = \dfrac{1}{2}$.

In addition to the properties cited by the theorem 2.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0,1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{2.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$$

So, instead of using the functional given by (2.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

2.1.2. A Equivalent statement to the Riemann Hypothesis. —

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 2.5. — . The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{2.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (2.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{2.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{2.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2.2. Proof that the zeros of $\eta(s)$ are on the critical line $\Re(s) = \frac{1}{2}$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of the function $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longrightarrow (1 - 2^{1-s})\zeta(s) = 0$. Let us denote $\zeta(s) = A + iB$, and $\theta = tLog2$, then :

$$(1-2^{1-s})\zeta(s) = \left[A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta\right] + i\left[B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta\right]$$

$(1 - 2^{1-s})\zeta(s) = 0$ gives the system:

$$A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta = 0$$
$$B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0$$

As the functions *sin* and *cos* are not equal to 0 simultaneously, we suppose for example that $sin\theta \neq 0$, the first equation of the system gives $B = \frac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, the second equation is written as :

$$\frac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0 \Longrightarrow A = 0$$

Then, $B = 0 \Longrightarrow \zeta(s) = 0$, it follows that:
(2.5)

s is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = A + iB = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:
(2.6)

s is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall \mathscr{N} > n_0, \ \left| \sum_{n=1}^{\mathscr{N}} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n \longrightarrow +\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}=0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}=0$$

Using the definition of the limit of a sequence, we can write:

(2.7) $\forall\epsilon_1>0 \quad \exists n_r,\forall N>n_r \quad |\Re(\eta(s)_N)|<\epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2<\epsilon_1^2$

(2.8) $\forall\epsilon_2>0 \quad \exists n_i,\forall N>n_i \quad |\Im(\eta(s)_N)|<\epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2<\epsilon_2^2$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma}<\epsilon_1^2$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma}<\epsilon_2^2$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

(2.9) $$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2$$

We can write the above equation as :

(2.10) $$0<\ \rho_N^2\ <2\epsilon^2$$

or $\rho(s)=0$.

2.2.1. Case $\sigma=\frac{1}{2}\Longrightarrow 2\sigma=1$. — We suppose that $\sigma=\frac{1}{2}\Longrightarrow 2\sigma=1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 2.6.*** — . *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.5-2.6), it follows the proposition :

> ***Proposition 2.7.*** — . There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{2.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

As $t_j > 0$, and that there is an infinity of zeros on the critical line, then the result of the formula given by (2.11) is independent of t_j. We return now to $s = \sigma + it$ one zero of $\eta(s)$ on the critical, let $\eta(s) = 0$. We take $\sigma = \frac{1}{2}$. Starting from the definition of the limit of sequences, applied above, we obtain:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

with any contradiction. From the proposition (2.5), it follows that $\zeta(s) = \zeta(\frac{1}{2} + it) = 0$. There are therefore zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$.

2.2.2. Case $0 < \Re(s) < \frac{1}{2}$. —

2.2.2.1. *Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.* — Using, for this case, point 4 of theorem (2.2), we deduce that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\dfrac{1}{2} < \sigma < 1$. Then, from the proposition (2.5), it follows that the function $\zeta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \dfrac{1}{2}$ and the **Riemann Hypothesis is true**.

2.2.2.2. *Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$.* — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (2.9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} = -\infty} \tag{2.12}$$

Again, the above result is independent of t.

2.2.3. Case $\dfrac{1}{2} < \Re(s) < 1$. — Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. According to point 4 of theorem 2.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$, is also a zero of the function $\eta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, that is $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{2.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma')$$

From the equation (2.13), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}>-\infty \tag{2.14}$$

Then, we have the 2 following cases:

1)- There exists an infinity of complex numbers $s_l=\sigma_l+it_l$ with $\sigma_l\in]0,1/2[$ such that $\eta(s_l)=0$. For each s'_l, the left member of the equation (2.14) above is finite and depends of σ'_l and t'_l, but the right member is a function only of σ'_l. Hence the contradiction, therefore, the function $\eta(s)$ has all its zeros on the critical line $\sigma=\dfrac{1}{2}$. It follows that **the Riemann hypothesis is verified**.

2)- There is at most a single zero $s_0=\sigma_0+it_0$ of $\eta(s)$ with $\sigma_0\in]0,1/2[,t_0>0$ such that $\eta(s_0)=0$. Let us call this zero *isolated zero* that we denote by (IZ). Therefore, the interval $]1/2,1[$ contains a single zero $s'_0=1-\sigma_0+it_0$. Since the critical line contains an infinity of zeros of $\zeta(s)=0$, it follows that all the nontrivial zeros of $\zeta(s)$ are on the critical line $\sigma=\dfrac{1}{2}$, except the 4 zeros relative to (IZ). Here too, we deduce that **the Riemann Hypothesis holds** except at most for the (IZ) in the critical band. □

2.3. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s),\quad s=\sigma+it$$

on the critical band $0<\Re(s)<1$, in obtaining:

- $\eta(s)$ vanishes for $0<\sigma=\Re(s)=\dfrac{1}{2}$;

- $\eta(s)$ does not vanish for $0<\sigma=\Re(s)<\dfrac{1}{2}$ and $\dfrac{1}{2}<\sigma=\Re(s)<1$ except at most for the (IZ) (with its symmetrical) inside the critical band.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0<\Re(s)<1$ vanish on the critical line $\Re(s)=\dfrac{1}{2}$ except at most at (IZ) (with its symmetrical). Applying the equivalent proposition to the Riemann Hypothesis 3.1, all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s)=\dfrac{1}{2}$ except at most at (IZ) (with its symmetrical) inside the critical band. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 2.8. — . *All nontrivial zeros of the function* $\zeta(s)$ *with* $s = \sigma + it$ *lie on the vertical line* $\Re(s) = \frac{1}{2}$, *except for at most four zeros of respective affixes* $(\sigma_0, t_0), (1-\sigma_0, t_0), (\sigma_0, -t_0), (1-\sigma_0, -t_0)$, *belonging to the critical band.*

BIBLIOGRAPHY

[1] Bombieri E.:, *The Riemann Hypothesis*, In The millennium prize problems. J. Carlson, A. Jaffe, and A. Wiles Editors. Published by The American Mathematical Society, Providence, RI, for The Clay Mathematics Institute, Cambridge, MA. (2006), 107–124.

[2] Borwein P., Choi S., Rooney B. and Weirathmueller A.: *The Riemann hypothesis - a resource for the afficionado and virtuoso alike.* 1st Ed. CMS Books in Mathematics. Springer-Verlag New York. 588p. (2008)

[3] Titchmarsh E.C., Heath-Brown D.R.: *The theory of the Riemann zeta-function.* 2sd Ed. revised by D.R. Heath-Brown. Oxford University Press, New York. 418p. (1986)

CHAPTER 3

IS THE *abc* CONJECTURE TRUE?

Abstract. — In this paper, we consider the *abc* conjecture. In the first part, we give the proof of the conjecture $c < rad^{1.63}(abc)$ that constitutes the key to resolve the *abc* conjecture. The proof of the *abc* conjecture is given in the second part of the paper, supposing that the *abc* conjecture is false, we arrive in a contradiction.

Résumé. — Dans cet article, nous considérons la conjecture *abc*. Dans la première partie, nous donnons la preuve de la conjecture $c < rad^{1.63}(abc)$ qui constitue la clé pour résoudre la conjecture *abc*. Dans la deuxième partie de l'article, la preuve de la conjecture *abc* est donnée en supposant qu'elle est fausse, nous arrivons à une contradiction.

3.1. Introduction and notations

Let a be a positive integer, $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as:

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{3.1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{3.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [**8**]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 3.1. — *(**abc Conjecture**): For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.3}$$

where K is a constant depending only of ϵ.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [**5**]. It concerned the best example given by E. Reyssat [**5**]:

$$2 + 3^{10}.109 = 23^5 \Longrightarrow c < rad^{1.629912}(abc) \tag{3.4}$$

A conjecture was proposed that $c < rad^2(abc)$ [**3**]. In 2012, A. Nitaj [**4**] proposed the following conjecture:

Conjecture 3.2. — *Let a, b, c be positive integers relatively prime with $c = a + b$, then:*

$$c < rad^{1.63}(abc) \tag{3.5}$$

$$abc < rad^{4.42}(abc) \tag{3.6}$$

Firstly, we will give the proof of the conjecture given by (3.5) that constitutes the key to obtain the proof of the *abc* conjecture. Secondly, we present in section three of the paper the proof that the *abc* conjecture is true.

3.2. A Proof of the conjecture $c < rad^{1.63}(abc)$, case $c = a + b$

Let a, b, c be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = rad(abc)$, $c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1$.

In a previous paper [**1**], we have given, for the case $c = a + 1$, the proof that $c < rad^{1.63}(ac)$. In the following, we will give the proof for the case $c = a + b$.

Proof. — If $c < rad(abc)$, then we obtain:

$$c < rad(abc) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (3.5) is satisfied.

If $c = rad(abc)$, then a, b, c are not coprime, case to reject. In the following, we suppose that $c > rad(abc)$ and a, b and c are not prime numbers.

$$c = a + b = \mu_a rad(a) + \mu_b rad(b) \overset{?}{<} rad^{1.63}(abc) \tag{3.7}$$

3.2.1. $\mu_a \leq rad^{0.63}(a)$**.** — We obtain :

$$c = a+b < 2a \leq 2rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow c < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

Then (3.7) is satisfied.

3.2.2. $\mu_c \leq rad^{0.63}(c)$**.** — We obtain :

$$c = \mu_c rad(c) \leq rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$$

and the condition (3.7) is satisfied.

3.2.3. $\mu_a > rad^{0.63}(a)$ **and** $\mu_c > rad^{0.63}(c)$**.** —

3.2.3.1. ***Case:*** $rad^{0.63}(c) < \mu_c \leq rad^{1.63}(c)$ ***and*** $rad^{0.63}(a) < \mu_a \leq rad^{1.63}(a)$: — We can write:

$$\left.\begin{array}{l} \mu_c \leq rad^{1.63}(c) \Longrightarrow c \leq rad^{2.63}(c) \\ \\ \mu_a \leq rad^{1.63}(a) \Longrightarrow a \leq rad^{2.63}(a) \end{array}\right\} \Longrightarrow ac \leq rad^{2.63}(ac) \Longrightarrow a^2 < ac \leq rad^{2.63}(ac)$$

$$\Longrightarrow a < rad^{1.315}(ac) \Longrightarrow c < 2a < 2rad^{1.315}(ac) < rad^{1.63}(abc)$$

$$\Longrightarrow \boxed{c = a + b < R^{1.63}}$$

3.2.3.2. ***Case:*** $\mu_c > rad^{1.63}(c)$ ***or*** $\mu_a > rad^{1.63}(a)$. — I- We suppose that $\mu_c > rad^{1.63}(c)$ and $\mu_a \leq rad^2(a)$:

I-1- Case $rad(a) < rad(c)$: In this case $a = \mu_a.rad(a) \leq rad^3(a) \leq rad^{1.63}(a) rad^{1.37}(a) <$
$rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.37}(c) < rad^{1.63}(abc) \Longrightarrow$ $\boxed{c < R^{1.63}}$.

I-2- Case $rad(c) < rad(a) < rad^{\frac{1.63}{1.37}}(c)$: As $a \leq rad^{1.63}(a).rad^{1.37}(a) < rad^{1.63}(a).rad^{1.63}(c) \Longrightarrow c < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < R^{1.63} \Longrightarrow$ $\boxed{c < R^{1.63}}$.

I-3- Case $rad^{\frac{1.63}{1.37}}(c) < rad(a)$:

I-3-1- We suppose $c \leq rad^{3.26}(c)$, we obtain:

$$c \leq rad^{3.26}(c) \Longrightarrow c \leq rad^{1.63}(c).rad^{1.63}(c) \Longrightarrow$$

$$c < rad^{1.63}(c).rad(a)^{1.37} < rad^{1.63}(c).rad(a)^{1.63}.rad^{1.63}(b) = R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$$

I-3-2- We suppose $c > rad^{3.26}(c) \Longrightarrow \mu_c > rad^{2.26}(c)$. We consider the case $\mu_a = rad^2(a) \Longrightarrow a = rad^3(a)$. Then, we obtain that $X = rad(a)$ is a solution in positive integers of the equation:

$$X^3 + 1 = c - b + 1 = c' \tag{3.8}$$

But it is the case $c' = 1 + a$. If $c' = rad^n(c')$ with $n \geq 4$, we obtain the equation:

$$rad^n(c') - rad^3(a) = 1 \tag{3.9}$$

But the solutions of the equation (3.9) are [**2**] :$(rad(c') = 3, n = 2, rad(a) = +2)$, it follows the contradiction with $n \geq 4$ and the case $c' = rad^n(c'), n \geq 4$ is to reject.

In the following, we will study the cases $\mu'_c = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (3.8) can be written as :

$$(X + 1)(X^2 - X + 1) = c' \tag{3.10}$$

Let δ any divisor of c', then:

$$X + 1 = \delta \tag{3.11}$$

$$X^2 - X + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3X \tag{3.12}$$

We recall that $rad(a) > rad^{\frac{1.63}{1.37}}(c)$.

I-3-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \Longrightarrow l < \mu'_c$. As δ is a divisor of c', then l is a divisor of μ'_c, we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3X)$, we obtain:

$$m = l^2 rad^2(c') - 3rad(a) \Longrightarrow 3rad(a) = l^2 rad^2(c') - m$$

A- Case $3|m \Longrightarrow m = 3m', m' > 1$: As $\mu'_c = ml = 3m'l \Longrightarrow 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

$$rad(a) = l^2 rad(c').\frac{rad(c')}{3} - m'$$

It follows that a,c' are not coprime, then the contradiction.

B - Case $m = 3 \Longrightarrow \mu'_c = 3l \Longrightarrow c' = 3lrad(c') = 3\delta = \delta(\delta^2 - 3X) \Longrightarrow \delta^2 = 3(1 + X) = 3\delta \Longrightarrow \delta = lrad(c') = 3$, then the contradiction.

I-3-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $n = 0 \Rightarrow c'' = \frac{A}{lrad(c')} \Rightarrow rad(c')|A$, then the contradiction with the hypothesis above $rad(c') \nmid A$. In the following, we suppose that $n > 0$. If $lrad(c') \nmid \mu_{c'}$ then the case is to reject. We suppose $lrad(c')|\mu_{c'} \Longrightarrow \mu_{c'} = m.lrad(c')$, then $\dfrac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \Longrightarrow \delta^2 - 3rad(a) = 1 \Longrightarrow (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \Longrightarrow \delta = 2 = l.rad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \Longrightarrow \delta = 3 = lrad^2(c')$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c') - m \Longrightarrow rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

I-3-2-3- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu'_c.rad(c') = lrad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$.

F - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let q be a prime that divides m, it follows $q|\mu'_c \Longrightarrow q = c'_{j'_0} \Longrightarrow c'_{j'_0}|\delta^2 \Longrightarrow c'_{j'_0}|3rad(a)$. Then $rad(a)$ and $rad(c')$ are not coprime. It follows the contradiction.

I-3-2-4- We suppose $\delta = \prod_{j\in J_1} c'^{\beta_j}_j$, $\beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $rad(c') \nmid \delta$. We can write:

$$\delta = \mu_\delta.rad(\delta), \quad rad(c') = m.rad(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1 \tag{3.13}$$

Then, we obtain:

$$c' = \mu'_c.rad(c') = \mu'_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_\delta.rad(\delta)(\delta^2 - 3X) \Longrightarrow$$
$$m.\mu'_c = \mu_\delta(\delta^2 - 3X) \tag{3.14}$$

- If $\mu'_c = \mu_\delta \Longrightarrow m = \delta^2 - 3X = (\mu'_c.rad(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Longrightarrow m > \delta \Longrightarrow rad(c') > m > \mu'_c.rad(\delta) > rad^3(c')$ because $\mu'_c > rad^{2.26}(c')$, it follows $rad(c') > rad^2(c')$. Then the contradiction.

- We suppose $\mu'_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu'_c.rad(\delta) - 1 > 0 \Longrightarrow rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow$$

$$c' > rad(ac') > c'.rad(\delta) - rad(c') > 0 \Longrightarrow 1 > rad(\delta) - \frac{rad(c')}{c'} > 0, \quad rad(\delta) \geq 2$$

$$\Longrightarrow \text{The contradiction} \tag{3.15}$$

- We suppose $\mu_\delta < \mu'_c$. In this case, from the equation (3.14) and as $(m, \mu_\delta) = 1$, it follows we can write:

$$\mu'_c = \mu_1.\mu_2, \quad \mu_1, \mu_2 > 1 \tag{3.16}$$

$$c' = \mu'_c rad(c') = \mu_1.\mu_2.rad(\delta).m = \delta.(\delta^2 - 3X) \tag{3.17}$$

$$\text{so that} \quad m.\mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \Longrightarrow \delta = \mu_2.rad(\delta) \tag{3.18}$$

** We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j0}$ so that $c'_{j0}|\mu_1$ and $c'_{j0}|\mu_2$. But $\mu_\delta = \mu_2 \Rightarrow c'^2_{j0}|\delta$. From $3X = \delta^2 - m\mu_1 \Longrightarrow c'_{j0}|3X \Longrightarrow c'_{j0}|X$ or $c'_{j0} = 3$.

- If $c'_{j0}|X$, it follows the contradiction with $(c', a) = 1$.

- If $c'_{j0} = 3$. We have $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Longrightarrow \delta^2 - 3\delta + 3 - m.\mu_1 = 0$. As $3|\mu_1 \Longrightarrow \mu_1 = 3^k\mu'_1, 3 \nmid \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0 \tag{3.19}$$

- We consider the case $k > 1 \Longrightarrow 3 \nmid (1 - 3^{k-1}m\mu'_1)$. Let us recall the Eisenstein criterion [**7**]:

> ***Theorem 3.3. — (Eisenstein Criterion)*** *Let $f = a_0 + \cdots + a_nX^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that $p \nmid a_n$, $p|a_i$, $(0 \leq i \leq n-1)$, and $p^2 \nmid a_0$, then f is irreducible in $\mathbb{Q}$.*

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

$$R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1) \tag{3.20}$$

then:

- $3 \nmid 1$, - $3|(-3)$,- $3|3(1 - 3^{k-1}m\mu'_1)$, and - $3^2 \nmid 3(1 - 3^{k-1}m\mu'_1)$.

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

- We consider the case $k = 1$, then $\mu_1 = 3\mu'_1$ and $(\mu'_1, 3) = 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - m\mu'_1) = 0 \tag{3.21}$$

* If $3 \nmid (1 - m.\mu'_1)$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

$$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)$$

and we find a contradiction with $R'(\delta) = 0$.

* We consider that $3|(1 - m.\mu'_1) \Longrightarrow m\mu'_1 - 1 = 3^i.h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. δ is an integer root of the polynomial $R'(Z)$:

(3.22)

$R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Rightarrow$ the discriminant of $R'(Z)$ is :$\Delta = 3^2 + 3^{i+1} \times 4.h$

As the root δ is an integer, it follows that $\Delta = l^2 > 0$ with l a positive integer. We obtain:

$$\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \tag{3.23}$$

$$\Longrightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \tag{3.24}$$

We can write the equation (3.21) as :

$$\delta(\delta - 3) = 3^{i+1}.h \Longrightarrow 3^3\mu'_1 \frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = 3^{i+1}.h \Longrightarrow \tag{3.25}$$

$$\mu'_1 \frac{rad(\delta)}{3}.\left(\mu'_1 rad(\delta) - 1\right) = h \tag{3.26}$$

We obtain $i = 2$ and $q^2 = 1 + 12h = 1 + 4\mu'_1 rad(\delta)(\mu'_1 rad(\delta) - 1)$. Then, q satisfies :

$$q^2 - 1 = 12h \Rightarrow \tfrac{(q-1)}{2}.\tfrac{(q+1)}{2} = 3h = (\mu'_1 rad(\delta) - 1).\mu'_1 rad(\delta) \Rightarrow \tag{3.27}$$

$$q - 1 = 2\mu'_1 rad(\delta) - 2 \tag{3.28}$$

$$q + 1 = 2\mu'_1 rad(\delta) \tag{3.29}$$

It follows that $(q = x, 1 = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.30}$$

with $N = 12h > 0$. Let $Q(N)$ be the number of the solutions of (3.30) and $\tau(N)$ is the number of suitable factorization of N, then we announce the following result concerning the solutions of the Diophantine equation (3.30) (see theorem 27.3 in [**6**]):

- If $N \equiv 2 (\bmod 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3 (\bmod 4)$, then $Q(N) = [\tau(N)/2]$.

- If $N \equiv 0 (\mathrm{mod}\, 4)$, then $Q(N) = [\tau(N/4)/2]$.
$[x]$ is the integral part of x for which $[x] \leq x < [x] + 1$.

Let (α', m'), $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (3.30), then $\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h$, but $q = x$ and $1 = y$ satisfy the equation (3.29) given by $x + y = 2\mu'_1 rad(\delta)$, it follows α', m' verify also $\alpha' + m' = 2\mu'_1 rad(\delta)$, that gives $\alpha' - m' = 2(\mu'_1 rad(\delta) - 1)$, then $\alpha' = x = q = 2\mu'_1 rad(\delta)$ and $m' = y = 1$. So, we have given the proof of the uniqueness of the solutions of the equation (3.30) with the condition $x + y = 2\mu'_1 rad(\delta)$. As $N = 12h \equiv 0(\mathrm{mod}\, 4) \Longrightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]$, the expression of $3h = \mu'_1 . rad(\delta).\, (\mu'_1 rad(\delta) - 1)$, then $Q(N) = [\tau(3h)/2] > 1$. But $Q(N) = 1$, then the contradiction and the case $3|(1 - m.\mu'_1)$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that δ is a root of the following polynomial :

$$R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 \tag{3.31}$$

The discriminant of $R(Z)$ is:

$$\Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with } q \in \mathbb{N}^* \quad \text{as } \delta \in \mathbb{N}^* \tag{3.32}$$

- We suppose that $2|m\mu_1 \Longrightarrow c'$ is even. Then $q^2 \equiv 5(\mathrm{mod} 8)$, it gives a contradiction because a square is $\equiv 0, 1$ or $4(\mathrm{mod}\, 8)$.

- We suppose c' an odd integer, then a is even. It follows $a = rad^3(a) \equiv 0(\mathrm{mod} 8) \Longrightarrow c' \equiv 1(\mathrm{mod} 8)$. As $c' = \delta^2 - 3X.\delta$, we obtain $\delta^2 - 3X.\delta \equiv 1(\mathrm{mod}\, 8)$. If $\delta^2 \equiv 1(\mathrm{mod}\, 8) \Longrightarrow -3X.\delta \equiv 0(\mathrm{mod}\, 8) \Longrightarrow 8|X.\delta \Longrightarrow 4|\delta \Longrightarrow c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4(\mathrm{mod} 8) \Longrightarrow \delta \equiv 2(\mathrm{mod} 8)$ or $\delta \equiv 6(\mathrm{mod}\, 8)$. In the two cases, we obtain $2|\delta$. Then, the contradiction with c' an odd integer.

It follows that the case $c > rad^{3.26}(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^{3.26}(c)$ and large, then $c = rad^3(c) + h, h > rad^3(c)$, h a positive integer and $\mu_a < rad^2(a) \Longrightarrow a + l = rad^3(a)$, $l > 0$. Then we obtain :

$$rad^3(c) + h = rad^3(a) - l + b \Longrightarrow rad^3(a) - rad^3(c) = h + l - b > 0 \tag{3.33}$$

as $rad(a) > rad^{\frac{1.63}{1.37}}(c)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h + l - b = m > 0 \tag{3.34}$$

Let $X = rad(a) - rad(c)$, then X is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{3.35}$$

To resolve the above equation, we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{3.36}$$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0$. The two real roots of (3.36) are:

$$t_1 = u^3 = \frac{m+\alpha}{2}, \quad t_2 = v^3 = \frac{m-\alpha}{2} \tag{3.37}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant Δ, it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.38}$$

with $N = 4rad^3(ac) > 0$. From the expression of Δ above, we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{3.39}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{3.40}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{3.41}$$

Let $Q(N)$ be the number of the solutions of (3.38) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph I-3-2-4- (case $3|(1 - m.\mu'_1)$), we obtain a contradiction.

It follows that the cases $\mu_a \leq rad^2(a)$ and $c > rad^{3.26}(c)$ are impossible.

II- We suppose that $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$:

II-1- Case $rad(c) < rad(a)$: As $c \leq rad^3(c) = rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow c < rad^{1.63}(c).rad^{1.37}(a) < rad^{1.63}(ac) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}$.

II-2- Case $rad(a) < rad(c) < rad^{\frac{1.63}{1.37}}(a)$: As $c \leq rad^3(c) \leq rad^{1.63}(c).rad^{1.37}(c) \Longrightarrow$

$$c < rad^{1.63}(c).rad^{1.63}(a) < rad^{1.63}(abc) \Longrightarrow \boxed{c < R^{1.63}}.$$

II-3- Case $rad^{\frac{1.63}{1.37}}(a) < rad(c)$:

II-3-1- We suppose $rad^{2.63}(a) < a \leq rad^{3.26}(a) \Longrightarrow a \leq rad^{1.63}(a).rad^{1.63}(a) \Longrightarrow a < rad^{1.63}(a).rad^{1.37}(c) \Longrightarrow c = a + b < 2a < 2rad^{1.63}(a).rad^{1.63}(c) < rad^{1.63}(abc) \Longrightarrow c < R^{1.63} \Longrightarrow \boxed{c < R^{1.63}}$.

II-3-2- We suppose $a > rad^{3.26}(a)$ and $\mu_c \leq rad^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a, c), we arrive at a contradiction. It follows that the case $\mu_c \leq rad^2(c)$ and $a > rad^{3.26}(a)$ is impossible.

Finally, we have finished the study of the case $rad^{1.63}(c) < \mu_c \leq rad^2(c)$ and $\mu_a > rad^{1.63}(a)$.

3.2.3.3. *Case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^{1.63}(a)$*. — Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$,
- $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c)$.

III-1- We suppose $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a) \Longrightarrow c > rad^3(c)$ and $a > rad^{2.63}(a)$. We can write $c = rad^3(c) + h$ and $a = rad^3(a) + l$ with h a positive integer and $l \in \mathbb{Z}$.

III-1-1- We suppose $rad(c) < rad(a)$. We obtain the equation:

$$rad^3(a) - rad^3(c) = h - l - b = m > 0 \tag{3.42}$$

Let $X = rad(a) - rad(c)$, from the above equation, X is a real root of the polynomial:

$$H(X) = X^3 + 3rad(ac)X - m = 0 \tag{3.43}$$

As above, to resolve (3.43), we denote $X = u + v$, It follows that u^3, v^3 are the roots of the polynomial $G(t)$ given by :

$$G(t) = t^2 - mt - rad^3(ac) = 0 \tag{3.44}$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4rad^3(ac) = \alpha^2, \quad \alpha > 0 \tag{3.45}$$

The two real roots of (3.44) are:

$$t_1 = u^3 = \frac{m+\alpha}{2}, \quad t_2 = v^3 = \frac{m-\alpha}{2} \tag{3.46}$$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the equation (3.45), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \tag{3.47}$$

with $N = 4rad^3(ac) > 0$. From the equations (3.46), we remark that α and m verify the following equations:

$$x + y = 2u^3 = 2rad^3(a) \tag{3.48}$$

$$x - y = -2v^3 = 2rad^3(c) \tag{3.49}$$

$$\text{then} \quad x^2 - y^2 = N = 4rad^3(a).rad^3(c) \tag{3.50}$$

Let $Q(N)$ be the number of the solutions of (3.47) and $\tau(N)$ is the number of suitable factorization of N, and using the same method as in the paragraph I-3-2-4- (case $3|(1 - m.\mu'_1)$), we obtain a contradiction.

III-1-2- We suppose $rad(a) < rad(c)$. We obtain the equation:

$$rad^3(c) - rad^3(a) = b + l - h = m > 0 \tag{3.51}$$

Using the same calculations as in III-1-1-, we find a contradiction.

It follows that the case $\mu_c > rad^2(c)$ and $\mu_a > rad^{1.63}(a)$ is impossible.

III-2- We suppose $\mu_a > rad^2(a)$ and $\mu_c > rad^{1.63}(c) \Longrightarrow a > rad^3(a)$ and $c > rad^{2.63}(c)$. We can write $a = rad^3(a) + h$ and $c = rad^3(c) + l$ with h a positive integer and $l \in \mathbb{Z}$.

The calculations are similar to those in case III-1-. We obtain the same results namely the cases of III-2- to be rejected.

It follows that the case $\mu_c > rad^{1.63}(c)$ and $\mu_a > rad^2(a)$ is impossible. □

We can state the following important theorem:

> ***Theorem 3.4.*** — *Let a, b, c positive integers relatively prime with $c = a + b$, then $c < rad^{1.63}(abc)$.*

3.3. *The Proof of the* abc *conjecture*

We note $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$. We recall the following proposition [**4**]:

Proposition 3.5. — Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:

$$lim_{\epsilon\to 0}K(\epsilon) = +\infty \tag{3.52}$$

3.3.1. Case : $\epsilon \geq 0.63$**.** — As $c < R^{1.63}$ is true, we have $\forall \epsilon \geq 0.63$:

$$c < R^{1.63} \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with\ \ K(\epsilon) = e^{\frac{1}{0.63^2}},\ \epsilon \geq 0.63 \tag{3.53}$$

Then the abc conjecture is true.

3.3.2. Case: $\epsilon < 0.63$**.** —

3.3.2.1. *Case:* $c > R$. — From the statement of the abc conjecture 3.1, we want to give a proof that $c < K(\epsilon)R^{1+\epsilon} \Longleftrightarrow Logc < LogK(\epsilon) + (1+\epsilon)LogR \Longleftrightarrow LogK(\epsilon)+(1+\epsilon)LogR - Logc > 0$. For our proof, we proceed by contradiction of the abc conjecture, so we assume that the conjecture is false:
(3.54)

$$\exists\, \epsilon_0 \in]0, 0.63[, \forall\, K(\epsilon) > 0, \quad \exists\, c_0 = a_0 + b_0 \quad \text{so that} \quad c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \Longrightarrow c_0 \text{ not a prime}$$

We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Let $Y_{c_0}(\epsilon) = \frac{1}{\epsilon^2} + (1+\epsilon)LogR_0 - Logc_0, \epsilon \in]0, 0.63[$. From the above explications, if we will obtain $\forall\epsilon \in]0, 0.63[, Y_{c_0}(\epsilon) > 0 \Longrightarrow Y_{c_0}(\epsilon_0) > 0$, then the contradiction with (3.54).

About the function Y_{c_0}, we have $lim_{\epsilon\longrightarrow 0.63}Y_{c_0}(\epsilon) = 1/0.63^2 + Log(R_0^{1.63}/c_0) > 0$ and $lim_{\epsilon\longrightarrow 0}Y_{c_0}(\epsilon) = +\infty$. The function $Y_{c_0}(\epsilon)$ has a derivative for $\forall\epsilon \in]0, 0.63[$, we obtain with $R_0 > 2977$:
(3.55)

$$Y'_{c_0}(\epsilon) = -\frac{2}{\epsilon^3} + LogR_0 = \frac{\epsilon^3 LogR_0 - 2}{\epsilon^3} \Rightarrow Y'_{c_0}(\epsilon) = 0 \Rightarrow \epsilon = \epsilon' = \sqrt[3]{\frac{2}{LogR_0}} \in]0, 0.63[$$

Discussion:
- If $Y_{c_0}(\epsilon') \geq 0$, it follows that $\forall\epsilon \in]0, 0.63[, Y_{c_0}(\epsilon) \geq 0$, then the contradiction with $Y_{c_0}(\epsilon_0) < 0 \Longrightarrow c_0 > K(\epsilon_0)R_0^{1+\epsilon_0}$. Hence the abc conjecture is true for

$\epsilon \in]0, 0.63[$.

- If $Y_{c_0}(\epsilon') < 0 \Longrightarrow \exists \epsilon_1, \epsilon_2$ satisfying $0 < \epsilon_1 < \epsilon' < \epsilon_2 < 0.63$, so that $Y_{c_0}(\epsilon_1) = Y_{c_0}(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We recall the following definition:

Definition 3.6. — The number ξ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1 x + \cdots + a_m x^m, \quad a_m \neq 0 \tag{3.56}$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \Longrightarrow \dfrac{c_0}{R} = \dfrac{\mu_c}{rad(ab)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1}$.

i) - We suppose that $\epsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\epsilon_1^2$ and $R_0 = \alpha_1$ are also algebraic numbers. We obtain:

$$\frac{\mu_c}{rad(ab)} = e^{\frac{1}{\epsilon_1^2}} R_0^{\epsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \tag{3.57}$$

From the theorem (see theorem 3, page 196 in [**9**]):

Theorem 3.7. — *$e^{\beta_0}\alpha_1^{\beta_1}\ldots\alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$.*

we deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (3.57) is transcendental, but the term $\dfrac{\mu_c}{rad(ab)}$ is an algebraic number, then the contradiction and the *abc* conjecture is true.

ii) - We suppose that ϵ_1 is transcendental, in this case there is also a contradiction, and the *abc* conjecture is true.

Remark 3.8. — - We obtain also that $K(\epsilon) > 1$ if $\epsilon \in]0, 0.63[$. If not, we consider the example $9 = 8 + 1$ with $9 > 2 \times 3$, we take $\epsilon = 0.2$, then $c < K(0.2)R^{1+.02} < 1.R^{1.2}$. But $c = 9 > 6^{1.2} \approx 8.58$, then the contradiction and $\boxed{K(\epsilon) > 1,\ \forall \epsilon \in]0, 0.63[}$.

3.3.2.2. *Case:* $c < R$. — In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \ K(\epsilon) > 1, \ 0 < \epsilon < 0.63 \tag{3.58}$$

The constant $K(\epsilon)$ is taken as for the case $c > R$ above, and the *abc* conjecture is true.

Then the proof of the *abc* conjecture is finished for all $\epsilon > 0$.

3.4. *Conclusion*

We have given an elementary proof of the *abc* conjecture. We can announce the important theorem:

Theorem 3.9. — *For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.59}$$

where K is a constant depending of ϵ.

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BIBLIOGRAPHY

[1] A. Ben Hadj Salem, *Progress in The Proof of The Conjecture* $c < rad^{1.63}(abc)$ *- Case :* $c = a+1$. Submitted to the Lithuanian Mathematical Journal. December 2020, 10 pages. 2020.

[2] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's Conjecture*. Journal für die Reine und Angewandte Mathematik, Vol. 2004, Issue **572**, pp 167-195, 2004.

[3] P. Mihăilescu, *Around ABC*. European Mathematical Society Newsletter, **N° 93**, Sept. pp 29-34. 2014.

[4] A. Nitaj, *Aspects expérimentaux de la conjecture abc*. Séminaire de Théorie des Nombres de Paris(1993-1994), London Math. Soc. Lecture Note Ser., Vol **n°235**. Cambridge Univ. Press, pp 145-156. 1996.

[5] B. De Smit, `https://www.math.leidenuniv.nl/ desmit/abc/`. [last accessed December 2020].

[6] B.M. Stewart, *Theory of Numbers*. 2^{sd} edition, The Macmillan Compagny, N.Y., pp 196-197. 1964.

[7] C. Touibi, *Algèbre Générale* (in French). Cérès Editions, Tunis, pp 108-109. 1996.

[8] M. Waldschmidt, *On the abc Conjecture and some of its consequences*, presented at The 6th World Conference on 21st Century Mathematics, Abdus Salam School of Mathematical Sciences (ASSMS), Lahore (Pakistan), March 6-9, 2013.

[9] A. Baker, 1971. *Effective Methods in Diophantine Problems.* Proceedings of Symposia in Pure Mathematics, Vol. **XX**, 1969 Number Theory Institute. AMS, pp 195-205. 1971.

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