Projective Hyperbolic Geometry of Electromagnetic Fields

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ABSTRACT

We show that the e.m. field of a point charge is the acceleration part of a geodesic equation, in the Beltrami-Klein Ball model of hyperbolic geometry,in 3dim velocity space.The result is obtained by assuming that the interval of interaction is timelike instead of null.This gives rise to a formal 4 velocity of interaction and a rest frame for an inertial observer moving with the formal velocity.The geodesic between two points in the tangent space is given by projective velocity 4 vectors or bivectors.The moment of Lienard-Wiechert potential is the geodesic between the velocity of the source charge and the formal velocity of interaction,in bivector form.The Lorentz force is proportional to the geodesic between the velocity of the source charge and that of the interaction,in the rest frame of the test charge.The energy- stress tensor, the field Lagrangian density, the e.m energy density, the e.m. momentum density also have geometric meanings.The inverse of the field is related to virtual,uniformly accelerated motions,described by a Rindler-like coordinate system.All e.m. entities are finite everywhere and Lorentz covariant.

1 Introduction.

In order to describe classical electrodynamics with a timelike interval of interaction instead of a null one as in Maxwell's theory (ref.1), we use a Minkowski spacetime foliation of hyperboloids of two sheets. Since the interval of interaction is timelike,there corresponds a formal 4 velocity associated with it.This enables us to obtain the hyperbolic distance between the formal 4 velocity of interaction and any other 4 velocity such as the 4 velocity of the source charge or the 4 velocity of the test charge.The 4 velocities are unit normals of all the hyperboloids of the foliation.There is an arc length between any two of these 4-velocities.The arc length is the angle between any two 4 velocities and represents a geodesic segment on the hyperboloid of unit radius.The hyperbolic distance between any two 4 velocities is the hyperbolic tangent of the angle (refs2,3).We can thus define a projective relative velocity 4 vector which is simply the directed hyperbolic distance in the tangent space starting at any two 4 velocities and ending at the other (ref.4).The projective relative 4 velocity is therefore a geodesic distance of the Beltrami-Klein ball model of hyperbolic geometry in 3 dim velocity space (refs2,3,5). However,we describe the 3 dim velocity space entirely in terms of 4 vectors so we are dealing with a projective form of the Beltrami-Klein model. The geodesic distance between any two 4-velocity is a line representation of the geodesic distance.It is well known that projective geometry has also an areal form of every line.This areal form is an antisymmetric second rank tensor representation of the hyperbolic distance in the tangent space.It turns out that the moment of Lienard-Wiechert potential(refs6,7) about the interval of interaction is a tensorial form of the geodesic distance between the 4 velocity of the source charge and that of the interaction or vice-versa.To each geodesic there corresponds an acceleration term in the geodesic eq, or equivalently, the Christoffel symbols term. We prove that the nonacceleration field of a point charge moving with arbitrary 3 velocity and 3 acceleration is the acceleration part of the geodesic between the 4 velocity of the source charge and the 4 velocity of interaction (or vice-versa) expressed in antisymmetric tensor form (ref5).The total field is also a geodesic acceleration living in a different hyperboloid.The total field has a conformal factor.Its geodesic is one between the 4 velocity of interaction and a special hyperbolic 4 velocity.We will present the results for the total field and the acceleration field in a separate publication due to the many subtleties involved in the derivation.We do show however that the derivative of the interval of interaction relative to the proper time of the source charge or the test charge produces entities which have the same form and the same properties as the real total field but use the 4 velocity of the test charge instead of the special hyperbolic 4 velocity. We show that the field Lagrangian density is the square of the geodesic acceleration term and that the energy stress tensor is the square of the acceleration term times a classical reflection operator. Its spatial components represent an inversion (ref3). The Lorentz force is proportional to a projective form of the expression for the Einstein composition law (addition or substraction) of two 3 velocities (refs2,7), the projective relative 4 velocity between that of the source and that of the interval of interaction. It is the acceleration part of the geodesic eq as viewed from an observer in the rest frame of the test charge. The absolute value of the electric field 4 vector described by the Lorentz force is essentially a projection of the geodesic into a certain transverse (meaning perpendicular) direction. A magnetic field 4 vector is also defined .Its absolute value is the acceleration part projected into the transverse direction. An energy density 4 vector and a Poynting momentum density 4 vector are obtained. The projective relative 4 velocity in the direction of the Poynting 4 vector is also obtained. The electric and magnetic field 4 vectors are orthogonal to each other. They are also orthogonal to the direction of the Poynting momentum 4 vector and to the 4 velocity of the test charge. Four mutually orthogonal directions are thus defined.This resembles ,but is not, a plane wave propagating.

There is another foliation of Minkowski spacetime involved.It is a foliation of hyperboloids of one sheet which we describe using a Rindler-like coordinate system (refs8,9).The foliation consists of timelike hyperbolic motions representing virtual motions at constant accelerations.The normals of the hyperboloids of two sheets become the initial and the final unit tangents of the hyperbolic motions.The angle between them is an arc length of the timelike hyperbolic motion and a geodesic arc segment of each of the hyperboloids of one sheet.The source charge is positioned on one of the hyperboloid at the retarded coordinate time and connects to another virtual trajectory at the field point or the test charge via the timelike interval of propagation or its projection onto the plane of the two trajectories.Thus is the influence transmitted.The virtual hyperbolic trajectories involve an inversion of the fields keeping the directions constant.The inverses are connected to the radii of the the hyperbolic motions and their duration, using coordinate times.

1A The notation.

4 vectors are in capital letters.3 vectors are not.Unit 4 vectors are written with a caret to their right,on top.4 vectors and 3 vectors have no arrows,only their size differentiates them.The metric used is [1,1,1,i],to avoid using covariant and contravariant components.Components are used sparingly,the formulas are written in a coordinate independent form.Whether an index appears as a subscript or a superscript is irrelevant.Indices ijkl run from 1 to 4.Indices abcd run from 1 to 3.The summation convention for repeated indices is used,unless specified otherwise.Dot products are specified by a dot as in A.B or a.b.Cross products are specified by an old fashioned x as in $(AxB)_{ij} = (A_iB_j\text{-}A_jB_i)$ or in $(wxv)_{ab} =(w_av_b -w_bv_a)$.When the meaning is clear, the indices are omitted as in (AxB). A^ ,B^ are the unit 4 vectors of A and B.The speed of light will be =1 only if this does not lead to confusion.It will be written as c because of the need to keep track of units in cumbersome equations. We use total derivatives to describe partial derivatives or variation, letting the context give which one is used.

1B Preliminary formulas.

The interval of interaction (of propagation,of influence) between the source charge q' and the field point or the test charge q is:

E= [R – R'(τ')] (1a); where R' is the retarded 4 position of q' and R is the 4 position of the field point or the test charge q.In the latter case R =R(τ). τ' and τ are the respective proper times.In terms of component: R= [r,ict] (1b); R' =[r'(t'),ict'] (1c); r is the 3 position of the field point or the instantaneous position of the test charge q, t is the instantaneous time. r'(t') is the retarded position of q', t' is the retarded time.The 4 velocity of q' is U' = dR'/dτ'. The 4 acceleration of q' is $d^2R'/d\tau'^2$ = (a'); Its 3 velocity is u'(t') and its 3 acceleration is a'(t').The 4 velocity of q and its 4 acceleration use the same notation without the primes.The same is true of the 3 velocity and the 3 acceleration of q. The interval of interaction is assumed to be timelike, not null.Thus:

$E.E= -l^2$ (1d) ; Here l is an invariant length(fundamental length) which is small enough so that it does not contradict experimental results(ref.1). Otherwise, it is unspecified. In Maxwell's theory E.E = 0 .

The Lienard-Wiechert (LW) 4 vector potential is:

A= -q'/(E.U') = [A,iϕ] ; (see refs6,7) Where A is the 3 vector potential and ϕ is the scalar potential.

The formal 4 velocity of the influence is:

E/l = W/c (1e) ; W.W/c^2 = - 1 (1f); Its components are:

[r-r'/l, ic(t-t')/l] (1g) ; W/c = $[(w/c)/(1-w^2/c^2)^{1/2}, i/(1- w^2/c^2)^{1/2}]$ (1h);

(r-r')/l = $(w/c)/(1- w^2/c^2)^{1/2}$ (1i); ic(t-t')/l = $i/(1- w^2/c^2)^{1/2}$ (1j);

Finally: w/c = r-r'/c(t-t') (1k); is the 3 velocity of interaction.It must be emphasized that w is only a formal velocity.The interaction does not move! What is true is that we can find an observer which moves in a frame which has that particular value of this 3 velocity. In the rest frame of such an observer w=0 has a meaning. It turns out that such a frame has great importance because all three fields, the total field, the acceleration field, and the non acceleration have E and therefore W in their cross products.

2 Projective relative 4 velocities.

2A Projective relative velocity 4 vectors.

Take a foliation of Minkowski spacetime with hyperboloids of two sheets.Take the

future oriented hyperboloids only. One of this hyperboloid will have unit normals.

These unit normals can be represented by 4 velocity unit vectors issued from an origin O. The chosen hyperboloid is said to have radius 1 and the normals are the Minkowski equivalent of a circle of unit radius with the radii being the unit 4 velocity vectors.The other hyperboloids of the foliation will simply have different radii.(An inverse radius being considered a radius also). The hyperbolic angle between any two normals i.e. any two unit 4 velocities, will be determined by the dot product of the two 4 velocities unit vectors. This will define the hyperbolic cosine of the hyperbolic angle.We will use the physicist definition of the hyperbolic angle. Mathematicians call it the half angle.

Take any two 4 velocities U'/c and U/c. We will be mostly interested in the 4 velocity of the source charge q' and of the test charge q but in what follows they could be any two 4 velocities. They don't even have to be derivatives of a trajectory. To 4 velocity U' corresponds a 3 velocity u' and to 4 velocity U corresponds a 3 velocity u. We have

$U'/c = [(u'/c)/(1- u'^2/c^2)^{1/2} , i/(1 - u'^2/c^2)^{1/2}]$ (1);

Using the metric [1,1,1,i] to avoid having to deal with covariant and contravariant components.

$U/c = [(u/c)/(1- u^2/c^2)^{1/2} , i/(1 - u^2/c^2)^{1/2}]$ (2) ;

$U'.U/-c^2 = [1 - u'.u/c^2]/(1 - u'^2/c^2)^{1/2} (1 - u^2/c^2)^{1/2} = \cosh\theta_{U'U}$ (3) ; $\theta_{U'U}$ is the hyperbolic angle between U' and U. We will use the convention that the angle will be the same whether we're going from U' towards U or U towards U'. so θ will run from zero to infinity and never be negative. There will be exceptions later.

We want to find "projective" versions of u' and u so that they are 4 vectors and not 3 vectors. The answer is:

$U'/c = [(V_{U'U}/c) /(1- V^2_{U'U}/c^2)^{1/2} + (U/c)/(1 - V^2_{U'U}/c^2)^{1/2}$ (4) ;

$U/c = [(V_{UU'}/c)/(1- V^2_{UU'}/c^2)^{1/2} + (U'/c)/(1- V^2_{UU'}/c^2)^{1/2}$ (5) ;

These expressions are independent of any components. They are "absolute" 4 vectors and scalars. The order U'U in $V_{U'U}$ means that the 4 velocity is centered at U so the normal to the hyperboloid from origin O to the hyperboloid is in the vertical timelike direction defined by U/c. A virtual motion from U/c to U'/c brings the normal to the position U'/c making an angle $\theta_{U'U}$ with U/c.Similarly the order UU' means that U'/c is now in the vertical direction and the virtual motion is from U'/c to U/c. The angle θ is the same.

$V_{U'U}/c = [(U'/c)(-c^2/U'.U) - U/c]$ (6) ; $V_{UU'}/c = [(U/c)(- c^2/U.U') - U'/c]$ (7) ;

Eq6 and 7 are the projective relative 4 velocities. They are spacelike 4 vectors.

$(V_{U'U}/c).U/c = 0$ (8a) ; $(V_{UU'}/c).U'/c = 0$ (8b) ; So they lie in the tangent space of the hyperboloid centered at U/c and U'/c respectively.

It is easy to check that when u=0 $(V_{U'U})_{u=0} = u' = [u',0]$ (9) ;

and that when u' =0 $(V_{UU'})_{u'=0} = u = [u, 0]$ (10) ; It is in that sense that they are projective relative 4 velocities.

When u=0 $(U'/c)[(-c^2)/U'.U] = [V_{U'U} + U]/c] = [u', ic]$ (10a) similarly :

$(U/c)[(-c^2)/U.U'] = [V_{UU'} + U']/c = [u, ic]$ (10b) when u'=0 ;

Thus $V_{U'U} = [u',ic] - [0,ic] = [u',0]$ (10c) when u=0;

$V_{UU'} = [u,ic] - [0,ic] = [u,0]$ (10d) when u'=0; this proves eqs9,10.

We can define the unit spacelike 4 vectors $V_{U'U}$^ and $V_{UU'}$^ by

$V_{U'U} = |V_{U'U}| V_{U'U}$^ and $V_{UU'} = |V_{UU'}| V_{UU'}$^

$V_{U'U}$^.$V_{U'U}$^ = $V_{UU'}$^.$V_{UU'}$^ = 1 ; $V_{U'U}$^.U =0 ; $V_{UU'}$^.U' = 0 ; (11)

We have :$U'/c = [|V_{U'U}/c| V_{U'U}\text{^} + (U/c)]/(1 - V^2_{U'U}/c^2)^{1/2}$ (12) ;

$U'.U/-c^2 = \cosh\theta_{U'U} = 1/(1 - V^2_{U'U}/c^2)^{1/2}$ (13) ; similarly

$U/c = [|V_{UU'}/c| V_{UU'}\text{^} + U'/c]/(1 - V^2_{UU'}/c^2)^{1/2}$ (14) ;

$U.U'/-c^2 = \cosh\theta_{UU'} = 1/(1 - V^2_{UU'}/c^2)^{1/2}$ (15) ; therefore

$\Theta_{UU'} = \theta_{U'U}$ (16) ; $\cosh\theta_{UU'} = \cosh\theta_{U'U}$ (17) ;

$\text{Sinh}\theta_{U'U} = \sinh\theta_{UU'}$ (18) ; $\tanh\theta_{U'U} = |V_{U'U}/c| = \tanh\theta_{UU'} = |V_{UU'}/c|$ (19);

$U'/c = \sinh\theta_{U'U} V_{U'U}\text{^} + \cosh\theta_{U'U} U/c$ (20);

$U/c = \sinh\theta_{UU'} V_{UU'}\text{^} + \cosh\theta_{UU'} U'/c$ (21);

Define two spacelike unit vectors a'^ and a^

such that a'^ = $\cosh\theta_{U'U} V_{U'U}\text{^} + \sinh\theta_{U'U} U/c$ (22) ;

and $a^\wedge = \cosh\theta_{UU'} V_{UU'}{}^\wedge + \sinh\theta_{UU'} U'/c$ (23) ;

We have $a'^\wedge . U'/c = 0$ (24a) ; $a^\wedge . U/c = 0$ (24b) ;

We also have $a'^\wedge = - V_{UU'}{}^\wedge$ (25a); $a^\wedge = - V_{U'U}{}^\wedge$ (25b) ;

For the explanation of the minus sign draw a diagram.

2B Projective relative velocity bivectors.

In projective geometry, quantities have a line segment representation.This was described in the previous section. They also have an aerial (antisymmetric tensor) representation. The line segment representation is a directed line segment, a Minkowski 4 vector. The areal representation is a directed area, i.e. a 4x4 antisymmetric Minkowski tensor. We call it a bivector.

$(U/c \times V_{U'U}/c)_{ij}$ (26) represents $V_{U'U}/c$. $(U'/c \times V_{UU'}/c)_{ij}$ (27) represents $V_{UU'}/c$.

$[U/c \times (V_{U'U}/c) /(1 - V^2_{U'U}/c^2)^{1/2}]_{ij}$ (28) represents $(V_{U'U}/c)/(1 - V^2_{U'U}/c^2)^{1/2}$.

$[U'/c \times (V_{UU'}/c)/(1 - V^2_{UU'}/c^2)^{1/2}]_{ij}$ (29) represents $(V_{UU'}/c)/(1 - V^2_{UU'}/c^2)^{1/2}$.

We observe that when $u=0$ eq (26) reduces to iu'/c and when $u' = 0$

eq(27) reduces to iu/c . The i is due to our choice of the metric [1,1,1,i].

Similarly eq28 reduces to $(iu'/c)/(1 - u'^2/c^2)^{1/2}$ when $u=0$.

Eq29 reduces to $(iu/c)/(1 - u^2/c^2)^{1/2}$ when $u' =0$.

Using the definitions of $V_{U'U}$ and $V_{UU'}$ (eqs 6 and 7) it is easy to prove that

$[U/c \times V_{U'U}/c]_{ij} = - [U'/c \times V_{UU'}/c]_{ij}$ (30) ; and that

$[U/c \times (V_{U'U}/c)/(1 - V^2_{U'U}/c^2)^{1/2}]_{ij} = - [U'/c \times (V_{UU'}/c)/(1 - V^2_{UU'}/c^2)^{1/2}]_{ij}$ (31) ;

3 The projective geometric meaning of the Lienard-Wiechert (LW) potential, its moment about the interval of propagation and other formulas.

3A The Lienard-Wiechert (LW) potential is given by:

$A_{LW} = [A, i\phi] = - q'U'/E.U'$ (1) ; (refs6,7) $E= lW/c$ (2) ; (ref.1)

$A_{LW} = q'(U'/c)/(E.U'/-c) = q'(U'/c)/l(W.U'/-c^2) = (q'/l)(U'/c)[-c^2/(U'.W)]$ (3);

$= (q'/l)[(U'/c)(-c^2/U'.W) – W/c + W/c] = (q'/l)[V_{U'W} /c + W/c]$;

$A_{LW} = (q'/l)[V_{U'W}/c + W/c]$ (4) ; with $V_{U'W}/c = [(U'/c)(-c^2/U'.W) – W/c]$ (5) ;

The LW potential is proportional to projective relative 4 velocity given by eq5 plus an extra term involving W/c. $A_{LW} = (q'/l)\text{sech}\theta_{U'W}(U'/c)$ (5a);

Compare with the examples discussed in the previous section. Recall that

$W/c = [(r-r'(t'))/l , ic(t-t')/l] = [w/c, ic]/(1- w^2/c^2)^{1/2}$; $w/c =(r- r'(t'))/c(t- t')$; see introduction.

When w=0 $V_{U'W} = [u',0]$ (6) ; when w=0 eq2.10a = $[u'/c, i]$ (7) ; so

$A_{LW} = (q'/l)[u'/c, i]$ (8); and $(U'/c)(-c^2/U'.W) = [u'/c , i]$ (9); when w = 0 .

We must be careful to interpret the meaning of w=0 correctly.

It does not mean that (r-r')/(t-t') = 0. It means that an inertial observer moving

with 3 velocity w would have w=0 in its rest frame.

As in the previous section,

$E.U'/-c =l(W.U'/-c^2) = l\cosh\theta_{U'W} = l\cosh\theta_{WU'} = l/(1- V^2_{U'W}/c^2)^{1/2}$ (10) ;

$|V_{U'W}/c| = |V_{WU'}/c| = \tanh\theta_{U'W} = \tanh\theta_{WU'}$ (11); $\theta_{U'W} = \theta_{WU'}$ (12) ;

$|V_{U'W}/c|/(1 – V^2_{U'W}/c^2)^{1/2} = |V_{WU'}/c|/(1- V^2_{WU'}/c^2)^{1/2} = \sinh\theta_{U'W}$ (13) ;

We also have: $V_{U'W} =|V_{U'W}|V_{U'W}$^ (14a) ; $V_{WU'} = |V_{WU'}| V_{WU'}$^ (14a) ;

$U'/c = [V_{U'W}/c + W/c]/(1- V^2_{U'W}/c^2)^{1/2}$ (15a) ;

$W/c = [V_{WU'}/c + U'/c]/(1- V^2_{WU'}/c^2)^{1/2}$ (15b) ;

$U'/c = \sinh\theta_{U'W} V_{U'W}$^ + $\cosh\theta_{U'W} W/c$ (15c) ;

$W/c = \sinh\theta_{WU'} V_{WU'}{}^\wedge + \cosh\theta_{WU'} U'/c$ (15d);

We can get 4 vectors reducing to r-r' since the interval E is a 4vector.

Since $E = [r-r'(t'), ic(t-t')] = l\sinh\theta_{WU'} V_{WU'}{}^\wedge + l\cosh\theta_{WU'} U'/c$, we expect that

$E =[\eta + (E.U'/c)U'/c]$ (16); where η is a 4vector orthogonal to U'/c. η therefore generalizes [r-r',0]. [E.U'/-c]U'/c generalizes [0, ic(t-t')]. We immediately have

$(E.U'/-c)U'/c = l\cosh\theta_{WU'} U'/c = [0,ic(t-t')]= l/(1- w^2/c^2)^{1/2}$ (17) when u' = 0 ;

We can write $E.U'/-c = c(T^{inst} - T^{ret})_{U'}$ (18) where inst stands for instantaneous and ret stands for retarded. We also write:

$E = R- R'(\tau') = \eta+ c(T^{inst} - T^{ret})_{U'} U'/c$ (19); η can be written as $\eta = R_{U'}{}^{inst} - R'$ (20);

Where $R_{U'}{}^{inst}$ means the position along the virtual trajectory of an observer moving with 4 velocity U'/c which is instantaneous with R in the rest frame of U'.

3B Projective meaning of the moment of LW potential.

The moment $[ExA]_{ij} = \{Ex [-q'U'/E.U']\}_{ij} = [Exq'(U'/c)/(E.U'/-c)]_{ij}$ (21) ;

LW is dropped from A_{LW}. $E=lW/c$; $(E.U'/-c) = l(U'.W/-c^2)$ (22) ;

$ExA = [Ex(q'/l)(U'/c)(-c^2/U'.W)] = l[(W/c)x (q'/l)(U'/c)(-c^2/U'.W)]$

$= q'[(W/c)x (U'/c)(-c^2/U'.W)] = q'[(W/c)x \{(U'/c)(-c^2/U'.W) - W/c +W/c\}]$

Finally $[ExA_{LW}]_{ij} = q'[(W/c)x V_{U'W}/c]_{ij} = -q'[(U'/c)x V_{WU'}/c]_{ij}$ (23) ; !!

We have transformed the moment of potential ,from which we can derive the total field $F^{tot}{}_{ij}$ (see our previous article in the references),into a relative velocity current in tensor (aerial) form.It is a projective relative 4 velocity current in antisymmetric tensor form.The moment has units of angular momentum per unit charge,except for a factor of 1/c. By multiplying eq23 by q/c we get a kind of projective angular momentum.

3C Alternate derivation of the projective meaning of the LW potential.

We first review some formula derived in a previous article.(ref.1).

From E = R-R'(τ') where R is the 4 position of the field point and R' is the retarded 4 position of the source charge q'.Varying the fundamental length l without varying the field point R, it was found, writing the variation as ordinary derivatives

$dc\tau'/dl^2 = c/(2E.U') = -1/(2E.U'/-c)$ (ref.1). From which we obtain:

$dc\tau'/dl^2 = -1/(2l\cosh\theta_{U'W}) = -(1- V^2_{U'W}/c^2)^{1/2}/2l$ (24) ; in projective form.

From $d(E.E)/dx^j = d(l^2)/dx^j = 0$ (writing the partial derivatives as ordinary derivatives) it was found :

$d\tau'/dx^j = E^j/E.U'$ (refs1,6) . So that

$d\tau'/dx^j = (lW^j/c)/(lW.U'/c) = -(W^j/c)/(W.U'/-c)$;

$-dc\tau'/dx^J = (W^j/c)(-c^2/W.U')$ (25) ;

$-dc\tau'/dx^j = [(W^j/c)(-c^2/W.U') - U'^j/c + U'^j/c] = [V^j_{WU'}/c + U'^j/c]$ (26);

We also found : $dx'^i/dx^j = E^jU'^i/E.U'$ (ref.1).

Therefore $dx'^i/dx^j = (lW^jU'^i/c)/(lW.U'/c)$;

$-dx'^i/dx^j = (W^jU'^i/c^2)(-c^2/W.U')$ (27); $dx'^J/dx^i = -(W^iU'^j/c^2)(-c^2/W.U')$ (28) ;

$-(dx'^i/dx^j - dx'^j/dx^i) = (-c^2/W.U')[U'^iW^j/c^2 - U'^jW^i/c^2]$ (29) ;

$(dx'^i/dx^j - dx'^j/dx^i) = (WxU'/c^2)_{ij}(-c^2/W.U') = [W/c \times V_{U'W}/c]_{ij} = -[U'/c \times V_{WU'}/c]_{ij}$ (30);

$q'(dx'^i/dx^j - dx'^j/dx^i) = q'[W/c \times V_{U'W}/c]_{ij} = -q'[U'/c \times V_{WU'}/c]_{ij} = [E \times A_{LW}]_{ij}$ (31) ;

We have used eq 23 for the moment of potential. Eq 31 shows the cameleon like nature of timelike electrodynamics.

4 The non-acceleration field.

4A Projective form of F_{ij}^{na}. F_{ij}^{na} as a Klein ball model geodesic acceleration.

The non acceleration field F_{ij}^{nonacc} of a single point charge q' moving with arbitrary velocity and acceleration is : $F_{ij}^{na} = q'[ExU'/c]_{ij}/[E.U'/-c]^3$ (1); (ref.1)
Eq1 can be written as:

$F_{ij}^{na} = q'[ExU'/c]_{ij} (-c^2/W.U')/l[E.U'/-c]^2$ (2);

$= q'[W/c\ x(U'/c)(-c^2/W.U')]_{ij}/[E.U'/-c]^2$ (3) ;

$= q'[(W/c)(-c^2/W.U')x\ U'/c]_{ij}/[E.U'/-c]^2$ (4);

$F_{ij}^{na} = q'\{(W/c)x\ [(U'/c)(-c^2/U'.W) – W/c +W/c]\}_{ij}/[E.U'/-c]^2$ or

$F_{ij}^{na} = -q'\{(U'/c)x[(W/c)(-c^2/W.U') – U'/c + U'/c]\}_{ij}/[E.U'/-c]^2$

$= q'\{(W/c)\ x[V_{U'W}/c +W/c]\}_{ij}/[E.U'/-c]^2$ or

$= -q'\{(U'/c)x[V_{WU'}/c + U'/c]\}_{ij}/[E.U'/-c]^2$; finally:

$F_{ij}^{na} = q'[(W/c)x\ V_{U'W}/c]/[E'U'/-c]^2 = -q'[(U'/c)x\ V_{WU'}/c]_{ij}/[E.U'/-c]^2$ (5) ;

We also have: $F_{ij}^{na} = [Ex\ A_{LW}]_{ij}/[E.U'/-c]^2$ (6); using eq31, section3 .

We have one final steps. $E.U'/-c = l/[1-V^2_{U'W}/c^2]^{1/2} = l/[1-V^2_{WU'}/c^2]^{1/2}$;

$F_{ij}^{na} = (q'/l^2)[(W/c)xV_{U'W}/c]_{ij}\ (1-V^2_{U'W}/c^2) =(-q'/l^2)[(U'/c)xV_{WU'}/c]_{ij}(1-V^2_{WU'}/c^2)$ (7);

This very important formula will be called the canonical projective form or representation of the nonacceleration field. Remember that the na field contains the acceleration implicitly,not explicitly,except for electrostatic or magnetostatic.

We observe that F_{ij}^{na} has a very interesting form. Besides its q'/l^2 factor,

It has $(W/c\ x\ V_{U'W}\hat{\ })_{ij}$ or $(U'/c\ x\ V_{WU'}\hat{\ })_{ij}$ as bivector "directions".Lastly it has a

factor $(V/c)[1- V^2/c^2]$, a cubic in V/c, where $V= V_{U'W}$ or $V_{WU'}$. Imagine a virtual motion starting at the unit normal W/c of a future or past hyperboloid of two sheets at an angle θ=0 then through a virtual motion reaching the final unit normal U'/c of the hyperboloid having moved through an angle $\theta_{U'W}$ on the hyperboloid along a geodesic segment $\theta_{U'W}$ of the hyperboloid. In the tangent space of the hyperboloid centered at W/c,the geodesic line segment is $\tanh\theta_{U'W}$

as is well known.(refs2,3,5). The same reasoning applies if we start the virtual motion at U'/c and move to W/c after making an angle $\theta_{WU'}$ .Let us write V(θ)/c = tanhθV^ to describe the virtual motion during the virtual transit.

$V(\theta)/c = \tanh\theta\, V\hat{}$ (8); $d(V/c)/d\theta = \mathrm{sech}^2\,\theta\, V\hat{} = (1- V^2/c^2)\, V\hat{}$ (9);

$d^2 (V/c)/d\theta^2 = -2(V/c)[1- V^2/c^2]$ (10); $d^2\tanh\theta/d\theta^2 = -2\tanh\theta/\cosh^2\theta$ (11) ;

$-(1/2)d^2\tanh\theta/d\theta^2\, V\hat{} = (V/c)[1 – V^2/c^2] = \tanh\theta/\cosh^2\theta\, V\hat{}$ (12)

We see that eqs8 to 12 must be evaluated at $\theta_{U'W}$ to give the eq 12 term which appears in F_{ij}^{na} . Eqs 10,11,12 suggest that we are dealing with a covariant derivative. The direction of the virtual motion in the tangent space is that of unit spatial 4vector $V_{U'W}$^ or $V_{WU'}$^ which reduce to a unit 3vector in the u'/c direction if w/c =0 or a unit 3vector in the w/c direction if u'/c = 0.The important thing is that the direction is in one direction only in the tangent plane.The question is:

Is $d^2(V/c)/d\theta^2 + 2(V/c)[1- V^2/c^2] = 0$ the eq of a geodesic ?

or equivalently is $d^2\tanh\theta/d\theta^2 + 2\tanh\theta/\cosh^2\theta = 0$ the eq of a geodesic.

The answer is to be found in Svante Janson Riemannian Geometry 122p 3/15/20.

In example 7.7 eq 7.92 he has the the geodesic eq for the 3dim Klein Ball model of hyperbolic geometry (ref.5):

$d^2\Upsilon^a/dt^2 = \{d[\log(1- \Upsilon^2)]/dt\}d\Upsilon^a/dt$ (13); a=1,2,3. t is a parameter, not necessarily time. The solution to eq13 is ϒ(t) = tanh(t)ϒ^ (14); The components are omitted since the direction is constant. In 3 dim any 3 velocity v/c = tanhα where α is the rapidity. Θ is the projective form of the rapidity and can be used as well.We can now verify that eq 13 for the geodesic is satisfied by tanhθ.

We have: ϒ(θ) = tanhθ ϒ^ (!5); $d\Upsilon(\theta)/d\theta = \mathrm{sech}^2\theta\, \Upsilon\hat{}= 1/\cosh^2\theta\, \Upsilon\hat{}$(16) ;

$d[\log(1- \Upsilon^2)]/d\theta = d[\log(1- \tanh^2\theta)]/d\theta = -2\tanh\theta$ (17) ;

$d^2\Upsilon/d\theta^2 = -2\tanh\theta/\cosh^2\theta\, \Upsilon\hat{} = d^2\tanh\theta/d\theta^2\, \Upsilon\hat{}$ (18); so we have proved it. Note that we have used contravariant component of ϒ in eq 13. It is instructive to double check that we have the correct Christoffel symbols and that we identify them.Svante Janson has :

$\Gamma^k_{ij} = [x_i\delta_{jk} + x_j\delta_{ik}]/(1- x^2)$ his eq 7.86 . ijk=1to 3 and correspond to abc here,x^2 is the Euclidean sum of the squares. Since the motion is rectilinear only the components i=j=k=1 survive. So the only Christoffel symbol that survive is Γ^1_{11}.

$\Gamma^1_{11} = [x_1\delta_{11} + x_1\delta_{11}]/(1- x^2) = 2x_1/(1- x^2)$ (19) ; we use v/c instead of x so

$\Gamma^1_{11} = 2(v_1/c)/(1- v.v/c^2)$ (20); but there is only one component of v. so

$\Gamma^1_{11} = 2|v/c|/(1- v^2/c^2)$ (21); v^2 is the Euclidean sum of the components $v.v = v_1^2$

Replacing |v/c| by tanhθ we get :

$\Gamma^1_{11} = 2\tanh\theta\cosh^2\theta$ (22); recall that the geodesic eq in velocity space of 3dim

Is given in general by: $d^2v^a/ds^2 + \Gamma^a_{bc}(dv^b/ds)(dv^c/ds) = 0$ (23);

In our notation $d^2\tanh\theta/d\theta^2 = -\Gamma^1_{11}/\cosh^4\theta = -2\tanh\theta/\cosh^2\theta$ (24);

$\Gamma^1_{11} = 2\tanh\theta\cosh^2\theta$ (25); which the same as eq22. So the results are consistent.

F_{ij}^{na} represents a bivector projective form of the contravariant component of the acceleration term of a geodesic of a hyperboloid of two sheet, in the tangent space of the hyperboloid, in the Klein ball model of 3dim hyperbolic space!!

Let: $[d^2(V(\theta)/c)/d\theta^2]$ evaluated at $\theta = \theta_{U'W} = [d^2|V_{U'W}/c|/d\theta_{U'W}{}^2]V\hat{}_{U'W}$

or $= [d^2|V_{WU'}/c|/d\theta_{WU'}{}^2]V\hat{}_{WU'}$ we have:

$F_{ij}^{na} = (-q'/2l^2)[(W/c)x\{d^2|V_{U'W}/c|/d\theta_{U'W}{}^2\}V\hat{}_{U'W}]_{ij}$ (26); or

$F_{ij}^{na} = (q'/2l^2)[(U'/c)x\{d^2|V_{WU'}/c|/d\theta_{WU'}{}^2\}V\hat{}_{WU'}]_{ij}$ (27) ;

4B Some projections and their geometric meanings.

Let a general unit bivector be defined as

$S_{ij} = [(U/c)x\ V^\wedge]_{ij}$ (28) with U/c being any unit 4 velocity and V^ be any spacelike unit vector orthogonal to it. We want $S_{ij}U_j/c$ and $S_{ij}V^\wedge_j$.

$S_{ij}V^\wedge_j = [(U/c)xV^\wedge]_{ij}\ V^\wedge_j = [U_I V^\wedge_j - U_J V^\wedge_i]V^\wedge_j = U_i/c$ (29);

$S_{ij}U_j/c = [(U/c)xV^\wedge]_{ij}U_j/c = [U_i V^\wedge_j - U_j V^\wedge_i]U_j/c = V^\wedge_I$ (30);

Choosing U/c = W/c or U'/c and $V^\wedge = V^\wedge_{U'W}$ or $V^\wedge_{WU'}$ we get

$[(W/c)x\ V^\wedge_{U'W}]_{ij}W_j/c = (V^\wedge_{U'W})_I$ (31) ; $[(U'/c)x\ V^\wedge_{WU'}]_{ij}U'_j/c = (V^\wedge_{WU'})_i$ (32);

$[(W/c)x\ V_{U'W}/c]_{ij}W_j/c = (V_{U'W}/c)_I$ (31a) ; $[(U'/c)xV_{WU'}/c]_{ij}U'_j/c = (V_{WU'}/c)_I$ (32a);

$[(W/c)x\ V^\wedge_{U'W}]_{ij}V^\wedge_{U'W\ j} = (W/c)_I$ (33); $[(U'/c)xV^\wedge_{WU'}]_{ij}V^\wedge_{WU'\ j} = (U'/c)_I$ (34);

$[(W/c)xV_{U'W}/c]_{ij}V^\wedge_{U'W\ j} = |V_{U'W}/c|(W/c)_I$ (33a);

$[(U'/c)xV_{WU'}/c]_{ij}V^\wedge_{WU'\ j} = |V_{WU'}/c|(U'/c)_I$ (34a);

These eqs give us a little intuition as to the meaning of the projections.

The moment of LW potential projections are very instructive.

$[ExA]_{ij}W_j/c = q'[(W/c)xV_{U'W}/c]_{ij}W_j/c = q'V_{U'W\ i}/c$ (35)

$= -q'[(U'/c)xV_{WU'}/c]_{ij}W_j/c$ (35);

$[ExA]_{ij}U'_j/c = -q'[(U'/c)xV_{WU'}/c]_{ij}U'_j/c = -q'V_{WU\ i}/c$ (36)

$= q'[(W/c)xV_{U'W}/c]_{ij}U'_j/c$ (36); Eqs35,36 represent projective currents in line form. Compare them with the same currents in bivector form given by $[ExA]_{ij}$.

$[ExA]_{ij}V^\wedge_{U'W\ j} = q'[(W/c)x\ V_{U'W}/c]_{ij}V^\wedge_{U'W\ j}/c = q'|V_{U'W}/c|W_i/c$ (37)

$= -q'[(U'/c)xV_{WU'}/c]_{ij}V^\wedge_{U'W\ j}$ (37);

$[ExA]_{ij}V^\wedge_{WU'\ j} = -\ q'[(U'/c)x\ V_{WU'}/c]_{IJ}V^\wedge_{WU'\ j} = -q'|V_{WU'}/c|U'_i/c$ (38)

$= q'[(W/c)x\ V_{U'W}/c]_{ij}V^\wedge_{WU'\ j}$ (38);

Note that adding expression like $V(\theta)/c + |V(\theta)/c|U/c$ the sum lies on the light cone. This allows us to visualize expressions like $|V(\theta)/c|U/c$. They can also be visualized as lying in the tangent space of a hyperboloid of one sheet. This will be further explored in a later section in connection with virtual timelike hyperbolic motions.

The field F_{ij}^{na} gives expressions similar to the previous ones such as:

$F_{ij}^{na}W_j/c = (q'/l^2)(V_{U'W}/c)_i (1- V^2_{U'W}/c^2)$ (39);

$F_{ij}^{na}U'_j/c = (-q'/l^2)(V_{WU'}/c)_i (1- V^2_{WU'}/c)$ (40);

$F_{ij}^{na}V\hat{}_{U'W\,j} = (q'/l^2)|V_{U'W}/c|(1- V^2_{U'W}/c^2)W_i/c$ (41) ;

$F_{ij}^{na}V\hat{}_{WU'\,j} = (-q'/l^2)|V_{WU'}/c|(1- V^2_{WU'}/c^2)U'_i/c$ (42) ;

So the projections of the field yield expressions of the form

$|d^2V(\theta)/d\theta^2|V\hat{}$ and $|d^2V(\theta)/d\theta^2|U/c$ whose sum is a null vector.

We also have the useful relations:

$[(W/c)x (V_{U'W}/c)/(1- V^2_{U'W}/c^2)^{1/2}]_{ij}W_j/c = (V_{U'W}/c)_i /(1- V^2_{U'W}/c^2)^{1/2}$ (43a);

$[(U'/c)x (V_{WU'}/c)/(1- V^2_{WU'}/c^2)^{1/2}]_{ij}U'_j /c = (V_{WU'}/c)_i /(1- V^2_{WU'}/c^2)^{1/2}$ (43b) ;

Putting $V\hat{}_{U'W}$ instead of $V_{U'W}$ in eq 43a we get $V\hat{}_{U'W}/(1- V^2_{U'W}/c^2)^{1/2}$ (44a) ;

Let c=1 in what follows except when required for clarity.

Putting $V\hat{}_{WU'}$ instead of $V_{WU'}$ in eq 43b we get $V\hat{}_{WU'}/(!- V^2_{WU'})^{1/2}$ (44b);

We also have:

$[WxV_{U'W}/(1- V^2_{U'W})^{1/2}]_{ij}V\hat{}_{U'W\,j} = |V_{U'W}/(1- V^2_{U'W})^{1/2}|W_i$ (45a);

$[U'x V_{WU'} /(1- V^2_{WU'})^{1/2}]_{ij}V\hat{}_{WU'\,j} = |V_{WU'} /(1- V^2_{WU'})^{1/2}|U'_i$ (45b);

Putting $V\hat{}_{U'W}$ instead of $V_{U'W}$ in eq 45a we get $W_i /(1- V^2_{U'W})^{1/2}$ (46a);

Putting $V\hat{}_{WU'}$ instead of $V_{WU'}$ in eq45b we get $U'_i/(1- V^2_{WU'})^{1/2}$ (46b);

Now that we have all these projections we want to express the following relations in terms of them:

$U' = [V_{U'W} + W]/(1- V^2_{U'W})^{1/2}]$ (47a); $W= [V_{WU'} + U']/(1- V^2_{WU'})^{1/2}$ (47b);

$-V\hat{}_{WU'} = [V\hat{}_{U'W} +|V_{U'W}|\ W] /(!- V^2_{U'W})^{1/2}$ (48a);

$-V\hat{}_{U'W} = [V\hat{}_{WU'} + |V_{WU'}|\ U']/(1- V^2_{WU'})^{1/2}$ (48b);

Eq 47a = eqs(43a + 46a) . Eq 47b = eqs(43b + 46b).

Eq 48a = eqs(44a +45a) . Eq 48b = eqs(44b +45b).

4C Quadratic expressions of projective bivectors.Their meaning.

Let $S_{ij} = [U/c \times V\hat{}]_{ij} = [U_iV\hat{}_j - U_jV\hat{}_i]/c$ (49); U/c and V^ are any two unit timelike and spacelike 4 vectors, respectively, orthogonal to each other.

$S_{ij}S_{ij} = [(U.U/c^2)V\hat{}.V\hat{} + (U.U/c^2)V\hat{}.V\hat{}\ ] = -2$ (50a);

$(-1/2)S_{ij}S_{ij} = 1$ (50b) ; Let $S^*_{ij} = [U/c \times V/c]_{ij}$ (51); $(-1/2)S^*_{ij}S^*_{ij} = V^2/c^2$ (52);

$S_{ij}S_{ik}= - [U_jU_k/-c^2 + V\hat{}_jV\hat{}_k] = -P_{jk}$ (53a); (-1;2) $S^*_{ij}S^*_{ik} = -P_{jk} V^2/c^2$ (53b);

$P_{jk} = [U_jU_k/-c^2 + V\hat{}_jV\hat{}_k]$ (54);

P_{jk} is an operator that projects any 4 vector A perpendicularly onto the U/c, V^ plane.

$P_{jk}A_k = [(A.U/-c)U/c + (A.V\hat{})V\hat{}]_j = (T_A)_j$ (55); T_A is the projected A vector. The subscript is just to specify that it is A that is projected. Note that it is (A.U/-c)U/c not (A.U/c)U/c that is the projection of A along U/c. The latter is a reflection. To see that note that $A= A_{perp} +A_{para}$ where perp refers to the component perpendicular to U/c and para is the component parallel to U/c.

$A-A_{para} = A_{perp} = [A – (A.U/-c)U/c] = [A +(A.U/c)U/c]$ (56a); To check: $A_{perp}.U/c =0$;

A_{para} = (A.U/-c)U/c (56b) ; One word of caution. One must not use

$P^R_{ij} = [U_iU_j/c^2 + V\text{^}_iV\text{^}_j]$ or $[U_iU_j/c^2 - V\text{^}_iV\text{^}_j]$ or $[U_iU_j/-c^2 - V\text{^}_iV\text{^}_j]$,they represent reflections.

Now let $Q_{ij} = \delta_{ij} - P_{ij} = \delta_{ij} - [U_IU_J/-c^2 + V\text{^}_iV\text{^}_j]$ (57); where δ_{ij} is the Kronecker delta in 4 dim. We can use it because we use the metric) [1,1,1,i].

Q_{ij} projects any 4 vector A perpendicularly out of the (U/c,V^) plane.

$Q_{ij}A_j$ = {A- [(A.U/-c)U/c + (A.V^)V^]}$_I$ = $(D_A)_i$ (58) ;

A= A_{out} + A_{in} (58a) ; in=in plane; out= out of plane. A-A_{in} = A_{out} (58b);

Since A_{out}= {A- [(A.U/-c)U/c + (A.V^)V^]} (58c) using eq55, it is proved.

S_{ij} is also an operator. It performs a similar useful function as the two previous ones.

$S_{ij}A_j$ = [U/c(A.V^) – V^(A.U/c)]$_I$ = S_A (59);

S_{ij} projects any 4 vector A onto the U/c,V^ plane so that S_A is perpendicular to A. It must therefore be orthogonal to the two previous projected 4 vectors T_A ,D_A .It is straightforward to show that $S_A.T_A = S_A.D_A = T_A.D_A = 0$; so we obtain 3 mutually orthogonal 4 vectors by operating on any 4 vector A with S_{ij} ,P_{ij} or Q_{ij} .

The piece de resistance is the operator $K_{ij} = \delta_{ij} - 2P_{ij}$ (60) which is a reflection operator. It is closely related to the energy stress tensor of F_{ij}^{na} as will be shown in the next subsections.

$K_{ij} A_j$ = {A – 2[(A.U/-c)U/c + (A.V^)V^]}$_I$ (61);

We have:

$S_{ij}U_j/c = V\text{^}_I$; $S_{ij}V\text{^}_j = U_i/c$; $P_{ij}U_j/c = U_i/c$; $P_{ij}V\text{^}_j = V\text{^}_I$; $Q_{ij}U_j/c = Q_{ij}V\text{^}_j = 0$;

$K_{ij}U_j/c = -U_i/c$; $K_{ij}V\text{^}_j = -V\text{^}_I$; as could have been expected..

4D Quadratic expressions of the moment of LW potential.

$[ExA]_{ij} = q'[W/c \times V_{U'W}/c]_{ij} = -q'[U'/c \times V_{WU'}/c]_{ij}$ (62);

$(-1/2)[ExA]_{ij}[ExA]_{ij} = q'^2V^2_{U'W}/c^2 = q'^2V^2_{WU'}/c^2$ (63) ;

$[EXA]_{ij}[ExA]_{ik} = -q'^2 (V^2_{U'W}/c^2)[W_jW_k/-c^2 + V\wedge_{U'W,j} V\wedge_{U'W,k}]$

$= -q'^2(V^2_{WU'}/c^2)[U'_JU'_k/-c^2 + V\wedge_{WU'j} V\wedge_{WU'k}]$ (64);

$(-1/2)\{[ExA]_{kl}[ExA]_{kl}\}\delta_{ij} + [ExA]_{mi}[ExA]_{mj} = q'^2V^2_{U'W}/c^2\{\delta_{ij}- [W_iW_j/-c^2 + V\wedge_{U'W\,I} V\wedge_{U'W\,j}]\}$

$= q'^2V^2_{WU'}/c^2\{\delta_{ij} - [U'_iU'_j/-c^2 + V\wedge_{WU'i}V\wedge_{WU'j}]\}$ (65);

Remember that $V^2_{U'W}V\wedge_{U'W\,i}V\wedge_{U'W\,j} = V_{U'W\,I}\, V_{U'W\,j}$. Now the reflection:

$(-1/2)\{[ExA]_{kl}[ExA]_{kl}\}\delta_{ij} + 2[ExA]_{mi}[ExA]_{mj} = q'^2(V^2_{U'W}/c^2)\{\delta_{ij} - 2[W_iW_j/-c^2+V\wedge_{U'Wi} V\wedge_{U'Wj}]\}$

$= q'^2(V^2_{WU'}/c^2)\{\delta_{ij} - 2[U'_iU'_j/-c^2 + V\wedge_{WU'i}V\wedge_{WU'j}]\}$ (66);

Note that these are operators times the square of the geodesics.

4E Quadratic expressions for the field. The Lagrangian field density.The energy stress tensor.

From eq 4.7, 4.26,27 we get for the field Lagrangian density.

$(-1/2)F^{na}_{ij}F^{na}_{ij} = (q'^2/l^4)(V^2_{U'W}/c^2)(1- V^2_{U'W}/c^2)^2 = (q'^2/l^4)(V^2_{WU'}/c^2)(1- V^2_{WU'}/c^2)^2$

$= (q'^2/4l^4)[d^2 \tanh\theta_{U'W}/d\theta^2_{U'W}]^2 = (q'^2/4l^4) [d^2\tanh\theta_{WU'}/d\theta^2_{WU'}]^2$ (67);

The field Lagrangian density is proportional to the square of the geodesic accelerations.

$F^{na}_{ij}F^{na}_{ik} = (-q'^2/4l^4)(d^2\tanh\theta_{U'W}/d\theta^2_{U'W})^2[W_jW_k/-c^2 + V\wedge_{U'W\,J}V\wedge_{U'W\,k}]$

$= (-q'^2/4l^4)(d^2\tanh\theta_{WU'}/d\theta^2_{WU'})^2[U'_jU'_k/-c^2 + V\wedge_{WU'j}V\wedge_{WU'k}]$ (68) ;

And now for the energy stress tensor. We have, using $V= V_{U'W}$ or $V_{WU'}$:

$\{(-1/2)(F^{na}_{kl}F^{na}_{kl})\delta_{ij} + 2F^{na}_{mi}F^{na}_{mj}\}$

$= (q'^2/l^4)(V^2/c^2)(1- V^2/c^2)^2\{\delta_{ij} - 2[W_iW_j/-c^2 + V\hat{}_{U'W\,i}V\hat{}_{U'W\,j}]\}$

$= (q'^2/l^4)(V^2/c^2)(1- V^2/c^2)^2\{\delta_{ij} - 2[U'_iU'_j/-c^2 + V\hat{}_{WU'\,i}V\hat{}_{U'W\,j}]\}$

$= (q'^2/4l^4)[d^2\tanh\theta_{U'W}/d\theta^2_{\;U'W}]^2\{\delta_{ij} - 2[W_iW_j/-c^2 + V\hat{}_{U'W\,i}V\hat{}_{U'W\,j}]\}$

$= (q'^2/4l^4)[d^2\tanh\theta_{WU'}/d^2\theta_{WU'}]^2\{\delta_{ij} - 2[U'_iU'_j/-c^2 + V\hat{}_{WU'\,i}V\hat{}_{WU'\,j}]\} = 2T_{ij}$ (69);

We recognize eq4.69 as being twice the energy stress tensor T_{ij}.(refs. 6,7,8).It is equal to $(q'^2/4l^4)$ times the square of the geodesic acceleration times ½ the reflection operator of eq4.69 which depends on whether the virtual motion is from W/c to U'/c or vice versa. The trace of T_{ij} is determined by the reflections.

Since δ_{ii} summed over i is =4 and $-2[U.U/-c^2 + V\hat{}.V\hat{}] = -4$, the trace is zero. In addition since the product of two reflections is the identity,

$T_{ij}T_{jk} = (q'^2/2l^4)^2[(V^2/c^2)(1- V^2/c^2)^2]^2\delta_{ik}$ (70) ; let $T_{ij}T_{ij} = T^2$ (71);$\delta_{ii} = 4$ (summed over i)

$T^2 = 4\,[(q'^2/2l^4)^2\,(V^2/c^2)(1- V^2/c^2)^2]^2$

$= (4/4)[(q'^2/4l^4)(d^2\tanh\theta/d\theta^2\,)^2]^2$ (72); see eq 4.69 for the ¼ factor.

$T = [(q'^2/l^4)(V^2/c^2)(1- V^2/c^2)^2] = [(q'^2/4l^4)(d^2\tanh\theta/d\theta^2)^2\,]$ (73); Let:

$U/c = W/c$ or U'/c , $V\hat{} = V\hat{}_{U'W}$ or $V\hat{}_{WU'}$, $V^2 = V^2_{U'W}$ or $V^2_{WU'}$, $\theta = \theta_{U'W} = \theta_{WU'}$

$T_{ij} = T\{\delta_{ij}/2 - [U_iU_j/-c^2 + V\hat{}_iV\hat{}_j] = [T(\delta_{ij}/2) - TP_{ij}]$ (74);

$T_{ij} - T(\delta_{ij}/2) = -T[U_iU_j/-c^2 + V\hat{}_iV\hat{}_j] = -TP_{ij}$ (75);

$F_{ij}F_{ik} = -TP_{ij}$ (76); $(-1/2)F_{ij}F_{ij} = T$ (77) ;

$T = T^{na}$; $T_{ij} = T^{na}_{ij}$; $F_{ij} = F^{na}_{ij}$;

5 The Lorentz force. The energy density. The Poynting 4 vector.

So far, we have dealt only with two 4 velocities, namely U' and W which represented two normal of a hyperboloid of two sheets. We only dealt with two geodesic segment $V_{U'W}$ and $V_{WU'}$ and we found geometric quantities involving them. The Lorentz force brings a third velocity into play, the 4 velocity U of a test charge q on which charge q' acts. This brings about a third tangent to the hyperboloid of two sheets.This means that geodesics between U and U' and between U and W will appear in the tangent space centered at U. Since U represent a third observer, the test charge, we need to investigate how the previous entities appear as viewed from U. New quantities appear in profusion and it will be necessary to simplify the notation when things get too messy.

5A Projective Einstein law of addition or substraction 4 vectors and tensors.Olinde Rodrigues-like expressions.

What is $U'.W/-c^2 = \cosh\theta_{U'W}$ as viewed from U?

Let $U' = [V_{U'U} + U]/(1- V^2_{U'U})^{1/2}$ (1) ; $W = [V_{WU} + U]/(1- V^2_{WU})^{1/2}$ (2);

we have omitted the c's. Putting eqs 1,2 into the cosh we get:

$U'.W/-c^2 = [1- V_{U'U}.V_{WU}]/(1- V^2_{U'U})^{1/2}(1- V^2_{WU})^{1/2}$

$= \cosh\theta_{U'U} \cosh\theta_{WU} - \sinh\theta_{U'U} \sinh\theta_{WU} V\text{^}_{U'U} V\text{^}_{WU}$ (3) ;

Compare this with $\cosh(\theta_A - \theta_B) = \cosh\theta_A \cosh\theta_B - \sinh\theta_A \sinh\theta_B$.

The geodesic segment $V_{U'W} = [U'(-c^2/U'.W) - W]$ becomes using eqs1,2,3

$V_{U'W}/c = [e^{u'w}{}_u + (e^{u'w}{}_u .W/c)W/c](1- V^2_{WU})^{1/2}$ (4);

With $e^{u'w}{}_u = (V_{U'U} - V_{WU})/[1- V_{U'U}.V_{WU}]$ (5); similarly:

$V_{WU'}/c = [e^{wu'}{}_u + (e^{wu'}{}_u .U'/c)U'/c](1- V^2_{U'U})^{1/2}$ (6);

With $e^{wu'}{}_u = (V_{WU} - V_{U'U})/[1- V_{WU} . V_{U'U}] = - e^{u'w}{}_u$ (7) ;

We can obtain the bivector form of the geodesics in term of U.

$(W x V_{U'W})_{ij} = [W x U'(-c^2/U'.W)]_{ij}$

$= [(U x\, e^{u'w}{}_u) + (V_{WU} x\, e^{u'w}{}_u)]_{ij}$ (8);

we have used eqs1and 4 to obtain this result and omitted the c whenever this cannot lead to confusion.Eq8 gives the geodesic segment from W to U'.

$(U' x V_{WU'}) = [U' x W(-c^2/W.U')] = [(U x\, e^{wu'}{}_u) + (V_{U'U} x\, e^{wu'}{}_u)]$ (9); since eq9 is the negative of eq8 and using the fact that $e^{u'w}{}_u = -\, e^{wu'}{}_u$ we have:

$[V_{WU} x\, e^{u'w}{}_u] = [V_{U'U} x\, e^{u'w}{}_u]$ (9a); we have omitted the subscript ij throughout.

Eq5 and 6 are projective forms of the Einstein law of addition or subtraction of velocities. See V.Fock and Landau and Lifschitz in references.

Write $V_{WU} = V_{WU\ pae} + V_{WU\ ppe}$ (9b) ;

where the subscript pae means parallel to $e^{u'w}{}_u$ or $e^{wu'}{}_u$ and ppe means perpendicular to it. We will prove in the next section that $V_{WU\ ppe} = V_{U'U\ ppe}$ (9b);

let $e_{ij} = (U/c x\, e^{u'w}{}_u)_{ij}$ (10); $b_{ij} = (V_{WU}/c \ x\, e^{u'w}{}_u)_{ij} = (V_{WU\ ppe}/c \ x\, e^{u'w}{}_u)_{ij}$ (11)

$(W x\, V_{U'W}) = e_{ij} + b_{ij}$ (12); $V^2_{U'W} = (-1/2)(e_{ij} + b_{ij})(e_{ij} + b_{ij})$ (13);

$[e_{ij} b_{ij}] = 0$ (14) ; $(-1/2)\, e_{ij}e_{ij} = e^2$ (15); $(-1/2)\, b_{ij}b_{ij} = -\, V^2_{WU\ ppe}\, e^2$ (16);

These important formulas are obtained by using the fact that U, $V_{WU\ ppe}$, $e^{u'w}{}_u$ are mutually orthogonal. We can call eqs 5.10 ,5.11 the electric and magnetic parts of the rotation tensor (or geodesic tensor) given by eq 5.12.

$V^2_{U'W} = e^2 - V^2_{WU\ ppe}\, e^2 = e^2\, [1 - V^2_{WU\ ppe}]$ (17); to simplify the notation write pe instead of ppe.

Let $[1 - V^2_{WU\ pe}/c^2] = 1/\cosh^2\theta_{pe}$ (18) ; $|V_{WUpe}/c| = \tanh\theta_{pe}$ (18a);

$V^2_{U'W}/c^2 = e^2/\cosh^2\theta_{pe} = \tanh^2\theta_{U'W}$ (19a); $(V^2_{U'W}/c^2)\cosh^2\theta_{pe} = e^2$ (19b);

$(-1/2)[e_{ij}e_{ij}] = e^2 = (V^2_{U'W}/c^2)\cosh^2\theta_{pe}$ (20);

$(-1/2)b_{ij}b_{ij} = -V^2_{WU\,pe}\,e^2 = -\tanh^2\theta_{pe}\cosh^2\theta_{pe}\,V^2_{U'W}/c^2 = -\sinh^2\theta_{pe}\,V^2_{U'W}/c^2$ (21);

We can define a magnetic pseudo 4vector $b_i = (1/2)\varepsilon^{ijkl}\,U_i V_{WU\,pe\,j}\,e_k$ (22) ;

We should have : $b.b = (V^2_{WUpe}/c^2)\,e^2 = \sinh^2\theta_{WUpe}\,V^2_{U'W}/c^2$ (23);

Then $V^2_{U'W}/c^2 = e^2 - b^2$ (24) ; $e.b = 0$ (25);

U/c, V_{WUpe}/c, e , b are 4 mutually orthogonal 4 vectors.

We find $e_{ij}\,U_j = e_i$ (26); $e^{dual}{}_{ij}\,U_j = b_i$ (27) ; $b_{ij}\,U_j = 0$ (28);

We recall from eq 3.31 that

$[ExA_{LW}] = q'[Wx\,V_{U'W}] = q'[dx'_i/dx_j - dx'_j/dx_i] = q'(e_{ij} + b_{ij})$ (29);

Eq29 shows nicely how all the quantities are related: the moment of potential, the geodesic segment in bivector form, the rotation tensor, and the geodesic segment in bivector form as viewed from U, decomposed into an electric field part and a magnetic field part. It is noteworthy that the latter bear a close resemblance with some of the Olindes Rodrigues formulas for the addition of rotations or the decomposition of rotations.

5B The Lorentz force and its dual.

We observe that to go from the geodesic to the geodesic accelerations we only have to multiply by the factor $(1- V^2/c^2)$ with $V = V_{U'W}$ or $V_{WU'}$ .The fields require the additional factor q'/l^2 . All the formulas of the previous section are therefore applicable with $\mathcal{E}_{ij} = e_{ij}\,(1- V^2)$ (30) and $B_{ij} = b_{ij}\,(1- V^2)$ (31);

$F_{ij}U_j = (q'/l^2)\,e_{ij}\,U_j\,(1- V^2) = \mathcal{E}_i$ (32); $F^{dual}{}_{ij}U_j = (q'/l^2)\,e^{dual}{}_{ij}\,U_j\,(1- V^2) = B_i$ (33) ;

Where Ɛ is the electric field 4 vector (not the 3 vector) and B is the magnetic field pseudo 4 vector. $B_i = (q'/l^2)\, b_i\, (1- V^2)$ (34) ; b is given by eq22;

The Lorentz Force is $F^{na}{}_i = qF^{na}{}_{ij}U_j/c = qq'\, e_i /(E.U'/-c^2)^2$ (34); use eq26.

$F^{dual}{}_i = qq'(1/2)\mathcal{E}^{ijkl}U_jV_{wu\,pe\,k}\, e_l /[E.U'/-c^2]^2$ (35); $\mathcal{E}^{ijkl}$ is the Levi-Civita symbol.

Of course $\mathcal{E} = q'\, e /[E.U'/-c^2]^2$ (36); $B_i = q' (1/2)\mathcal{E}^{ijkl} U_jV_{WU\,pe\,k}\, e_i /[E.U'/-c^2]^2$ (37);

$\mathcal{E}.B = 0$ (38);

5C The field Lagrangian density.

$\mathcal{E}^2 = (q'^2/l^4)\, V^2_{U'W} (1- V^2_{U'W})^2\cosh^2 \theta_{WU\,pe}$ (39a) ;

$B^2 = (q'^2/l^4)V^2_{U'W} (1- V^2_{U'W})^2 \sinh^2 \theta_{WU\,pe}$ (39b);

$\mathcal{E}^2-B^2 = (q'^2/l^4)V^2_{U'W}(1- V^2_{U'W})^2$

$= q'^2 (-1/2)[dx'_l /dx_j – dx'_j /dx_i][dx'_i/dx_j- dx'_j/dx_i]/(E.U'/-c^2)^2$ (39c) ;using eq29.

5D The energy density

$(1/2)[\mathcal{E}^2 + B^2] = (q'^2/2l^4)V^2_{U'W}(1- V^2_{U'W})^2 [\cosh^2 \theta_{WU\,pe} + \sinh^2 \theta_{WU\,pe}]$

$= (q'^2/2l^4)V^2_{U'W} (1- V^2_{U'W})^2 \cosh 2\theta_{WU\,pe}$ (40a)

$= (q'^2/2l^4)[(1/4)(d^2\tanh\theta/d\theta^2)^2] \cosh 2\theta_{WU\,pe}$ (40b); $\theta = \theta_{WU\,pe}$

5E The Poynting 4 vector.

The Poynting vector in Maxwell's theory is $(e x h)_{ab}$ with a,b=1,2,3, and e and h are the electric and magnetic field respectively. The Poynting vector is orthogonal to e and h and is a pseudo-3-vector.The generalization of the Poynting vector to 4dim with a timelike ray of influence will involve both Ɛ and B with a 4dim cross product which will point in a direction orthogonal to U, Ɛ , B . The only direction left is ihe direction of the unit 4 vector $V\hat{}_{WU\ pe}$. The 4 dim Poynting vector is a pseudo-4-vector. We should have:

$P_{oyn} = |\mathcal{E}xB|V\hat{}_{WU\ pe} = |\mathcal{E}||B|V\hat{}_{WU\ pe}$ (41); the direction could be $-V\hat{}_{WU\ pe}$ of course this is just a heuristic argument.

$P_{oyn} = (q'^2/l^4)V^2_{U'W}(1- V^2_{U'W})^2 \cosh\theta_{WU\ pe} \sinh\theta_{WU\ pe}\ V\hat{}_{WU\ pe}$ (42);

The ratio of the Poynting vector to the energy density should give a projective relative 4 velocity.

$P_{oyn} /(1/2)[\mathcal{E}^2 + B^2]= 2\sinh\theta_{WU\ pe} \cosh\theta_{\ WU\ pe} /\cosh2\theta_{WU\ pe}\ V\hat{}_{WU\ pe}$

$= \tanh2\theta_{WU\ pe}\ V\hat{}_{WU\ pe}$ (43);

$P_{oyn} = (1/2)[\mathcal{E}^2 + B^2] \tanh2\theta_{WU\ pe}\ V\hat{}_{WU\ pe}$ (44);

The projective relativ 4 velocity is : $V(2\theta)/c = \tanh2\theta_{WU\ pe}\ V\hat{}_{WU\ pe}$ (45);

We are dealing with a different model of hyperbolic geometry.

We can form : $V(2\theta)/(1 - V^2 (2\theta)^{1/2} = \sinh2\theta\ V\hat{}_{WU\ pe}$ (46);

$(1-V^2(2\theta))^{-½} = \cosh2\theta$ (47) ; $U(2\theta) = [V(2\theta) + U]/(1- V^2(2\theta))^{1/2}$ (48);

U is the 4 velocity of test charge q.

The unit 4 vector $V\hat{}(2\theta)$ orthogonal to $U(2\theta)$ is:

$V\hat{}(2\theta) = [\cosh2\theta\ V\hat{}_{WU\ pe} + \sinh2\theta\ U/c]$ (49);

To check : $[U(2\theta). V\hat{}(2\theta)] =$

$[\sinh2\theta_{WU\ pe} V\hat{}_{WU\ pe} + \cosh2\theta\ U/c].[\cosh2\theta\ V\hat{}_{WU\ pe} + \sinh2\theta\ U/c] = 0$ (50);

It is interesting to contruct 4 vectors with the Poynting vector and the energy density.

$P_{oyn} = (1/2)[\mathcal{E}^2 + B^2]\tanh2\theta_{\ WU\ pe} V\hat{}_{WU\ .\ pe}$

Form: $|P_{oyn}|V\hat{}_{WU\ pe} + (1/2)[\mathcal{E}^2 + B^2]\ U/c = P_{em}$ (51); P_{em} is a total electromagnetic momentum density. Its square should yield the negative of an e.m. mass density squared .

Form: $(1/2)[\mathcal{E}^2 + B^2]V\hat{}_{WU\ pe} + |P_{oyn}|U/c = W_{em}$ (52) ; $P_{em}.\ W_{em} = 0$;

5F The energy stress tensor and its projections in the U, $V\hat{}_{WU\ pe}$, $\mathcal{E}\hat{}$,$B\hat{}$ directions.

Using the eqs 12 to16 and 29to31 of the previous subsections we have:

$2\ T_{ij} = \{(-1/2)[\mathcal{E}_{lm}\,\mathcal{E}_{lm} + B_{lm}\,B_{lm}\,]\delta_{ij} + 2[(\mathcal{E}_{ik} + B_{ik})(\mathcal{E}_{jk} + B_{jk})]\}$ (53);

$[(\mathcal{E}_{ik} + B_{ik})(\mathcal{E}_{jk} + B_{jk}) = [(\mathcal{E}_{ik}\mathcal{E}_{jk}) + (B_{ik}B_{jk}) + (\mathcal{E}_{ik}B_{jk} + \mathcal{E}_{jk}B_{ik})]$ (54) ;

$\mathcal{E}_{ik}B_{jk} = U_iV\hat{}_{pe\ j}\ |V_{\ pe}|\mathcal{E}^2$ (55a); $\mathcal{E}_{jk}B_{ik} = U_j\ V\hat{}_{pe\ I}\ |V_{pe}|\mathcal{E}^2$ (55b) ;

$(\mathcal{E}_{ik}B_{jk} + \mathcal{E}_{jk}B_{ik}) = [\ U_i\ V\hat{}_{pe\ j} + U_j\ V\hat{}_{pe\ i}\ ]\,|V_{pe}|\mathcal{E}^2$ (56) ;

$(\mathcal{E}_{ik}\mathcal{E}_{jk}) = -[\ U_iU_j/\text{-}c^2 + \mathcal{E}\hat{}_i\mathcal{E}\hat{}_j\ ]\mathcal{E}^2$ (57); $(B_{ik}B_{jk}) = [\ V\hat{}_{pe\ i}\ V\hat{}_{pe\ j} + \mathcal{E}\hat{}_i\mathcal{E}_j]V^2_{pe}\ \mathcal{E}^2$ (58);

$(-1/2)\mathcal{E}_{lm}\mathcal{E}_{lm} = \mathcal{E}^2$ (59a); $(-1/2)B_{lm}B_{lm} = -\ (V^2_{WU\ pe}/c^2)\mathcal{E}^2 = -\ B^2$ (59b) ;

$(-1/2)(\mathcal{E}_{lm}\mathcal{E}_{lm} + B_{lm}B_{lm}) = (\mathcal{E}^2 - B^2)$ (59c) ;

$2T_{ij} = (\mathcal{E}^2\text{-}\ B^2)\delta_{ij}\ + 2\{-[(U_iU_j/\text{-}c^2) + \mathcal{E}\hat{}_i\mathcal{E}\hat{}_j]\mathcal{E}^2 + [V\hat{}_{pe\ I}\ V\hat{}_{pe\ j} + \mathcal{E}\hat{}_i\mathcal{E}\hat{}_j]V^2_{\ pe}\ \mathcal{E}^2$

$+\ [U_iV\hat{}_{pe\ j}\ + U_jV\hat{}_{pe\ i}\ ]\,|V_{pe}/c|\mathcal{E}^2\ \}$ (60);

We need to take every projections in the U, $V_{WU\ pe}$, $\mathcal{E}$ and B directions to get a feeling for the meaning of the energy stress tensor and see whether we get back the results of the previous subsections.

$T_{ij} = (\mathcal{E}^2\ \text{-}\ B^2)\delta_{ij}\ /2\ \ +\{\ -[(U_iU_j/\text{-}c^2) + \mathcal{E}\hat{}_i\mathcal{E}\hat{}_j]\mathcal{E}^2 + [V\hat{}_{pe\ I}\ V\hat{}_{pe\ j} + \mathcal{E}\hat{}_I\ \mathcal{E}\hat{}_j\ ]V^2_{pe}\ \mathcal{E}^2$

$+\ [\ (U_i\ /c)V\hat{}_{pe\ j} + (U_j/c)V\hat{}_{pe\ I}\ ]\,|V_{pe}\ /c|\mathcal{E}^2\ \}$ (61);

The B^ projections are the easiest since they involve only the delta function.

$T_{ij}\ B\hat{}_j = (1/2)[\mathcal{E}^2 - B^2]B\hat{}_I$ (62a); $T_{ij}\ B\hat{}_j\ B\hat{}_I = (1/2)[\mathcal{E}^2 - B^2\,]$ (62b) ;

$T_{ij}\ U_j/c = \{(1/2)[\mathcal{E}^2 - B^2]U_i/c + [-(U_i U_j/-c^2)(U_j/c)\mathcal{E}^2 + (U_j U_j/c^2)V\hat{}_{pe\ i}|V_{pe}|\ \mathcal{E}^2]\}$

$= (1/2)[\mathcal{E}^2 - B^2]U_i/c + [-\mathcal{E}^2 U_i/c - V\hat{}_{pe\ i}\ |V_{pe}/c^2|\mathcal{E}^2]$

$= [(1/2)\mathcal{E}^2 - \mathcal{E}^2]U_i/c + (-1/2)B^2 U_i/c - V\hat{}_{pe\ i}[|V_{pe}/c^2|\mathcal{E}^2]$

$= (-1/2)\mathcal{E}^2 U_i/c + (-1/2)B^2 U_i/c - V\hat{}_{pe\ i}[|V_{pe}/c^2|\mathcal{E}^2]$

$= (-1/2)[\mathcal{E}^2 + B^2]U_i/c - V\hat{}_{pe\ i}[|V_{WU\ pe}/c^2\ |\ \mathcal{E}^2]$ (63); we recognize the first term as the energy density in the direction $- U_i/c$ what about the second term?

Recall that $|\mathcal{E}x\ B|V\hat{}_{WU\ pe\ I}$ is $P_{oyn\ i}$, the Poynting vector (4dim not 3dim);

The final result is:

$T_{ij}U_j/c = (-1/2)[\mathcal{E}^2 + B^2]U_i/c - |\mathcal{E}xB|V\hat{}_{WU\ pe\ I}$ (64);

this is exactly the negative of eq51 which we called P_{em}.

$T_{ij}\ U_j/c = -P_{em\ I}$ (65); $T_{ij}(U_j U_i/c^2) = (1/2)[\mathcal{E}^2 + B^2]$ (66);

In eq66 should we use $U_j U_i/-c^2$ instead $U_j U_i/c^2$? Not clear at this time.

The projections in the $\mathcal{E}\hat{}$ directions are easier.

$T_{ij}\ \mathcal{E}\hat{}_j = (1/2)[\mathcal{E}^2 - B^2]\mathcal{E}\hat{}_i + [-\mathcal{E}^2\ \mathcal{E}\hat{}_I + B^2\ \mathcal{E}\hat{}_i]$;

$T_{ij}\ \mathcal{E}\hat{}_j = (-1/2)[\mathcal{E}^2 - B^2]\mathcal{E}\hat{}_i$ (67) ;

Eq 67 brings the field Lagrangian density into the -$\mathcal{E}\hat{}$ direction.

$T_{ij}\ \mathcal{E}\hat{}_j\ \mathcal{E}\hat{}_I = (-1/2)[\mathcal{E}^2 - B^2]$ (68);

$T_{ij}\ V\hat{}_{pe\ j} = (1/2)[\mathcal{E}^2 - B^2]V\hat{}_{pe\ i} + V\hat{}_{pe\ i}\ B^2 + (U_i/c)|\mathcal{E}xB|$;

$T_{ij}\ V\hat{}_{pe\ j} = (1/2)[\mathcal{E}^2 + B^2]V\hat{}_{pe\ i} + (U_i/c)|\mathcal{E}xB|$ (69) ;

$T_{ij}\ V\hat{}_{pe\ j}(U_i/c) = -|\mathcal{E}xB|$ (70) ;

Notice that we are not getting $|\mathcal{E}xB|V\hat{}_{pe}$ directly, from these projections.

Note also that eq69 equals W_{em} of eq 52 and that eq64 equals $-P_{em}$ they are perpendicular.

It is interesting to get the double angle velocities in terms of the single angle ones.

Let $V(2\theta_{WU\,pe})/c = |V''|V\^{}_{pe}$ and $V_{WU\,pe}/c = |V|V\^{}_{pe}$

$\tanh 2\theta = 2|V|/(1 + V^2) = |V''|$ (71a) ;

$\cosh 2\theta = (1 + V^2)/(1- V^2)= 1/(1- V''^2)^{1/2}$ (71b);

$\sinh 2\theta = 2|V|/(1- V^2) = |V''|/(1- V''^2)^{1/2}$ (71c) ;

$U(2\theta)/c = \{[2|V| /(1- V^2)] V\^{}_{pe} + [(1+ V^2)/(1- V^2)](U/c) \}$ (72) ;

$V\^{}(2\theta) = \{ [(1 + V^2)/(1- V^2)]V\^{}_{pe} + [2|V|/(1- V^2)](U/c) \}$ (73) ;

Compare with eq 48,49.

6 Introducing the 4 velocity U_o as a special observer to simplify formulas.

6A Obtaining U_o as a projection of U onto the W, U' plane. Various formulas.

It turns out that all the formulas derived in the previous sections simplify by projecting the 4 velocity U perpendicularly onto the W, U' plane. The projection gives us the direction of a new 4 velocity U_o and we have:

$U = [V_{U,Uo} + U_o]/(1- V^2_{U,Uo})^{1/2}$ (1);

We are interested in simplifying $e^{u'w}{}_u = (V_{U'U} – V_{WU})/[1- V_{U'U} .V_{WU}]$. It turns out that U_o will do the job but the derivation is somewhat intricate and the geometry a little difficult. The method is perfectly general and does not depend on e.m.We first separate the relative 4 velocities into components parallel (pa) and perpendicular (pp) to $e^{u'w}{}_u$.

$V_{WU} = [V_{WU\,pp} + V_{WU\,pa}]$ (2a); $V_{U'U} = [V_{U'U\,pp} + V_{U'U\,pa}]$ (2b) ;

$[V_{U'U} – V_{WU}] = [V_{U'U\ pa} – V_{WU\ pa}]$ (3a); $V_{U'U\ pp} = V_{WU\ pp}$ (3b);

Eq (3b) is very important.

$[1- V_{U'U}.V_{WU}] = [1- (V_{U'U\ pp} + V_{U'U\ pa}).(V_{WU\ pp} + V_{WU\ pa})]$

$= [1 – V_{U'U\ pp} V_{WU\ pp} – V_{U'U\ pa} V_{WU\ pa}] = [1 – V^2_{WU\ pp} – V_{U'U\ pa} V_{WU\ pa}]$

$= [1- V^2_{WU\ pp}][1 - V_{U'U\ pa} V_{WU\ pa} /(1- V^2_{WU\ pp})]$ (4) ;

So $e^{u'w}{}_u = (V_{U'U\ pa} - V_{WU\ pa})/\{(1- V^2_{WU\ pp})[1- V_{U'U\ pa} V_{WU\ pa}/(1- V^2_{WU\ pp})]\}$ (5) ;

This suggest that the expressions $V_{U'U\ pa} /(1- V^2_{WU\ pp})^{1/2}$ and $V_{WU\ pa}/(1- V^2_{WU\ pp})^{1/2}$ have special significance.

To get the connection with the projection that gave us U_o observe that

$[U+V_{WU}]= W(-c^2/W.U)$ (6a); $[U + V_{U'U}]= U'(-c^2/U'U)$ (6b);

This was discussed in the section on the Lienard-Wiechert potential.

(6b) – (6a) = $V_{U'U} – V_{WU} = V_{U'U\ pa} – V_{WU\ pa}$. This means that eq6a and 6b lie on a line in the U', W plane in the direction of $e^{u'w}{}_u$ .That line must intersect U_o at a point kU_o/c.

To find k note that $U/c + V_{WU\ pp} /c = kU_o/c$ (7); a little thought makes us realize that $V_{WU\ pp} = V_{Uo,U}$ (8a) ; $V_{U'U\ pp} = V_{Uo,U}$ (8b) ;

Since $U/c + V_{Uo,U} /c = U_o (-c^2/U_o .U)$ (9) ; $k = (-c^2/U.U_o)$ (10) ;

We must have: $U_0(-c^2/U_o .U) + V_{WU\ pa} = W(-c^2/W.U)$ (11) ;

$W.U/-c^2 = [1- V_{WUo} .V_{UUo}]/(1- V^2_{WUo})^{1/2} (1- V^2_{UUo})^{1/2}$ (12);

$V_{WUo} .V_{UUo} = 0$ (13); don't confuse $V_{Uo,U}$ with $V_{U,Uo}$ They both lie in the same plane which is perpendicular to the plane wherein V_{WUo} lies.

$W.U/-c^2 = (1- V^2_{WUo})^{-1/2} (1- V^2_{UUo})^{-1/2}$ (14); eq11 becomes

$U_o (1- V^2_{Uo,U})^{1/2} + V_{WU\ pa} = W(1- V^2_{Uo,U})^{1/2} (1-V^2_{WUo})^{1/2}$ (14a); the right hand term is

$=[V_{WUo} + U_o](1- V^2_{WUo})^{1/2} (1- V^2_{Uo,U})^{1/2} /(1- V^2_{WUo})^{1/2} = [V_{WUo} + U_o](1- V^2_{Uo,U})^{1/2}$ (15);

Putting eq15 into eq 14a. $V_{WU\ pa} = V_{WUo} (1- V^2_{Uo,U})^{1/2}$ (16);

$V_{WUo} = V_{WU\ pa} / (1- V^2_{Uo,U})^{1/2}$ (17); in the same way,we get:

$V_{U'Uo} = V_{U'U\ pa} /(1- V^2_{Uo,U})^{1/2}$ (18);

which is what we guessed earlier.see eq5,8a,8b.

Eq5 becomes: $e^{u'w}{}_{u} = \{(V_{U',Uo} – V_{WUo})/[1- V_{U'Uo} V_{WUo}]\}/(1- V^2_{Uo,U})^{1/2}$ (19) ;

We recognize eq19 as representing: $e^{u'w}{}_{u} = e^{u'w}{}_{uo} /(1- V^2_{Uo,U})^{1/2}$ (20);

With $e^{u'w}{}_{uo} = (V_{U'Uo} – V_{WUo})/[1 – V_{U'Uo} V_{WUo}]$ (21); and

$1/(1 – V^2_{Uo,U})^{1/2} = 1/(1- V^2_{WU\ pp})^{1/2} = 1/(1 – V^2_{U'U\ pp})^{1/2}$ (22) ;

We also have : $V\hat{}_{U'Uo} = V\hat{}_{WUo} = e^{u'w}{}_{u}\hat{} = e^{u'w}{}_{uo}\hat{}$ (23);

The 4 directions are the same. The angle between U and U_o is :

$\Theta_{Uo,U} = \theta_{WU\ ppe} = \theta_{U'U\ ppe}$ (24); where we have put back ppe instead of pp to remind us that we are dealing with the mysterious angle of section 5 involving the Poynting vector, the energy density, the electric field 4 vector and the magnetic field pseudo 4 vector. It is simply the angle between U and U_o which is also the angle between U and the W, U' plane.

Eq 21 can be rewritten as :

$(e^{u'w}{}_{uo}) = (\tanh\theta_{U'Uo} - \tanh\theta_{WUo})V\hat{}_{WUo} /[1- \tanh\theta_{U'Uo} \tanh\theta_{WUo}]$

$= \tanh(\theta_{U'Uo} – \theta_{WUo})V\hat{}_{WUo}$ (25a);

Also $e^{w,u'}{}_{uo} = (\tanh\theta_{WUo} -\tanh\theta_{U'Uo})V\hat{}_{WUo} /[1- \tanh\theta_{WUo} \tanh\theta_{U'Uo}]$

$= \tanh(\theta_{WUo} - \theta_{U'Uo})V\hat{}_{WUo}$ (25b);

we have kept the direction the same for eq 25a and 25b so it is the angle differences which are the negative of each other.

Eq 20 can be rewritten as: $e^{u'w}{}_{u} = \cosh\theta_{Uo,U} \tanh(\theta_{U'Uo} - \theta_{WUo})V\hat{}_{WUo}$ (26);

$U'.W/-c^2 = [1- V_{U'Uo} V_{WUo}]/(1- V^2{}_{U'Uo})^{1/2} (1- V^2{}_{WUo})^{1/2}$

$= \cosh\theta_{U'Uo} \cosh\theta_{WUo} - \sinh\theta_{U'Uo} \sinh\theta_{WUo} = \cosh(\theta_{U'Uo} - \theta_{WUo})$ (27);

$(U'.W/-c^2)\, e^{u'w}{}_{uo} = \sinh(\theta_{U'Uo} - \theta_{WUo})V\hat{}_{WUo}$ (28);

What does the geodesic line segment look like as viewed from U_o ?

$V_{U'W} = [e^{u'w}{}_{uo} + (e^{u'w}{}_{uo} .W)W](1- V^2{}_{WUo})^{1/2}$ (29);

$V_{U'W} = \tanh(\theta_{U'Uo} - \theta_{WUo})[V\hat{}_{U'Uo} + (V\hat{}_{U'Uo}.W)W]/\cosh\theta_{WUo}$ (30);

this is the appearance of the geodesic segment from W to U' as viewed from U_o .

The geodesic bivector gives a simple result.

$[W x V_{U'W}] = (U_o x\, e^{u'w}{}_{uo}) + (V_{WUo} x\, e^{u'w}{}_{uo})$ (31);

the 2nd term is zero because both terms are in the V_{WUo} direction.

$[W x V_{U'W}] = (U_o x\, e^{u'w}{}_{uo}) = (U_o x V\hat{}_{WUo}) \tanh(\theta_{U'Uo} - \theta_{WUo})$ (32a);

$[U' x V_{WU'}] = (U_o x\, e^{wu'}{}_{uo}) = (U_0 x V\hat{}_{WU0}) \tanh(\theta_{WUo} - \theta_{WUo})$ (32b);

Again we have kept the same direction $V\hat{}_{WUo}$ in eq 32a and 32b and let the angles give the negative sign.The reader may wonder under what conditions do the angles add instead of subtract.We have seen so far expressions like:

$U'.W/-c^2 = [1- V_{U'U}.V_{WU}]/(1- V^2{}_{U'U})^{1/2} (1- V^2{}_{WU})^{1/2}$; with

$U' = [V_{U'U} + U]/(1- V^2{}_{U'U})^{1/2}$; $W= [V_{WU} + U]/(1- V^2{}_{WU})^{1/2}$;

Take $U'_R = [-V_{U'U} + U]/(1- V^2_{U',U})^{1/2}$ (33a); $V_{U'R,U} = -V_{U'U}$ (33b);

$U'_R .W/-c^2 = [1 + V_{U'U} .V_{WU}/c^2]/(1- V^2_{U'U} /c^2)^{1/2} (1- V^2_{WU})^{1/2}$ (34) ;

This is a projective form of Einstein law of addition of velocities (see V.Fock). We dealt so far with Einstein's law of substraction of velocities.Note that

V_{WU} reduces to [w,0] when u=0 and $-V_{U'U}$ reduces to [–u',0] when u=0, so the interpretation is reasonable. As a second example take:

$U'_{R*} = [-V_{U'Uo} + U_o]/(1- V^2_{U'Uo})^{1/2}$ (35a); $W= [V_{WUo} + U_o]/(1- V^2_{WUo})^{1/2}$ (35b);

$U'_{R*} .W/-c^2 = [1 + V_{U'Uo} V_{WUo} /c^2]/(1- V^2_{U'Uo} /c^2)^{1/2} (1- V^2_{WUo})^{1/2}$ (36a);

$U'_{R*} .W/-c^2 = \cosh(\theta_{U'Uo} + \theta_{WUo})$ (36b) ; $V_{UR*,Uo} = -V_{U'Uo}$ (36c);

$e^{u'r,w}{}_{u} = [V_{UR,U} - V_{WU}]/[1- V_{U'R,U} .V_{WU}]$ (37a);

$e^{u'r,w}{}_{u} = - [V_{U'U} + V_{WU}]/[1 + V_{U'U}.V_{WU}]$ (37b);

$e^{u'r*,w}{}_{uo} = [V_{U'R*,Uo} – V_{WUo}]/[1- V_{U'R*,Uo} .V_{WUo}]$ (38a);

$e^{u'r*,w}{}_{uo} = - [V_{U'Uo} + V_{WUo}]/[1 + V_{U'Uo} V_{WUo}]$ (38b);

$e^{u'r*,w}{}_{uo} = - \tanh(\theta_{U'Uo} + \theta_{WUo})V\hat{}_{WUo}$ (38c);

6B The LW potential as viewed from W, U, U_o .

$A = q'(U'/c) /(E.U'/-c) = (q'/l)[V_{U'W} + W]/c$ (39a); this was obtained in an earlier section and gives the potential as viewed from W. As viewed from U we have

$A = (q'/l)\{[V_{U'U} + U]\cosh\theta_{U'U} /c \cosh\theta_{U'W} \}$ (39b) ;

compared to eq39a , the form is the essentially same but there is the factor

$\cosh\theta_{U'U}/\cosh\theta_{U'W}$. As viewed from U_o we have:

$A = (q'/l)[V_{U'Uo} + U_o]\cosh\theta_{U'Uo} /\cosh(\theta_{U'Uo} – \theta_{WUo})$ (39c) ; we used eq6.27 here.

It is sometimes useful to absorb the inverse of the factors in l in eq 39a and b and pretend that the length is modified by “conformal” factors used to keep the form the same.

6C The geodesic bivector as viewed from U_o .Quadratic expressions.T_{ij} projections

From eq6.32a, $[WxV_{U'W}]_{ij} = [U_o \times e^{u'w}{}_{uo}]_{ij} = S^*{}_{ij}$;

$(-1/2)S^*_{ij} S^*_{ij} = V^2_{U'W}/c^2 = e^{2\ U'W}{}_{Uo} = \tanh^2(\theta_{U'Uo} - \theta_{WUo}) = \tanh^2 \theta_{U'W}$ (40);

$\tanh\theta_{U'W} = \tanh(\theta_{U'Uo} - \theta_{WUo})$ (41); $\theta_{U'W} = \theta_{U'Uo} - \theta_{WUo}$ (42);

$|V_{U'W}/c| = |\ e^{u'w}{}_{uo}\ |$ (43); $|\ e^{u'w}{}_{u}\ | = \cosh\theta_{UUo}\ |\ e^{u'w}{}_{uo}\ |$ (44); this surprising result tells us there is no magnetic field term,as viewed from U_o because the problem is 2 dimensional just as it is with the W, U’plane so it is in the U_o ,V_{WUo} plane,which are the same plane. We can now compute the quadratic eqs for the fields.

$(-1/2)F_{ij}F_{ij} = [q'^2/(E.U'/-c)^4]V^2_{U'W}/c^2 = [q'^2/(E.U'/-c)^4]\tanh^2(\theta_{U'Uo} - \theta_{WUo})$ (45);

it is also equal to $\mathcal{E}^2 - B^2$ as viewed from U, of course,and we have 4 mutually orthogonal direction in that case. In this case only 2 directions are involved.

Since $T= [q'^2/(E.U'/-c)^4]\tanh^2(\theta_{U'Uo} - \theta_{WUo})$ (45a); (we used eq 4.73) .

$F_{ij}F_{kj} = T[U_o \times V^\wedge{}_{WUo}]_{ij} [U_o \times V^\wedge{}_{WUo}]_{kj} = -T[(U^i{}_o U^k{}_o /-c^2 + V^\wedge{}_{WUo}{}^i V^\wedge{}_{WUo}{}^k]$ (46);

$2T_{ij} = T\{\delta_{ij} -2[(U^i{}_o U^j{}_o/-c^2) + V^\wedge{}_{WUo}{}^i V^\wedge{}_{WUo}{}^j]\}$ (47);

$2T_{ij}U^j{}_o/c = - TU^i{}_o/c$ (48a); $T_{ij}U^j{}_o /c = (-1/2)TU^i{}_o/c$ (48b);

$2T_{ij} U^j{}_o U^i{}_o/c^2 = - TU^i{}_o U^i{}_o/c^2 = T$ (49a); $T_{ij}U^j{}_o U^i{}_o/c^2 = (1/2) T$ (49b);

$2T_{ij}V^\wedge{}_{WUo}{}^j = -TV^\wedge{}_{WUo}{}^i$ (50a) ; $T_{ij}V^\wedge{}_{WUo}{}^j = (-1/2)TV^\wedge{}_{WUo}{}^i$ (50b);

$2T_{ij}V^\wedge{}_{WUo}{}^j V^\wedge{}_{WUo}{}^i = -T$ (51a); $T_{ij}V^\wedge{}_{WUo}{}^j V^\wedge{}_{WUo}{}^i = (-1/2)T$ (51b);

$2T_{ij}V^\wedge{}_{WUo}{}^j U^i{}_o/c = 0$ (52); no mixed components

The projection of $2T_{ij} = T\{\delta_{ij} - 2[W^iW^j/-c^2 + V\^_{U'W}{}^i V\^_{U'W}{}^j]\}$ of section4, eqs4.69 and 4.74 have exactly the same form and give similar results, the problem is similar and involves only the geodesic between W and U'. No magnetic term appears.

6D The Lorentz force

We already know that the Lorentz force will involve the projection of the geodesic acceleration between the points described by the initial normal W and the final normal U'. The Lorentz force is the projection in the U direction. Writing everything in terms of U_o , V_{WUo} , V_{UUo} we get:

Using $K = [q'/(E.U'/-c)^2] \tanh(\theta_{U'Uo} - \theta_{WUo})$ (53);

$F_{ij}U_j/c = K [U_o \times V\^_{WUo}]_{ij} [V_{UUo} + U_o]_j /(1- V^2_{UUo})^{1/2}$ (54);

$= K[U_o \times V\^_{WUo}]_{ij} V\^_{UUo\, j} \sinh\theta_{UUo} + K[U_o \times V\^_{WUo}]_{ij} U_{o\, j} \cosh\theta_{UUo}$ (55);

the first term is =0 because $[U_o \times V\^_{WUo}]_{ij}V\^_{UUo\, j} = U_o{}^i (V\^_{WUo} .V\^_{UUo}) = 0$ (56a); since $V\^_{WUo}$ is orthogonal to V_{UUo}. The second term

$K[U_o \times V\^_{WUo}]_{ij} U_{o\, j} \cosh\theta_{UUo} = KV\^_{WUo\, i} \cosh\theta_{UUo}$ (56b);

$qF_{ij}U_j = [qq'/(E.U'/-c)^2]\tanh(\theta_{U'Uo} - \theta_{WUo})\cosh\theta_{UUo} V\^_{WUo}$ (57);

The last result is consistent with the previous results.

6E Quadratic expressions involving U_o , V_{UUo} , $V\^_{WUo}$, $b\^_i = (1/2)\varepsilon^{ijkl} U_{oj} V\^_{UUo\, k} V\^_{WUo\, l}$ .The energy stress tensor T''_{ik} and its projections.

The previous subsection leaves us with a puzzling problem,namely, how do we construct quantities involving the four mutually orthogonal directions U_o , $V\^_{UUo}$ $V\^_{WUo}$, $b\^$, and how to get expressions resembling the electric and magnetic field. Try $D_{ij} = [(U_o + V_{UUo})\times V\^_{WUo}]_{ij}$ (58); D_{ij} is made up of 3 orthogonal 4 vectors.

$D_{ij} \cosh\theta_{UUo} |V_{U'W}| = [U \times V\^_{WUo}]_{ij}|V_{U'W}|$ (59) ;

The 4 vector $|V_{U'W}|V\^_{WUo}$ lies in the U, $V\^_{WUo}$ plane and represents a new relative 4 velocity . To that new relative 4 velocity corresponds a new 4 velocity.

$U''/c = [\,|V_{U'W}|V\hat{}_{WUo} + U/c]/(1- V^2_{U'W}/c^2)^{1/2}$ (60);

$V\hat{}_{U''U} = V\hat{}_{WUo}$ (61); $U'' = [\,V_{U''U} + U\,]/(1- V^2_{U''U}\,)^{1/2}$ (62);

$V_{U''U} = |V_{U'W}|V\hat{}_{WUo}$ (63); $e\hat{}^{u'w}{}_{u} = e\hat{}^{u'w}{}_{uo} = V\hat{}_{U'Uo} = V\hat{}_{WUo} = V\hat{}_{U''U}$ (64);

Eq 64 shows the equivalences getting more and more crowded.

We want to form $T^{*\prime}_{ik} = \{(-1/2)(D_{lm}D_{lm})\delta_{ik}/2 + D_{ij}D_{kj}\}V^2_{U'W}/c^2$ (65);

$\{[U_o \times V\hat{}_{WUo}]_{ij} + [V_{UUo} \times V\hat{}_{WUo}\,]_{ij}\}\{[U_o \times V\hat{}_{WUo}\,]_{kj} + [V_{UUo} \times V\hat{}_{WUo}\,]_{kj}\}$

$= \{-\,[(U_o^i\,U_o^k/-c^2) + V\hat{}_{WUo}{}^i\,V\hat{}_{WUo}{}^k\,] + [\,V\hat{}_{UUo}{}^i\,V\hat{}_{UUo}{}^k + V\hat{}_{WUo}{}^i\,V\hat{}_{WUo}{}^k\,]V^2_{UUo}/c^2$

$+ [(U_o^i/c)V\hat{}_{UUo}{}^k + (U_o^k/c)V\hat{}_{UUo}{}^i\,]\,|V_{UUo}/c|\} = D_{ij}D_{kj}$ (66);

Setting i=k we get: $(-1/2)D_{lm}D_{lm} = [1 - V^2_{UUo}/c^2] = 1/\cosh^2\theta_{UUo}$ (67);

So we do need the $\cosh\theta_{UUo}$ of eq 59 in the definition of T'_{ik}. Define:

$T^{*\prime\prime}_{ik} = \{(-1/2)(D_{lm}D_{lm})\delta_{ik}/2 + D_{ij}D_{kj}\}\cosh^2\theta_{UUo}\,V^2_{U'W}/c^2\}$ (68); the term:

$-[(U_o^i\,U_o^k/-c^2) + V\hat{}_{WUo}{}^i\,V\hat{}_{WUo}{}^k\,]\cosh^2\theta_{UUo}\,V^2_{U'W}/c^2$ (69a); is the real electric field squared when multiplied by $q'^2/(E.U'/-c)^4 = K'$ times a P_{ik}. The term:

$\{[V\hat{}_{UUo}{}^i\,V\hat{}_{UUo}{}^k + V\hat{}_{WUo}{}^i V\hat{}_{WUo}{}^k\,]\tanh^2\theta_{UUo}\,\cosh^2\theta_{UUo}\,V^2_{U'W}/c^2$ (69b); is the real magnetic field term squared when multiplied by K' times another P_{ik}. The term:

$\{[(U_o^i/c)V\hat{}_{UUo}{}^k + (U_o^k/c)V\hat{}_{UUo}{}^i\,]\tanh\theta_{UUo}\,\cosh^2\theta_{UUo}\,V^2_{U'W}/c^2$ (69c); is the absolute value of the real Poynting 4 vector when multiplied by K' the operator described in eq69c.

Let $T''_{ik} = K'T^{*\prime\prime}_{ik}$ (70); be the full energy stress tensor.

We are now ready to find the projections using eqs69a,69b,69c and 67, and the fact that $K'[(-1/2)D_{lm}D_{lm}\cosh^2\theta_{UUo}V^2_{U'W}/c^2\,]\delta_{ik}/2 = [\mathcal{E}^2 - B^2]\delta_{ik}/2$

$T''_{ik}U_o^k/c = (-1/2)[\mathcal{E}^2 + B^2]U_o^i/c - \mathcal{E}BV\hat{}_{UUo}{}^i$ (70a); using eq69a,69c

$T''_{ik} U_o^k/c = (-1/2)[\mathcal{E}^2 + B^2]U_o^i/c - |\mathcal{E}x B|V\hat{}_{UUo}^i$ (70b);

$T''_{ik}U_o^kU_o^i/c^2 = (1/2)[\mathcal{E}^2 + B^2]$ (70c);

$T''_{ik} (U_o^k/c)V\hat{}^i_{UUo} = - |\mathcal{E}xB|$ (70d);

$T''_{ik} V\hat{}_{UUo}^k = (1/2)[\mathcal{E}^2 - B^2]V\hat{}_{UUo}^i + B^2 V\hat{}_{UUo}^i + \mathcal{E}BU_o^i/c$ (71a); using 69b,69c

$T''_{ik} V\hat{}_{UUo}^k = (1/2)[\mathcal{E}^2 + B^2]V\hat{}_{UUo}^i + |\mathcal{E}xB|U_o^i/c$ (71b);

$T''_{ik} V\hat{}_{UUo}^kV\hat{}_{UUo}^i = (1/2)[\mathcal{E}^2 + B^2]$ (71c);

$T''_{ik} V\hat{}_{UUo}^k U_o^i/c = - |\mathcal{E}xB|$ (71d); same as eq70d of course.

$T''_{ik} V\hat{}_{WUo}^k = (1/2)[\mathcal{E}^2 - B^2]V\hat{}_{WUo}^i + B^2 V\hat{}_{WUo}^i - \mathcal{E}^2 V\hat{}_{WUo}^i$ (72a);

$T''_{ik} V\hat{}_{WUo}^k = (-1/2)[\mathcal{E}^2 - B^2]V\hat{}_{WUo}^i$ (72b); using eqs69b,69a in (72a).

$T''_{ik} V\hat{}_{WUo}^k V\hat{}_{WUo}^i = (-1/2)[\mathcal{E}^2 - B^2]$ (72c);

We also have the two obvious projections:

$T''_{ik} B\hat{}_{uo}^k = (1/2)[\mathcal{E}^2 - B^2]B\hat{}_{Uo}^i$ (73a);

$T''_{ik} B\hat{}_{Uo}^K B\hat{}_{Uo}^i = (1/2)[\mathcal{E}^2 - B^2]$ (73b);

where $B\hat{}_{Uo}^i = (1/2)\mathcal{E}^{ijkl} (U_o^j/c)V\hat{}_{UUo}^k V\hat{}_{WUo}^l$ (74a);

To differentiate it from $B\hat{}_U^i = (1/2)\mathcal{E}^{ijkl} (U^j/c)V\hat{}_{UoU}^k V\hat{}_{WUo}^l$ (74b);

A comparison between the present projections and those of the

ordinary energy stress tensor T_{ik} of section5F shows that they are the same with U replaced by U_o, $V\hat{}_{UoU}$ is replaced by $V\hat{}_{UUo}$, $V\hat{}_{WUo}$ is the same.

$B\hat{}_U$ is replaced by $B\hat{}_{Uo}$. Moreover the electric and magnetic fields have

The same magnitude in both cases. $\mathcal{E}^{\wedge} = V\hat{}_{WUo}$ so it has the same direction

In both cases.

6F Total e.m. momentum density and its dual, e.m mass density, revisited.

In section 5, we defined an electromagnetic momentum density 4 vector:

$P_{em} = (1/2)\{[Ɛ^2 + B^2]U/c +2|ƐxB|V\wedge_{UoU} \}$ (75a); and its "dual"

$W_{em} = (1/2)\{[Ɛ^2 + B^2]V\wedge_{UoU} +2|ƐxB|U/c \}$ (75b);

We want to define a em mass density such that

$P_{em} .P_{em} = -M_{em}^2 c^4/V_3^2$ (76a); $W_{em} .W_{em} = M_{em}^2 c^4/V_3^2$ (76b);

$[Ɛ^2 + B^2] = (q'^2/l^4)[\cosh 2\theta_{UoU} (V^2_{U'W}/c^2)(1- V^2_{U'W} /c^2)^2]$ (77a);

$2|ƐxB| = (q'^2/l^4)[\sinh 2\theta_{UoU} (V^2_{U'W}/c^2)(1- V^2_{U'W}/c^2)^2]$ (77b);

$W_{em .} W_{em} = (1/4)(q'^2/l^4)^2[\cosh^2 2\theta_{UoU} - \sinh^2 2\theta_{UoU}]\{V^2_{U'W} (1- V^2_{U'W})^2 \}^2$

$= (1/4)T^2 = M_{em}^2 c^4/V_3^2$ (78 a); $T/2 = M_{em} c^2/V_3$ (78b);

$T = 2M_{em} c^2/V_3 = (q'^2/l^4)V^2_{U'W} (1- V^2_{U'W})^2$ (79);

We don't have much choice here if we want to specify V_3 . If we are willing to believe that $V_3 = l^3$ and $q'^2/2l$ which has the right units is also to be chosen then:

$M_{em} c^2/l^3 = T/2$ (80); $M_{em} c^2 = (T/2)l^3 = (q'^2/2l)V^2_{U'W} (1- V^2_{U'W})$ (81); write:

$M_{em} c^2/l^3 = \rho c^2 = (q'^2/2l^4)V^2_{U'W} (1- V^2_{U'W})^2 = T/2$ (82) ;

The desired em mass density is ρ . We can also have :

$P''_{em} = (1/2)\{[Ɛ^2 + B^2]U_o/c + 2|ƐxB|V\wedge_{UUo} \}$ (83a);

$W''_{em} = (1/2)\{[Ɛ^2 + B^2]V\wedge_{UUo} +2|ƐxB|U_o/c \}$ (83b);

Fortunately the previous derivations show that

$T'' =T$ (84a); $M''_{em} = M_{em}$ (84b) ; $\rho'' = \rho$ (84c) ; $V''_3 = V_3 = l^3$ (84d);

We can write:

$2T_{ij} = \rho c^2\{\delta_{ij} -2[W^iW^j/-c^2 + V\wedge_{U'W}^i V\wedge_{U'W}^j]\}$ (85a);

$2T''_{ij} = \rho c^2 \{\delta_{ij} – 2[U_o^i U_o^j /-c^2 + V\wedge_{WUo}^i V\wedge_{WUo}^j]\}$ (85b);

It is worthy of note that that the $2P_{ij}$ terms in eqs85a,85b can be written

In terms of null vectors.

Let $V\^ = V\^_{U'W}$;

$(W^i/c + V\^^l)(W^k/c + V\^^k) = [W^i W^k/c^2 + V\^^l V\^^k] + [(W^i/c) V\^^k + (W^k/c) V\^^l]$ (86);

We want to eliminate the second term and get the negative of the first.

$[W^i/c - V\^^i][W^k/c + V\^^k] = [W^i W^k/c^2 - V\^^i V\^^k] + [(W^i/c)V\^^k - (W^k/c)V\^^i]$ (87);

$[-W^i/c - V\^^i][-W^k/c + V\^^k] = [W^i W^k/c^2 - V\^^i V\^^k] + [(-W^i/c)V\^^k + (W^k/c)V\^^l]$ (88);

Adding eqs87 and 88 we get:

$[W^i/c - V\^^i][W^k/c + V\^^k] + [-W^i/c - V\^^i][-W^k/c + V\^^k] = -2[W^i W^k/-c^2 + V\^^i V\^^k]$ (89);

$[W^i/c - V\^^i][W^k/c + V\^^k] + [W^i/c + V\^^i][W^k/c - V\^^k] = -2[W^i W^k/-c^2 + V\^^i V\^^k]$ (90);

This is the desired result.

Let $N^{++}_{U'W} = [W/c + V\^_{U'W}]$ (91a); $N^{+-}_{U'W} = [W/c - V\^_{U'W}]$ (91b);

$2T_{ij} = \rho c^2[\delta_{ij} + N^{i++}_{U'W} N^{j+-}_{U'W} + N^{j++}_{U'W} N^{i+-}_{U'W}]$ (92);

$N^{++}_{WUo} = [U_o/c + V\^_{WUo}]$ (93a); $N^{+-}_{WUo} = [U_o/c - V\^_{WUo}]$ (93b);

$2T''_{ij} = \rho c^2 [\delta_{ij} + N^{i++}_{WUo} N^{j+-}_{WUo} + N^{j++}_{WUo} N^{i+-}_{WUo}]$ (94);

7 Foliation of hyperboloids of one sheet in Minkowski spacetime.Virtual timelike hyperbolic trajectories.

In the previous sections it was shown that the nonacceleration field represents the acceleration term of a geodesic in the tangent space of a hyperboloid of two sheets belonging to a foliation of hyperboloids of 2 sheets in Minkowski space time. A virtual motion, from one normal to another, traced a geodesic on a

hyperboloid of two sheets .It also traced a geodesic in the tangent space of either normal. Those are geodesics of the 3dim Klein Ball model of hyperbolic geometry in velocity space. The purpose of this section is to show that these geodesics have a counterpart in a foliation of Minkowski spacetime by hyperboloids of one sheet. Two virtual timelike hyperbolic trajectories, obtained using an inversion of the field, are connected by the interval of propagation or a projection of it. This gives a different way to interpret the way curvature is transmitted from the source charge to the field point or the test charge.

7A Preliminaries. Review of timelike hyperbolic trajectories with constant acceleration.

We have: $u(t)/(1- u^2/c^2)^{1/2} = gt$ (1) ; $g = d\,[dr(t)/d\tau]/dt$ (2);

τ is the proper time. g is a constant of the motion. The motion is rectilinear.Let the initial time $t_i = 0$; the initial 3 velocity $u_i = 0$;

$g= [u(t)/t]/(1- u^2/c^2)^{1/2}$ (3); for all t's including the initial time.We will be mostly interested in the various quantities evaluated at the final time t_f .In particular

$g = u_f\,(t_f)/(1- u_f^2/c^2)^{1/2}$ (4); from eq1, it is easy to see that :

$[1/(1- u^2)^{1/2}\,] = [1+ (gt/c)^2\,]^{1/2}$ (5); $u(t) = gt/[1+ (gt/c)^2\,]^{1/2}$ (6) ;

Let $a = du/dt = d^2r/dt^2$ be the 3 acceleration. From eq6 we get:

$a(t) = gt/[1+ (gt/c)^2]^{3/2}$ (7); the initial acceleration a_i =0.

We will be interested in the final acceleration $a_f = gt_f/[1+ (gt_f/c)^2]^{3/2}$ (7a);

We can express a in terms of u. $a(t) = (u(t)/t)[1- u^2/c^2]$ (8);

Eq8 is the important formula. We will be mostly interested in

$a(t_f\,) = (u_f/t_f)[1- u_f^2/c^2]$ (8a);

since $u= dx/dt$ (writing x(t) instead of r(t)).

$dx(t)/dt = gt/[1+ (gt/c)^2]^{1/2}$. $x(t) = (c^2/g)\{[1 + (gt/c)^2]^{1/2} -1\}$ (9); eq9 implies that the initial position x_i =0;

from eq9 we have: $[1 + (g/c^2)x(t)] = [1 + (gt/c)^2\,]^{1/2}$ (10);

$[1 + (g/c^2)x]^2 = [1 + (gt/c)^2]$ (10a); $[1 + (g/c^2)x]^2 - (gt/c)^2 = 1$ (10b) ;

$\cosh\theta = |[1 + (g/c^2)x]|$ (11a) ; $\sinh\theta = [gt/c]$ (11b) ;

we can also have : $x^2 - (ct)^2 = (c^2/g)^2$ (12a);

$(gx/c^2)^2 - (gt/c)^2 = 1$ (12b); $(gx/c^2)^2 = [1 + (gt/c)^2]$ (12c);

$[gx/c^2] = [1 + (gt/c)^2]^{1/2}$ (12d); eqs12a to 12d imply $x_i = c^2/g$.

We can also set , using eq12b. $gx/c^2 = \cosh\theta'$ (13a) ; $(gt/c) = \sinh\theta'$ (13b);

Since $\cosh\theta'$ > or = to 1, x > or = c^2/g which is consistent. We also have:

$x = (c^2/g)\cosh\theta'$ (13c); $ct = (c^2/g)\sinh\theta'$ (13d);

7B Identifying the elements of the trajectories.

$F_{ij} = (q'/l^2)[(W/c)x\, V_{U'W}/c]_{ij}\,(1 - V^2_{U'W}/c^2) = q'[(W/lc)x\,(V_{U'W}/lc)](1 - V^2_{U'W}/c^2]ij$ (14a);

Similarly : $F_{ij} = -q'[(U'/lc)x\, V_{WU'}/lc]_{ij}(1 - V^2_{WU'}/c^2)$ (14b);

We recognize the expressions as familiar from the previous subsection.

Let $\alpha_W/c^2 = (V_{U'W}/lc)(1 - V^2_{U'W}/c^2)$ (15a);

And $\alpha_{u'}/c^2 = (V_{WU'}/lc)(1 - V^2_{WU'}/c^2)$ (15b);

Compare eqs 15a and 15b with eq8a for a_f i.e. $a_f/c^2 = (u_f/ct_f)(1 - u_f^2/c^2)$ (15c);

We identify $ct_f = l$ (16a); $u_f = V_{U'W}$ (16b); in eq15a. $u_f = V_{WU'}$ (16c); in eq 15b.

And $a_f = \alpha_W$ (17a) ; in eq15a. $a_f = \alpha_{u'}$ (17b); in eq15b.

We also have $u_i = V_{WW} = 0$ (18a); in eq 15a. $u_i = V_{U'U'} = 0$ (18b); in eq15b.

To get $a_i = 0$ simply put V_{WW} in eq15a and $V_{U'U'}$ in eq 15b. Therefore:

$(\alpha_{wi}) = (\alpha_{u'i}) = 0$ (18c); note that the two α 's have the same magnitudes but different directions. We have identified a timelike hyperbolic motion in projective form hidden within the nonacceleration field.

$ct_f = l$; $t_f = l/c$ (19); finally

$g/c^2 = (V_{U'W}/lc)/(1- V^2_{U'W}/c^2)^{1/2} = \sinh\theta_{U'W} V\hat{}_{U'W} /l$ (20a);

or $g/c^2 = (V_{WU'}/lc)/(1- V^2_{WU'}/c^2)^{1/2} = \sinh\theta_{WU'} V\hat{}_{WU'} /l$ (20b);

7C Constructing the timelike hyperbolic motions of the foliation.

Eqs 12c,13a,b which describe hyperbolic motion involve c^2/g .We will use a Rindler-like coordinate system. (refs8,9)

We want timelike hyperbolic trajectories, which are part of a foliation, with the following properties.The virtual motion from W to U' or from U' to W will be described on the hyperboloid of one sheet as initial and final unit tangents during the virtual motion.The tangents of the hyperboloid of two sheets become the normals of the hyperboloid of one sheet. The hyperbolic angle traversed will be the same for both types of hyperboloid.To find the radius of each of the two timelike hyperbolic motions ,which are two geodesic segments of two hyperboloids of one sheet ,is not obvious and requires subtle maneuvering. Another requirement is that we should have the interval of propagation E connect one hyperboloid at 4 position R'(τ') of charge q' to another hyperboloid at position R of the field point or R(τ) of a test charge q.This is possible in the rest frame of q'.In general,it is not possible.

Since the field only involves the starting and ending points of each entities and not the details of the virtual motion we will write θ_f instead of $\theta_{U'W}$ or $\theta_{WU'}$.

$F_{ij} = q'[(W/lc)xV\hat{}_{U'W} \tanh\theta_f/l\cosh^2 \theta_f]_{ij}$ (21a);

$-F_{ij} = q'[(U'/lc)x V\hat{}_{WU'} \tanh\theta_f /l\cosh^2 \theta_f]_{ij}$ (21b);

We have only distributed the 1/l term among the two directions ,one timelike and the other spacelike in each eq. It turns out that the physical meaning of the trajectories as viewed from U', the rest frame of q', is much easier to understand than those in the rest frame of the influence,i.e. W. Eq21b is the best to start with. We begin with inversion of F_{ij} .Notice the minus sign.It is easier to work with.We invert without changing the directions.

$-F_{ij}^{inversion} = (1/q')[(lU'/c)x\ V\hat{}_{WU'}\ l\cosh^2\theta_f /\tanh\theta_f]_{ij}$ (22a);

$F_{ij}^{inversion} = (1/q')[(lW/c)x\ V\hat{}_{U'W}\ l\cosh^2\theta_f /\tanh\theta_f]_{ij}$ (22b);

$-F_{ij}^{inv} = -F_{ij}/[(-1/2)F_{mn}F_{mn}]$ (23a); $F_{ij}^{inv} = F_{ij}/[(-1/2)F_{mn}F_{mn}]$ (23b);

We could use eqs22a,b to obtain a hyperbolic trajectory which would have a radius of $l\cosh^2\theta_f/\tanh\theta_f$ with initial tangent U'/c or W/c and final tangent W/c or U'/c respectively and with the initial unit normal $V\hat{}_{WU'}$ or $V\hat{}_{U'W}$ respectively.The problem with that is we want two hyperbolic tractories,one for the origin of the influence at q', the other for the field point or the test charge receiving the influence.The only exception is in the case of self effects,in that case only one trajectory is needed in general because one part of the trajectory influences the other part via the ray of influence.We also want the times [E.U'/-c] i.e. c(t-t') to be somehow involved in the duration of the hyperbolic motions.

Try to distribute the two terms in a different way namely:

$F_{ij} = q'[(W/cl\cosh\theta_f)x\ V\hat{}_{U'W}\ (\tanh\theta_f/l\cosh\theta_f)]_{ij}$ (24a);

$-F_{ij} = q'[(U'/cl\cosh\theta_f)x\ V\hat{}_{WU'}(\tanh\theta_f/l\cosh\theta_f]_{ij}$ (24b);

$-F_{ij}^{inv} = (1/q')[l\cosh\theta_f\ U'/c\ x\ (l\cosh\theta_f/\tanh\theta_f)V\hat{}_{WU'}]_{ij}$ (25a);

$F_{ij}^{inv} = (1/q')[l\cosh\theta_f\ W/c\ x\ (l\cosh\theta_f/\tanh\theta_f)V\hat{}_{U'W}]_{ij}$ (25b);

Note that $l\cosh\theta_f$ = [E.U'/-c] the covariant form of c(t-t').This will turn out to be the correct way to get the desired trajectories.

7D Setting up to construct the needed trajectories. a) point of view of q'.

We will start with the trajectories as viewed from the source charge q'.

Take a point O as an origin. Use the label R_c to describe O. C means center.

R_c will be the center of the foliation of hyperboloids of one sheet.Let the vertical axis be in the direction U'/c which will also be the initial unit tangent of all the hyperboloids. U'/c represents the initial unit tangent U_{hi}/c of the foliation.Only the future oriented part of the hyperboloids will be needed.The horizontal axis will be in the direction of the unit spacelike vector $V\hat{}_{WU'}$ .This unit vector is the initial spatial direction $a_{hi}\hat{}$ of the foliation. It is the direction of the initial 4 acceleration of the virtual motion for the entire foliation.The final spatial direction will be labeled $a_{hf}\hat{}$. It is also the same for all members of the foliation.

Remember that $-V\hat{}_{U'W} = a_{hf}\hat{}$; since $-V\hat{}_{U'W}$ is confusing to use, better use $a_{hf}\hat{}$.We now have the center,the horizontal and vertical axes.The angle between the initial horizontal axis and the final axis is θ_f for the entire foliation.This looks very similar to a Rindler coordinate system but it is used in a different way.The inverse of the acceleration g/c^2 will be the radius of the first trajectory.The second trajectory will use the inverse of a g^*/c^2 acceleration, the third trajectory will use g^{**}/c^2,the 4th will use g^{***}/c^2 and so on. The following list will be used.

$g/c^2 = \sinh\theta_f/l$; $g^*/c^2 = \tanh\theta_f/l$; $g^{**}/c^2 = \tanh\theta_f/l\cosh\theta_f$ (26a,b,c);

$g^{***}/c^2 = \tanh\theta_f/l\cosh^2\theta_f$ (26d); The accelerations remain constant throughout the virtual motions from initial to final. The absolute values of the g's are used.

The radii will be the inverses of these gs. Let R_{hi} ,R^*_{hi} ,R^{**}_{hi} ,R^{***} the initial position of the 1st, 2nd,3rd,4th ,hyperbola respectively all on the horizontal axis.

$c^2/g = l/\sinh\theta_f$; $c^2/g^* = l/\tanh\theta_f$; $c^2/g^{**} = l\cosh\theta_f/\tanh\theta_f$; (27a,b,c);

$c^2/g^{***} = l\cosh^2\theta_f/\tanh\theta_f$ (27d);

We have: $|R_{hi} - R_c| = l/\sinh\theta_f$; $|R_{hi}^* - R_c| = l/\tanh\theta_f$; (28a,b);

$|R^{**}_{hi} - R_c| = l\cosh\theta_f/\tanh\theta_f$; $|R^{***}_{hi} - R_c| = l\cosh^2\theta_f/\tanh\theta_f$; (28c,d);

We have omitted the directions in eqs28a,b,c,d .They are all in direction $V\hat{}_{WU'}$.

$|R_{hf} - R_c| = l/\sinh\theta_f$; $|R^*_{hf} - R_c| = l/\tanh\theta_f$; (29a,b);

$|R^{**}_{hf} - R_c| = l\cosh\theta_f/\tanh\theta_f$; $|R^{***}_{hf} - R_c| = l\cosh^2\theta_f/\tanh\theta_f$; (29c,d);

Eqs29a,b,c,d are all in the same direction $a\hat{}_{hf} = -V\hat{}_{U'W}$ (30a);

as was mentioned before. $a\hat{}_{hf}$ = [$\sinh\theta_f\, a\hat{}_{hi}$ + $\cosh\theta_f$ U'/c] (30b);

$a\hat{}_{hi}$ = $V\hat{}_{WU'}$ (30c); as was also mentioned before.We now need the coordinate times during which the virtual motions occur on the vertical U' axis.

Let T_{hf} be the final time and T_{hi} be the initial time for the first hyperbola.Let the symbols *,**, *** refer to the times of the 2nd,3rd,4th, hyperbolas respectively.The difference in time during which the virtual motion last for the first hyperbola.

$T_{hf} - T_{hi} = T_{fi}$ (31); we have:

$cT_{fi} = l$; $cT^*_{fi} = l\cosh\theta_f$; $cT^{**}_{fi} = l\cosh^2\theta_f$; $cT^{***}_{fi} = l\cosh^3\theta_f$; (32a,b,c,d);

We notice that the times are stretched by the factor $\cosh\theta_f$ .The proper times of each hyperbola are obtained from the relation:

$\Theta = (g/c)\tau$ (33); so $c\tau_{fi}$ = $[l/\sinh\theta_{fi}]\theta_f$; $c\tau^*_{fi} = [l/\tanh\theta_f]\theta_f$;

$c\tau^{**}_{fi}$ = $[l\cosh\theta_f/\tanh\theta_f]\theta_f$; $c\tau^{***}_{fi} = [l\cosh^2\theta_f/\tanh\theta_f]\theta_f$;etc. (34a,b,c,d).

It may seem strange to use a radius such as $l\cosh\theta_f/\tanh\theta_f$ as a constant radius since it is a function of θ but there is no problem. During the virtual motion,we have: $c\tau = [l\cosh\theta_f/\tanh\theta_f]\theta$ (35) ; and so on. We vary τ and θ but not $\cosh\theta_f$ or $\tanh\theta_f$. We notice that the proper times are also stretched by the factor $\cosh\theta_f$. This is suggestive of red shifts or something akin to it.

We still don't know where to put the source charge, on which hyperbola, nor do we know on which hyperbola to place the field point or the point charge. We only know that they must be connected by the ray of influence E = lW/c. The answer is not difficult to find. Try $R^*_{hf} - R_{hi}$.

$R^*_{hf} - R_{hi} = [R^*_{hf} - R_c] - [R_{hi} - R_c]$. (36); $[R_{hi} - R_c] = [l/\sinh\theta_f]V\hat{}_{WU'}$ (36a);

$[R^*_{hf} - R_c] = [R^{**}_{hi} - R_c] + [l\cosh\theta_f]U'/c$

$= [l\cosh\theta_f/\tanh\theta_f]V\hat{}_{WU'} + [l\cosh\theta_f]U'/c$ (36b);

$[R^*_{hf} - R_{hi}] = \{[l\cosh\theta_f/\tanh\theta_f - l/\sinh\theta_f]V\hat{}_{WU'} + [l\cosh\theta_f]U'/c\}$ (37);

$= \{[l\cosh^2\theta_f / \sinh\theta_f - l/\sinh\theta_f]V\wedge_{WU'} + [l\cosh\theta_f]U'/c \}$ (37a);

$= [l\sinh^2 \theta_f / \sinh\theta_f]V\wedge_{WU'} + [l\cosh\theta_f]U'/c$;

$[R^*_{hf} - R_{hi}] = [l\sinh\theta_f V\wedge_{WU'} + l\cosh\theta_f U'/c] = lW/c = E$ (38)

This is the desired result. It shows that we must place $R'(\tau')$, the 4 position of source charge q', at R_{hi} . We must place the field point or the test charge q at R^*_{hf} .

Thus: $R'(\tau') = R_{hi}$ (39a); R or $R(\tau) = R^*_{hf}$ (39b);

Note that the trajectory is traversed in the time $E.U'/-c^2$ which is what we expected.

7D b)Virtual timelike trajectory construction. Point of view of the ray of influence.

We have the same construction as before except that the vertical axis is in the W/c direction.The initial unit tangent is W/c.The final unit tangent is U'/c.The horizontal axis is in the $V\wedge_{U'W}$ direction and is an initial unit normal to the hyperbola.It is also a line of constant initial time. The final normal to the trajectories is now in the $-V\wedge_{WU'}$ direction and is also a line of constant final time, not to be confused with final coordinate time. Time in the Rindler coordinate sense.

We take a center R_c as before. We have the same initial and final angle θ_f . We have the same g, g*,g**,g***, … divided by c^2. We have the same proper times τ,τ*,τ**, etc. We have R_{hi} ,R^*_{hi} ,R^{**}_{hi} ,R^{***}_{hi} whose distances from the center R_c are the same as before.Only the direction in the $V\wedge_{U'W.}$ is different. Same thing with R_{hf} , R^*_{hf} , R^{**}_{hf} ,etc are the same distance as before, only the final direction is $-V\wedge_{WU'}$ is different. Lastly, the coordinate times during which the various trajectories are in virtual motion, T_{fi} ,T^*_{fi} , T^{**}_{fi} , etc. , are the same as before but in the direction W/c. This innocuous looking change makes a very big difference however, since it makes the physical interpretation difficult. Geometrically everything is fine.

Possible Interpretations. 1) use $[R_{hi} - R_c] = [l/\sinh\theta_f]V\wedge_{U'W}$ (40a);

$[R_{hf} – R_c] = [R^*_{hi} – R_c] + lW/c = [l/\tanh\theta_f]V\wedge_{U'W} + lW/c$ (40b);

The construction requires: $R^*_{hi} = R'(\tau')$ (40c); $R_{hf} = R$ or $R(\tau)$ (40d);

This means that a virtual trajectory leaves R_{hi} and arrives at R_{hf} at coord time l Along the $V\text{^}_{U'W}$ axis, which is the coord time zero axis, the virtual motion is from R_{hi} to R^*_{hi} . At the same time as the first hyperbola leaves R_{hi} , a second hyperbola leaves R^*_{hi} and through a virtual motion reaches R^*_{hf} at coord time $l\cosh\theta_f$ while on the horizontal axis the virtual motion is from R^*_{hi} to R^{**}_{hi} .This means that a first hyperbola leave R_{hi} to influence the field point or the test charge q at R, while at the same time a second hyperbola leaves the position R' of the source charge q' and arrive at a later coord time , of duration $l\cosh\theta_f = E.U'/-c$,reaches the point R^*_{hf} to give exactly the right curvature because

$[R^*_{hf} – R_c] = [l\cosh\theta_f /\tanh\theta_f]V\text{^}_{U'W} + l\cosh\theta_f W/c$. Such a way for q' to influence R is difficult to understand physically even though it gives something like a correct answer.

2) Observe that if we take the difference $R^*_{hf} - R_{hi}$ just as we did in section 7D eq 38 except that instead of $lW/c = E$, we get lU'/c. We cannot interpret that as a ray of influence leaving R_{hi} and reaching R^*_{hf} having length l because R_{hi} is not $R'(\tau')$ and R^*_{hf} is not R. There is no obvious connection with the physics, it is just one of the many rays in the foliation.

3) We could simply choose $R'(\tau') = R_{hi}$; $R= R_{hf}$; with $R_{hf} – R_{hi} = lW/c$ .Then there is no need for hyperbolas it is just a vertical line of length l, but then $E.U'/-c$ does not appear .If we want it to appear we need to take $R^*_{hf} – R^{**}_{hi} = l\cosh\theta_f W/c$.But then we can't identify R' and R.Without the timelike hyperbolas the connection with the field curvatures is also lost. There is no clear way of deciding between these possibilities at this time.Only experience with a variety of situations can decide.

7D c) Point of view of U_o/c.

Since W and U' both lie in the U_o/c, $V^\wedge_{WUo}$ plane it is reasonable to ask whether one can form virtual hyperbolic trajectory which might give some sensible results.

Take the vertical time axis be in the U_o direction and the horizontal axis in the direction $V^\wedge_{WUo}$. Let Let the angle be θ and the final angle θ_f be θ_{WUo} .Let the initial unit tangent be U_o and the final unit tangent be W/c.

Set the initial hyperbolas be R_{hi} , R^*_{hi} ,R^{**}_{hi}, etc. Let the final positions be as before : R_{hf}, R^*_{hf} , R^{**}_{hf} , etc. all at the final time given by the final direction

$-V^\wedge_{UoW}$ each taken from a center R_c. The radii are $l/\sinh\theta_{WUo}$, $l/\tanh\theta_{WUo}$,

$l\cosh\theta_{WUo}/\tanh\theta_{WUo}$ etc. By the same procedure described in section 7D eq 38

we get: $R^*_{hf} - R_{hi} = l\sinh\theta_{WUo} V^\wedge_{WUo} + l\cosh\theta_{WUo} U_o/c = lW/c$ (41);

This is reasonable if we assume that, from the point of view of U_o , R_{hi} takes the place of R'(τ') and R^*_{hf} takes the place of R or R(τ);

One can be tempted to use the angle $\theta_{WU'} = \theta_{U'Uo} - \theta_{WUo}$ instead of θ_{WUo} keeping the same axes U_o , $V^\wedge_{WUo}$, $- V^\wedge_{UoW}$, as before we get:

$R^*_{hf} - R_{hi} = l\sinh\theta_{WU'} V^\wedge_{WUo} + l\cosh\theta_{WU'} U_o/c$ (42); this is not lW/c so the interpretation does not work.

It is interesting to derive the following useful formula which gives some insight into the situation. We want to show that :

$[(U'/c)x\, V^\wedge_{WU'}] = [(U_o/c)x\, V^\wedge_{U'Uo}]$ (43);

Since $V^\wedge_{WU'} = [V^\wedge_{U'Uo} + (V^\wedge_{U'Uo} .U')U'/c^2]/\cosh\theta_{U'Uo}$ (44);

$[U'/c \times V^\wedge_{WU'}] = [U'/c \times V^\wedge_{U'Uo}]/\cosh\theta_{U'Uo}$

$= \{[\sinh\theta_{U'Uo} V^\wedge_{U'Uo} + \cosh\theta_{U'Uo} U_o/c]x\, V^\wedge_{U'Uo}\}/\cosh\theta_{U'Uo} = [U_o/c \times V^\wedge_{U'Uo}]$;

So eq 43 is proved. Since $V^\wedge_{U'Uo} = V^\wedge_{WUo}$. We also have:

$[U'/c \times V\hat{}_{WU'}] = [U_o/c \times V\hat{}_{WUo}]$ (45);

7E Inversion of the energy stress tensor.

7E a) Various inversions.

Since $T_{ij} = [(-1/2)F_{mn}F_{mn}]\delta_{ij}/2 + F_{ik}F_{jk}$, it is reasonable to define its inversion by replacing its F_{ij} with $F^{inv}{}_{ij} = F_{ij}/[(-1/2)F_{mn}F_{mn}]$ from eq7.23a .

$T^{inv}{}_{ij} = \{[(-1/2)F_{mn}F_{mn}]\delta_{ij}/2 + F_{ik}F_{jk}\}/[(-1/2)F_{mn}F_{mn}]^2$ (46);

$T^{inv}{}_{ij} = T_{ij}/[(-1/2)F_{mn}F_{mn}]^2$ (46a); since $T = (-1/2)F_{mn}F_{mn}$,

$T^{inv}{}_{ij} = T_{ij}/T^2$ (46b); The inversion of the field Lagrangian density is $1/T$ (47);

Inversions of the form: $1/|T_{ij}U^iU^j/c^2|$, $1/|T_{ij}V\hat{}_iV\hat{}_j|$, $1/|T_{ij}V\hat{}_iU_j/c|$ with $V\hat{} = V\hat{}_{UoU}$ will give the inverse of the energy density or the absolute value of the Poynting vector for example. If we want to impart directions we can use expressions such as $T_{ij}V\hat{}_{UoU\,j}/[T_{mn}V\hat{}_{UoU\,m}U_n/c]^2$ for example. All the projections of T^{inv} can easily be obtained by dividing those of T_{ij} by T^2 . The electromagnetic momentum density:

$P_{em} = (1/2)[\mathcal{E}^2 + B^2]U/c + |\mathcal{E}\times B|V\hat{}_{UoU}$ gives $P_{em}{}^{inv} = P_{em}/-P_{em}.P_{em}$;

Its “dual” W_{em} gives $W_{em}{}^{inv} = W_{em}/W_{em}.W_{em}$.

$T_{ij}{}^{inv} = T_{ij}/T^2 = T_{ij}/4\rho^2c^4$ (48); using eq6.85a .

Another interesting inverse is :

$[\mathcal{E}^2 + B^2]/2|\mathcal{E}\times B| = 1/\tanh 2\theta_{UoU}$ (49); using eqs 6.77a and 77b .

So $[\tanh 2\theta_{UoU} V\hat{}_{UoU}]^{inv} = [1/\tanh 2\theta_{UoU}]V\hat{}_{UoU}$ (50);

7E b) Generators of inversions.

Let v be a 3 vector. Its inverse u is given by : $u = v/v.v$ (51a);

In components : $u^a = v^a/v.v$ (51b) ; a=1to 3 $u\hat{}^a = v\hat{}^a$ (51c); we can also reverse the formulas. $v = u/u.u$ (52a) ; $v^a = u^a/u.u$ (52b); we can also write $u = v^{inv}$ (53a); $v = u^{inv}$ (53b);

$du^a/dv^b = \delta_{ab}/v.v \; -2[v^a v^c\, dv^c/dv^b]/(v.v)^2$ (54a) ; $dv^c/dv^b = \delta_{bc}$ (54b);

$du^a/dv^b = \delta_{ab}/v.v \; -2v^a v^b/(v.v)^2 = [\delta_{ab} - 2v\hat{}^a\, v\hat{}^b]/(v.v)$ (55); $v\hat{} = v/|v|$.

$du^a/dv^b = u.u[\,\delta_{ab} - 2u\hat{}^a\, u\hat{}^b\,] = dv^{a\,inv}/dv^b$ (56);

similarly: $dv^a/du^b = [\delta_{ab} - 2u\hat{}^a u\hat{}^b]/(u.u)$ (57); $u\hat{} = u/|u|$.

$dv^a/du^b = v.v[\delta_{ab} - 2v\hat{}^a v\hat{}^b\,] = du^{a\,inv}/du^b$ (58);

we can easily prove : $du^a/dv^b = du^b/dv^a$ (59a); $dv^a/du^b = dv^b/dv^a$ (59b);

We are now ready to generalize these results using projective velocity 4 vectors and then compare the results with the expressions for T_{ij} .

Let $V = V_{U'W}$ (60a) ; $V^{inv} = V_{U'W}/V_{U'W}.V_{U'W}$ (60b);

Let $U_{ij} = [V_{U'W}.V_{U'W}\,\delta_{ij} - 2V_{U'W\,I}\, V_{U'W\,j}\,]/c^2$ (61) ; using $K = q'^2/[E.U'/-c]^4$

$2T_{ij} = KU_{ij}$ (62); $U_{ab} = [V_{U'W}.V_{U'W}\,\delta_{ab} - 2V^a{}_{U'W}\, V^b{}_{U'W}\,]/c^2$ (63);

In a frame moving with 3 velocity w eq 63 becomes:

$(U_{ab})_{w=0} = [u'.u'\,\delta_{ab} - 2u'^a\, u'^b\,]/c^2 = (u'.u')[\delta_{ab} - 2\, u'\hat{}_a\, u'\hat{}_b\,]/c^2$ (63a);

Eq63a is identical with eqs56 and 58.

let $U^{inv}{}_{ij} = U_{ij}/[V_{U'W}.V_{U'W}]^2 = [\delta_{ij} - 2V\hat{}^i{}_{U'W}\, V\hat{}^j{}_{U'W}]/[V_{U'W}.V_{U'W}]$

$= [\delta_{ij} - 2V\hat{}^{inv}{}_i V\hat{}^{inv}{}_j\,]V^{inv}.V^{inv}$ (64); $U^{inv}{}_{ab} = [\delta_{ab} - 2V\hat{}^a{}_{U'W}\, V\hat{}^b{}_{U'W}/[V_{U'W}.V_{U'W}]$ (64a);

$(U^{inv}{}_{ab})_{w=0} = [\delta_{ab} - 2u'\hat{}_a\, u'\hat{}_b\,]/[u'u'/c^2] = u'^{inv}.u'^{inv}[\delta_{ab} - 2u'^{inv}{}_a u'^{inv}{}_b\,]/c^2$ (64b);

$u'^{inv}/c = (u'/c)/[u'.u'/c^2]$ (64c);

$U_{4a} = [V_{U'W}.V_{U'W}\,\delta_{4a} - 2V_{U'W}{}^4 V_{U'W}{}^a\,]/c^2$ (65);

$(U_{4a})_{w=0} = 0$ (65a); since both δ_{4a} and $V_{U'W\,4}$ vanish. So $(U^{inv}{}_{4a})_{w=0} = 0$ (65b)

$U_{44} = [V_{U'W}.V_{U'W}\,\delta_{44} - 2V_{U'W\,4}V_{U'W\,4}\,]/c^2$ (66); $(U_{44})_{w=0} = (u'.u'/c^2)$ (66a);

$U^{inv}{}_{44} = [\delta_{44} - 2V\hat{}_{U'W\,4}V\hat{}_{U'W\,4}]/[V^2{}_{U'W}/c^2]$ (67); $(U^{inv}{}_{44})_{w=0} = 1/(u'^2/c^2)$ (67a);

So we have an extra term the U_{44} and its inverse.If we want to get rid of it we must multiply by a projection operator.

Let $h_{ij} = [\delta_{ij} + W_i W_j/c^2]$ (68);

$h_{ij}U_{jk} =[\delta_{ij} + W_i W_j/c^2]U_{jk} = [U_{ik} + (U_{jk}W_j)W^i/c^2]$ (69);

$U_{jk}W_j/c = [V^2{}_{U'W}\ \delta_{jk} - 2\ V_{U'W\ j}V_{U'W\ k}]W_j/c^3 = (V^2{}_{U'W}/c^2)W_k /c$ (69a);

$U_{jk}W_jW_i/c^2 = (V^2{}_{U'W}/c^2)W_kW_i/c^2$ (69b);

$h_{ij}U_{jk} =\{(V^2{}_{U'W}/c^2)[\delta_{ik} +W_iW_k/c^2]-2V_{U'W\ i}V_{U'W\ k}\}$ (70);$h_{ij}U^{inv}{}_{jk} = [h_{ij}U_{jk}]/(V^2{}_{U'W}/c^2)$ (70a)

so we have simply replaced the original delta function by h_{ik} . Nothing is changed with the spatial components a,b when w=0. Nothing is changed with the 4,a comonents when w=0. The 44 components however now give δ_{44} +icic/c^2 =0 so we have eliminated the 44 component.

$T_{ij} = (K/2)U_{ij}$ (71); $T^{inv}{}_{ij} = (1/2K)U^{inv}{}_{ij}$ (72); $U_{ij}\ U^{inv}{}_{jk} = \delta_{ik}$ (73a); $T_{ij}T^{inv}{}_{jk} = \delta_{ik}$ (73b);

$h_{ij}U_{jk}\ h_{kl}U^{inv}{}_{lm} = [(\delta_{ik} + W_iW_k/c^2) -2V\hat{}_{U'W\ i}V\hat{}_{U'W\ k}][(\delta_{km} + W_kW_m/c^2) -2V\hat{}_{U'W\ k}V\hat{}_{U'W\ m}]$

$= (\delta_{ik} + W_iW_k/c^2)(\delta_{km} + W_kW_m/c^2) = (\delta_{im} + W_iW_m/c^2)$ (73c); this reduces to δ_{ab} when w=0 as expected. We could of course have used eqs63a and 64b

$[(U_{ab})(U^{inv}{}_{bc})]_{w=0} = (u'.u'/c^2)[\delta_{ab} -2u\hat{}_a u\hat{}_b][\delta_{bc} -2u'\hat{}_b u'\hat{}_c]/(u'.u'/c^2) = \delta_{ac}$ (74);

With T_{ij} , $V^2{}_{U'W}/c^2$ is replaced by $(V^2{}_{U'W}/c^2)(1- V^2{}_{U'W}/c^2)^2$ and its inverse for $T^{inv}{}_{ij}$.

This means that eq56 $du^a /dv^b = u.u[\delta_{ab} - 2u\hat{}_a u\hat{}_b]$ should be interpreted as:

$du^a/dv^b = (u'.u'/c^2)[1- u'.u'/c^2]^2\{\delta_{ab} - 2u'\hat{}_a\ u'\hat{}_b\}$ (75); to make this more clear:

$\Upsilon = \Upsilon^{inv}/(\Upsilon^{inv}.\Upsilon^{inv})$ (76a) $\Upsilon^{inv} = \Upsilon/(\Upsilon.\Upsilon)$ (76b);

$\Upsilon = (V_{U'W}/c)[1- V^2{}_{U'W}/c^2]$ (76c); $\Upsilon^{inv} = V\hat{}_{U'W}/(|V_{U'W}/c|)[1- V^2{}_{U'W}/c^2]$ (76d);

$(\Upsilon)_{w=0} = (u'/c)[1- u'.u'/c^2]$ (77a); $(\Upsilon^{inv})_{w=0} = (u'\hat{}/c)/|u'/c|[1- u'.u'/c^2]$ (77b);

eq56 becomes $d(\Upsilon)^{a}_{w=0}/d(\Upsilon^{inv})^{b}_{w=0} = du^{a}/dv^{b}$ (78); hopefully this makes the procedure clearer. If we use $V_{WU'}$ instead of $V_{U'W}$,u' is replaced by w, u'^ by w^, $[\delta_{ij}+W_iW_j/c^2]$ by $[\delta_{ij} + U'_iU'_j/c^2]$.

7F Inverse of the Lorentz force.

$F_i = qF_{ij}U_j/c$ (79). $F^{inv}_i = F_i /F.F = [F_{ij}U_j/c]/q[F_{lm}U_mF_{ln}U_n/c^2]$ (80);

$F^{inv}_i = \{(qq'/l^2)|V_{U'W}/c|(1- V^2_{U'W}/c^2)\cosh\theta_{uo\,u} V\wedge_{WUo\,i}\}/(F.F)$ (81a);

$F^{inv}_i = V\wedge_{WUo\,i}/(qq'/l^2)[|V_{U'W}/c|(1- V^2_{U'W}/c^2)\cosh\theta_{UoU}]$ (81b); but

$[qF_{ij}U_j/c]/q^2[(-1/2)F_{lm}F_{lm}] = \cosh\theta_{UoU} V\wedge_{WUo}/(qq'/l^2)[|V_{U'W}/c|(1- V^2_{U'W}/c^2)]$

(82); so one must be careful.

8 The interval of propagation E connects the trajectories of the source charge and the test charge for all times.

8A Some curious projective formulas involving dE/cdτ' and d(W/c)/cdτ'.

Let E = R(τ) – R'(τ') . Since this formula must be true during the existence of q' and q the proper times must be related. Thus τ' is a function of τ and vice versa.We are interested in taking derivatives of E without changing the length l.We will be interested in taking dE/dτ', dE/dτ , dR/dτ', dR/dτ , dR'/dτ', dR'/dτ, dτ'/dτ, dτ/dτ',dW/dτ', dW/dτ. We will not take second derivatives because they will be discussed in a future article on the total field and the acceleration field.

dE/dτ' = [dR/dτ' – dR'/dτ']= [(dR/dτ)dτ/dτ' – U'] = [U(dτ/dτ') – U'] (1);

d(E.E)/dτ' = 0 = 2E.dE/dτ' = 2E.[U(dτ/dτ') – U'] (2); E.U(dτ/dτ') = E.U' (3);

dτ/dτ' = E.U'/E.U (4); dτ'/dτ = E.U/E.U' (4a); (E.U)dτ = (E.U')dτ' (5);

dE/dτ' = U[(E.U')/(E.U)] – U' = [U(E.U') – U'(E.U)]/[E.U] (6);

$dE_i /d\tau' = [U\text{x}\, U']_{ij}E_j /(E.U) = [(U/c)\text{x}(U'/c)]_{ij}E_j/(E.U/c^2)=[(U'/c)\text{x}(U/c)]_{ij}E_j/(E.U/-c^2)$ (7);

with $U' = [V_{U'W} + W]/(1- V^2_{U'W})^{1/2}$ and $U= [V_{UW} + W]/(1- V^2_{UW})^{1/2}$ eq 7 becomes:

$dE_i/cd\tau' = [V_{U'W} +W]x[V_{UW} +W]_{ij}W_j(U'.W/-c^2)(U.W/-c^2)/(U.W/-c^2)$ (8);

$dE_i/cd\tau'=\{[Wx (V_{UW} -V_{U'W})]_{ij}W_j /(1-V^2_{U'W})^{1/2}$

$+ [V_{U'W} x (V_{UW} – V_{U'W})]_{ij}W_j/(1- V^2_{U'W})^{1/2} \}$;

$dE_i/cd\tau' = [V_{UW} – V_{U'W}]/(1- V^2_{U'W})^{1/2}$ (9);

$d(W/c)/dc\tau' = [V_{UW} – V_{U'W}]/cl(1-V^2_{U'W} /c^2)^{1/2}$ (10);

eq10 is full of hidden meaning as will be explained shortly,but before we derive another equally meaningful equivalent expression.

$(-c^2/U.U')(dE_i/dc\tau') = [(U'/c)x (U/c)(-c^2/U.U')]_{ij} (W_j /c) (- c^2/U.W)$

$= [U'x V_{UU'}]_{ij} W_j /\cosh\theta_{UW}$ (10); $dE_i/cd\tau' = \{[U'xV_{UU'}]_{ij}W_j \}\cosh\theta_{UU'}/\cosh\theta_{UW}$ (11);

$d(W/c)/cd\tau' = \{[(U'/c) x (V_{UU'}/c)/l\cosh\theta_{UW} (1- V^2_{UU'}/c^2)^{1/2}]\}_{ij}W_j/c$ (12);

eq12 is also filled with hidden meaning. We now proceed to explain.

Define $\alpha^{uu'}{}_w /c^2 (1- V_{U'W}/c^2)^{3/2} = [V_{UW} – V_{U'W}]/cl(1- V^2_{U'W}/c^2)^{1/2} = (dW/c)/dc\tau'$ (13);

So $\alpha^{uu'}{}_w /c^2 = [V_{UW} –V_{U'W}](1- V^2_{U'W}/c^2)/cl = (1-V^2_{U'W}/c^2)^{3/2} (dW/c)/dc\tau'$ (14);

Eq14 resembles the difference of two geodesic accelerations.The term in $V_{U'W}$ is a geodesic acceleration part for the geodesic $V_{U'W}$. The V_{UW} doesn't have quite the right form.

$V_{UW} (1- V^2_{UW}) = V_{UW} (1-V^2_{UW})[(1- V^2_{U'W})/(1- V^2_{UW})]$ (15);

Eq15 is a geodesic acceleration with a "conformal" factor

$k^2 = [\cosh\theta_{UW}/\cosh\theta_{U'W}]^2$ (16); let us go back to eq12.

Define : $a^{uu'} /c^2 = (V_{UU'}/c)/l\cosh\theta_{UW} (1- V^2_{UU'}/c^2)^{1/2}$ (16);

$d(W/c)/cd\tau' = [(U'/c)x\, a^{uu'}/c^2]_{ij}W_j/c = [\alpha^{uu'}{}_w /c^2]/(1- V^2_{U'W}/c^2)^{3/2}$ (17);

we also want to prove: $a^{uu'}/c^2 = (1/l)\{(V_{UW} - V_{U'W})/c + [(V_{UW} - V_{U'W}).U'/c^2\ ]U'/c\}$ (17a); but first we explain below the reasons for all these definitions.

The definitions offered by eq14 and 16 are not arbitrarily given . The true 4 acceleration $d^2R/d\tau^2 = [a/(1- u^2/c^2) + (a.u/c^2)u/(1- u^2/c^2)^2\ , i(a.u/c)/(1- u^2/c^2)^2\ ]$;If we replace u by $V_{UU'}$ we get:

$d^2R/d\tau^2 = \{[a/(1- V^2_{UW}/c^2) + (a.V_{UW}/c^2)V_{UW}/(1- V^2_{UW}/c^2)^2\ ]$

$+[(W/c)(a.V_{UW}/c)\ /(1- V^2_{UW}/c^2)^2\ ]\}$ (18); this expression cannot be correct since a is a 3vector whereas the other terms are 4 vectors which reduce to u and i when w=0. We need a covariant generalization of a, a 4 vector which reduces to a when w=0. This 4 vector must be orthogonal to W.

Let $\alpha^u{}_w$ be this 4 vector. It is a 3-acceleration 4 vector!

$d^2R/d\tau^2 = \{[\ \alpha\ /(1- V^2_{UW}/c^2) + (\ \alpha.V_{UW}/c^2)V_{UW}/(1- V^2_{UW}/c^2)^2]$

$+ (W/c)[(\ \alpha.V_{UW}/c)/(1- V^2_{UW}/c^2)^2]\}$ (19); it is not difficult to verify that:

$d^2R/d\tau^2 = [\ \alpha + (\alpha.U/c^2)U/c]/(1- V^2_{UW}/c^2)$ (20); we have omitted the superscript and subscript of α . We could just as easily have used $V_{UU'}$ and U' in eqs19 and 20 with $\alpha^u{}_{u'}$. Using eq 20, it is easy to show that :

$[(U/c)x\ d^2R/d\tau^2]_{ij}W_j/c\ \ =\ \ \alpha^u{}_w/(1- V^2_{UW}/c^2)^{3/2}$ (21);

$d(W/c)/cd\tau'$ has the form of eq21 with $d^2R/d\tau^2$ replaced by $a^{uu'}$. See eq17. To be sure that $a^{uu'}$ represents an acceleration it should also satisfy eq 20 with $a^{uu'}{}_w$ of eq14 inserted in eq20 . It should also satisfy eq 17a.

$V_{UU'}/c = [\ e^{uu'}{}_w\ +(\ e^{uu'}{}_w\ .U'/c^2\ )U'](1- V^2_{U'W}/c^2)^{1/2}$ (22a);

$(V_{UU'}/c)/(1- V^2_{UU'}/c^2)^{1/2}\ \ = [\ e^{uu'}{}_w + (\ e^{uu'}{}_w\ .U'/c^2)U'\ ][1-V_{UW}.V_{U'W}/c^2]\cosh\theta_{UW}$

$= \{(V_{UW}\ -V_{U'W})/c + [(V_{UW} - V_{U'W}).U'/c^2]U'\}\cosh\theta_{UW}$ (22b);

$[(V_{UU'}/c)/(1- V^2_{UU'}/c^2)^{1/2}]/l\cosh\theta_{UW} = \{(V_{UW} - V_{U'W})/c + [(V_{UW} - V_{U'W}).U'/c^2]U'\}/l$

$= [\alpha^{uu'}{}_{w} + (\alpha^{uu'}{}_{w}.U'/c^2)U'] /c^2 (1- V^2_{U'W}/c^2) = a^{uu'}{}_{w}/c^2$ (23); eq17a is proved.

The most important thing about section 8 is that if we replace U by a different velocity U_h which can be obtained from the total field all the expressions obtained thus far are identical to those which relate to the true total field, the true acceleration field, the true nonacceleration field.This is strange and must have some deep meaning. The full properties of the total field and the acceleration field will be developed in a future article but unexpectedly, they are duplicated here by changing U_h into U.

Before we proceed further, we want to prove an interesting formula namely:

$(e^{uu'}{}_{w}) = [l(\alpha^{uu'}{}_{w}/c^2)/(1- V^2_{U'W}/c^2)][\cosh\theta_{UW}\cosh\theta_{U'W}]/\cosh\theta_{UU'}]$ (24);

$d(W/c)/d\tau' = \{[V_{UW} - V_{U'W}]/cl\}\cosh\theta_{U'W} = [\cosh\theta_{UU'}/l\cosh\theta_{UW}](e^{uu'}{}_{w})$ (25);

$d(W/c)/d\tau' = (\alpha^{uu'}{}_{w}/c^2)\cosh^3\theta_{U'W} = [\cosh\theta_{UU'}/l\cosh\theta_{UW}](e^{uu'}{}_{w})$ (26);

$(\alpha^{uu'}{}_{w}/c^2)\cosh^2\theta_{U'W} = [\cosh\theta_{UU'}/l\cosh\theta_{U'W}\cosh\theta_{UW}](e^{uu'}{}_{w})$ (27); same as eq24.

8B Field like entities.Geodesic accelerations.

Define: $F^{total\ uu'}{}_{ij} = (q'/l^2)[(W/c) x(V_{UW}/c) (1- V^2_{U'W}/c^2)]_{ij}$ (28a);

$F^{nonacc\ uu'}{}_{ij} = (q'/l^2)[(W/c) x(V_{U'W}/c)(1- V^2_{U'W}/c^2)]_{ij}$ (28b);

$F^{acc\ uu'}{}_{ij} = (q'/l)[(W/c) x(\alpha^{uu'}{}_{w}/c^2)]_{ij}$ (28c);

These definitions,as explained before,are made because they have exactly the same form as the real total field,nonacceleration field and acceleration field.

$F^{tot\ uu'}{}_{ij} - F^{na\ uu'}{}_{ij} = F^{acc\ uu'}{}_{ij}$ (30); can be rewritten as:

$(q'/l^2)\{(W/c) x[(V_{UW} - V_{U'W})/c](1- V^2_{U'W}/c^2)\}_{ij} = (q'/l)[(W/c)x(\alpha^{uu'}{}_{w}/c^2)]_{ij}$ (31);

This is a fundamental relation when dealing with the real fields.

The moment of $dE/dc\tau'$ is:

$[Ex\ dE/dc\tau'] = l^2[(W/c)xd(W/c)/dc\tau'] = l^2[(W/c)x(V_{UW} - V_{U'W})/lc]\cosh\theta_{U'W}$

$= [(W/c)x(V_{UW} - V_{U'W})/c]l\cosh\theta_{U'W}$ (32);

$[ExdE/dc\tau'](-c/E.U')^3 = (1/l^2)[(W/c)x(V_{UW} - V_{U'W})/c](1 - V^2_{U'W}/c^2)$

$= (1/l)[(W/c)x\ (\alpha^{uu'}{}_{W}/c^2)]$ (33) ;

$q'[ExdE/cd\tau'](-c/E.U')^3 = F^{tot\ uu'}{}_{ij} - F^{na\ uu'}{}_{ij} = F^{acc\ uu'}{}_{ij}$ (34);

These remarkable formulas show that something deep is going on and the importance of taking moments.It suggests that something similar might be possible with the true fields but it is not yet clear . We have:

$F^{tot\ uu'}{}_{ij} = [q'/(-c/E.U')^2]\ [(W/c)xV_{UW}/c]_{ij} = q'(-c/E.U)^2[(W/c)xV_{UW}/c](E.U/E.U')^2$ (35a);

The geodesic acceleration term is $(q'/l^2)(V_{UW}/c)\ (1 - V^2_{UW}/c^2)$ times the factor $(E.U/E.U')^2$ .That factor shows that it does not lie on the same hyperboloid of two sheets as the non acceleration field.Apparently, the acceleration field makes it jump to a new member of the foliation as well as changing its plane from $WxV_{U'W}$ to $Wx\ V_{UW}$.

$F^{nonacc\ uu'}{}_{ij} = q'(-c/E.U')^2[(W/c)x\ (V_{U'W}/c)]_{ij}$ (35b) also given by (28b);

Note that $F^{na}{}_{ij} = F^{nonacc\ uu'}{}_{ij}$ (36); the two fields are identical! Only the total field and the acceleration field differ from the real ones .

There is another geodesic and therefore a geodesic acceleration that we need to deal with. $V_{UU'}$ and $(U'x\ V_{UU'})$ should give rise to another field $F^{UU'}$.

$F^{UU'}{}_{ij} = (q'/l^2)[(U'/c)x\ (V_{UU'}/c)(1 - V^2_{UU'}/c^2)]_{ij}$ (37); eq37 is different from the total,acceleration or nonacceleration fields but it does represent a geodesic acceleration.It does have a counterpart with real fields when U is replaced with U_h.It plays an important part which will be explain in the future. It is interesting to project eq37 in the U direction as we did to get the Lorentz force.The problem is very different however.

$F^{UU'}{}_{ij}U_j/c = (q'/l^2)[\tanh\theta_{UU'}/\cosh^2\theta_{UU'}][(U'/c)(V\wedge_{UU'}.U/c)+(U'.U/-c^2)V\wedge_{UU'}]_I$ (38);

It brings the geodesic acceleration in a direction in the U' , $V\wedge_{UU'}$ plane orthogonal to U. Doing the same thing to $F^{tot\ uu'}{}_{ij}$ and $F^{na\ uu'}{}_{ij}$ give similar results.

$F^{tot\ uu'}{}_{ij}U_j/c = (q'/l^2)[\tanh\theta_{UW}/\cosh^2\theta_{U'W}][(W/c)(V\wedge_{UW}.U/c)+(W.U/-c^2)V\wedge_{UW}]_I$ (39);

$F^{na\ uu'}{}_{ij}U_j/c = (q'/l^2)[\tanh\theta_{U'W}/\cosh^2\theta_{U'W}][(W/c)(V\wedge_{U'W}.U/c)+(W.U/-c^2)V\wedge_{U'W}]_I$ (40);

Eq40 times q is of course the real Lorentz force.We also have from previous sections.

$F^{na}{}_{ij}U_j/c = (q'/l^2)(e^{u'w}{}_u)_i/\cosh^2\theta_{U'W} = (q'/l^2)[\tanh\theta_{U'W}/\cosh^2\theta_{U'W}]\cosh\theta_{UoU}\ V\wedge_{U'Uo}$;

So: $[(W/c)(V\wedge_{U'W}.U/c)+(W.U/-c^2)V\wedge_{U'W}]=\cosh\theta_{UoU}\ V\wedge_{U'Uo}$ (41);

Since $(W\times V\wedge_{U'W}) =(U_o \times V\wedge_{U'Uo}]$ and $V\wedge_{U'Uo} .V\wedge_{UUo}=0$, eq 41 is easily proved as we showed previously. We can also project $F^{acc\ uu'}{}_{ij.}$:

$F^{acc\ uu'}{}_{ij}U_j/c = [(\alpha^{uu'}{}_w/lc^2)][(W/c)(\alpha^{uu'}{}_w\wedge.U/c)+(W.U/-c^2)(\alpha^{uu'}{}_w\wedge)]_I$ (42);

It is possible to simplify many results by projecting W onto the U,U' plane to get a new 4 velocity U^*_o which must not be confused with U_o which is the projection of U onto the U', W plane as discussed in the previous sections.

8C Virtual timelike hyperbolic trajectories in the U,U' plane.

The Rindlerlike construction will be a little different than in the previous sections.We take the center O of the coordinate and call it R_c . The vertical time direction will be in direction U'/c.The horizontal spacelike direction will be $V\wedge_{UU'.}$The final spacelike direction will be in the direction $-V\wedge_{UU'}$ and represent a line of final equal time whereas $V\wedge_{UU'}$ represent a line of initial time.The initial and final lines will make an angle $\theta_{UU'}$ between them. U' will be the initial tangent to the timelike hyperbolas of the foliation.U will be its final tangent. $V\wedge_{UU'}$ will also represent the initial 4acceleration unit 4 vector $a^{uu'}\wedge$, $-V\wedge_{U'U}$ will be the direction of the final acceleration unit 4vector.The angle $\theta_{UU'}$ is labeled using the initial and final 4 velocities tangents instead of the angle between the initial and final acceleration unit 4 vectors.Both angles are the same. The accelerations are normal to the timelike hyperbolas of the foliation.The first hyperbola will start at a position R_{hi} on the $V\hat{}_{UU'}$,a distance $c^2/(a^{uu'})$ from the center R_c.

$|R_{hi} - R_c| = c^2/|a^{uu'}| = l\cosh\theta_{WU}/\sinh\theta_{UU'}$ (43); $R_{hi} = R'(\tau')$ (44);

the final position on this hyperbola is labeled R_{hf}. $|R_{hf} - R_c| = |R_{hi} - R_c|$(45); the coordinate time difference is : $c(T_{hf} - T_{hi}) = l\cosh\theta_{WU} = E.U/-c$ (46) ; the second hyperbola will start at position R^*_{hi} and end at R^*_{hf} .We want the interval of propagation to start at $R' = R_{hi}$ .Can the interval end at R^*_{hf} ? No! this is because E does not necessarily lie in the U, U' plane (which is the U', $V\hat{}_{UU'}$ plane).What we need is the projection of E onto the U',$V\hat{}_{UU'}$ plane.

Let $T^{uu'} = (E.V\hat{}_{UU'})V\hat{}_{UU'} + (E.U'/-c)U'/c$ (46a); the position R(τ) is projected to the point R^*_{hf}. We have: $R_{hi} - R_c + T^{uu'} = R^*_{hf} - R_c$ (46b); $E = T^{uu'} + D^{uu'}$ (47); $D^{uu'}$ is perpendicular to the U', $V\hat{}_{UU'}$ plane. The coordinate time difference of $R^*_{hf} - R_c$ must be: $c(T^*_{hf} - T^*_{hi}) = (E.U'/-c)$ (48); $|R^*_{hi} - R_c| = |R^*_{hf} - R_c|$ (49); the ratio $(E.U'/-c)/(E.U/-c) = k = \cosh\theta_{U'W}/\cosh\theta_{UW}$ (50); $|R^*_{hi} - R_c|/|R_{hi} - R_c| = k$ (51)
$|R^*_{hi} - R_c| = k|R_{hi} - R_c| = [(E.U'/-c)/(E.U/-c)][(E.U/-c)/\sinh\theta_{UU'}] = (E.U'/-c)/\sinh\theta_{UU'}$(52)
the acceleration at the second trajectory is: $|a^{uu'*}/c^2| = \sinh\theta_{UU'}/(E.U'/-c)$ (53a); at the first trajectory as we know, it is: $|a^{uu'}/c^2| = \sinh\theta_{UU'}/(E.U/-c)$ (53b);again, the reason for going through all these tedious steps is because the true total field and acceleration field will require them.

8D New 4 velocity U^*_o from projecting W onto the U, U' plane.

This new 4 velocity will simplify the results in the same way as the projection of U onto the U',W plane which gave us U_o .It turns out that it is also closely related to the virtual timelike hyperbolas of the previous subsection.

$d(W/c)/cd\tau' = [V_{UW} - V_{U'W}]\cosh\theta_{U'W}/cl = (\alpha^{uu'}{}_w/c^2)/(1 - V^2_{U'W}/c^2)^{3/2}$

$= [\cosh\theta_{UU'}/l\cosh\theta_{UW}](e^{uu'}{}_w)$; using eq24,25. A unit vector in the direction of $dW/d\tau'$ which is in direction of unit vectors $\alpha^{uu'}{}_w\hat{} = e^{uu'}{}_w\hat{}$. We will have:

$V_{UW\,perp} = V_{U'W\,perp} = V_{U*o\,W}$ (54); $V_{U'W\,par} = U'/\cosh_{U'W} - U*_o/\cosh\theta_{U'U*o}$ (55);

Since $\cosh\theta_{U'W} = \cosh\theta_{U'U*o}\cosh\theta_{WU*o}$ (56); because $V_{U'U*o}.V_{WU*o} = 0$ (57);

We obtain : $V_{UW\,par} = [V_{U'U*o}/\cosh\theta_{WU*o}]$ (58); in the same way:

$\cosh\theta_{UW} = \cosh\theta_{UU*o}\cosh\theta_{WU*o}$ (59); $V_{UW\,par} = V_{UU*o}/\cosh_{WU*o}$ (60);

The unit vectors $V\hat{}_{UU*o} = V\hat{}_{U'U*o} = \alpha^{uu'}{}_w\hat{} = e^{uu'}{}_w\hat{} = e^{uu'}{}_{u*o}\hat{}$ (61);

$d(W/c)/dc\tau' = [V_{UU*o} - V_{U}{}'_{U*o}]\cosh\theta_{U'U*o}\cosh\theta_{WU*o}/lc\cosh\theta_{WU*o}$ (62a);

$d(W/c)/cd\tau' = [V_{UU*o} - V_{U'U*o}]/cl(1 - V^2{}_{U'U*o}/c^2)^{1/2}$ (62b);

$d(W/c)/dc\tau' = [\tanh\theta_{UU*o} - \tanh\theta_{U'U*o}]V\hat{}_{U'U*o}/l(1 - V^2{}_{U'U*o}/c^2)^{1/2}$ (62c)

$d(W/c)/dc\tau' = \{\tanh(\theta_{UU*o} - \theta_{U'U*o})V\hat{}_{U'U*o}[\cosh\theta_{UU'}/l\cosh\theta_{UU*o}]\}$ (63);

The accelerations yield important results.

$(a^{uu'}{}_w/c^2) = [\alpha^{uu'}{}_w/c^2 + (\alpha^{uu'}{}_w.U'/c^2)U']/(1 - V^2{}_{U'W}/c^2)$

$= [\alpha^{uu'}{}_{U*o}/c^2 + (\alpha^{uu'}{}_{U*o}.U'/c^2)U']/(1 - V_{U'U*o}/c^2)$ (64);

$[(U'/c)x(a^{uu'}{}_w/c^2)]_{ij}U*_{o\,j}/c = [(U'/c)x(\alpha^{uu'}{}_w/c^2)/(1 - V^2{}_{U'W}/c^2)]_{ij}U*_{o\,j}/c$

$= [(U'/c)x(\alpha^{uu'}{}_{u*o}/c^2)/(1 - V^2{}_{U'U*o}/c^2)]_{ij}U*_{o\,j}/c$ (65);

Using $U' = (V_{U'U*o} + U*_o)\cosh\theta_{U'U*o}$ and the fact that $V_{U'U*o}$ is parallel to $(\alpha^{uu'}{}_w)$

$[(U'/c)x(\alpha^{uu'}{}_{u*o}/c^2)/(1 - V^2{}_{U'U*o}/c^2)]_{ij}U*_{o\,j}/c = (\alpha^{uu'}{}_{u*o}/c^2)_i/(1 - V^2{}_{U'U*o})^{3/2}$ (66);

$[(U'/c)x(\alpha^{uu'}{}_w/c^2)/(1 - V^2{}_{U'W}/c^2)]_{ij}U*_{o\,j}/c = (\alpha^{uu'}{}_w/c^2)\cosh^2\theta_{U'W}\cosh\theta_{U'U*o}$ (67);

$(\alpha^{uu'}{}_{u*o}/c^2)\cosh^3\theta_{U'U*o} = (\alpha^{uu'}{}_w/c^2)\cosh^2\theta_{WU*o}\cosh^2\theta_{U'U*o}\cosh\theta_{U'U*o}$ (68);

$(\alpha^{uu'}{}_w/c^2)\cosh^2\theta_{WU*o} = (\alpha^{uu'}{}_{u*o}/c^2)$ (69); note that $\cosh\theta_{U*o\,W}$ is a constant during the virtual motion from U' to U along the straight line in the direction $V\hat{}_{U'U*o}$ which is perpendicular to $V\hat{}_{U*o\,W}$.Multiplying eq69 by $\cosh^2\theta_{U'U*o}$ we get:

$(\alpha^{uu'}{}_{w}/c^2)/(1- V^2{}_{U'W}/c^2) = (\alpha^{uu'}{}_{u*o}/c^2)/(1- V^2{}_{U'U*o}/c^2)$ (70);

This is an important formula.As in the previous section with U_o the angle

$\theta_{UU'} = \theta_{UU*o} - \theta_{U'U*o}$ (71); $|V_{UU'}/c| = |e^{uu'}{}_{u*o}|$ (72);

$|V_{UU'}/c|\cosh\theta_{U*o\,W} = |e^{uu'}{}_{w}|$ (73); we are now able to understand the virtual trajectories in the U', $V\hat{}_{UU'}$. The projection of W perpendicular to the U,U' plane which is of course the U', $V\hat{}_{UU'}$ plane is in the direction $U*_o$ as mentioned previously.The projection is: $T^{UU'} = (E.V\hat{}_{UU'})V\hat{}_{UU'} + (E.U'/-c^2)U'$ eq45.

$T^{UU'} = l\cosh\theta_{U*oW}\, U*_0/c$ (74a);$D^{UU'} = l\sinh\theta_{U*o\,W}\, V\hat{}_{WU*o}$ (74b);

$E= l[\sinh\theta_{U*oW} V\hat{}_{WU*o} + \cosh\theta_{U*oW}\, U*_o/c\,]$ (74c);

The meaning of $T^{UU'} + D^{UU'} = E$ of eq47 is therefore clarified.

8E Virtual timelike trajectories in the U,W plane for the total field $F^{uu'tot}{}_{ij}$.

Until now, we have dealt only with virtual trajectories in the U',W plane or in the U',U plane and while the ray of influence did not necessarily connect with the trajectories,at least the ray started in the present and ended in the future.In the U',W plane for example it left the point charge q' at R' to influence a future field point R or a test charge q at R. The trajectory that we want to describe now have a different character because from the point of W, U is into its future, i.e the influence of q' issuing from W reaches q at U in the future of W . From the point of view of U (charge q), W is in its past,coming from charge q' in its past.The formalism,however makes no distinction as we will presently show,so describing the trajectories correctly is delicate.There is also a problem arising from the conformal factors which appear.

$F^{uu'\,tot}{}_{ij} = (q'/l^2)[WxV_{UW}(1- V^2{}_{U'W})] = -(q'/l^2)[UxV_{WU}(1- V^2{}_{WU'})]$ (75); we will only deal with the right hand side of eq75 and only describe the virtual hyperbolas from the point of view of q at U and not the first term on the left of eq75.

$F^{uu'\,tot}{}_{ij} = (-q'/l^2)[UxV_{WU}(1- V^2{}_{WU})(1- V^2{}_{WU'})/(1- V^2{}_{WU})]$ (76);

Let k = $\cosh\theta_{WU'}/\cosh\theta_{WU}$ (77a); let $l^* = kl$ (77b);

$F^{uu'\ tot}{}_{ij} = (-q'/l^{*2})[U x V_{WU} (1- V^2{}_{WU})]$ (78); eq78 has the same form as that of the nonacceleration field $F^{na}{}_{ij}$ whose trajectories were discussed in detail earlier.The nonacc field had U' instead of U and l instead of l*.We do not necessarily imply that l is transformed into l*, rather it should conservatively looked at as a gimmick to get eq78 to resemble the eq for the nonacc field whose trajectories we know how to obtain and interpret physically.The absorption of the conformal factor into l makes it more difficult to interpret because E=lW/c .

$E^* = l^*W/c$ (79); cannot really be interpreted as beginning at R' and ending at R.The second problem has already been mentioned, the fact that U must look back at q' via –E or – E* . There is also the – sign in front of q'. We will ignore all these problems and use the same procedure as for the nonacc field.We will see how we can adjust, later.

Take the vertical time axis in the U direction and the horizontal initial time direction as V_{WU}^. Take a center R_c as the origin along the horizontal V_{WU}^ direction. The final direction will be in the $-V_{UW}$^ making an angle θ_{WU} with V_{WU}^. The angle θ_{WU} will be the final angle θ_f of the trajectory. The initial tangent to the hyperbola of the foliation is U/c.The final tangent is W/c.The initial unit normal to the hyperbolas is V_{WU}^ .The final unit normal is $-V_{UW}$^. The motion on the first hyperbola starts at R_{hi} on the horizontal axis and ends at R_{hf} on the $-V_{UW}$^ axis.The motion on the second starts at R^*_{hi} ends at R^*_{hf}.The motion on the 3rd starts at R^{**}_{hi} ends at R^{**}_{hf} etc.We have using the results for the nonacc field:

$|R_{hi} - R_c| = l^*/\sinh\theta_f$ (80a); $|R^*_{hi} - R_c| = l^*/\tanh\theta_f$ (80b);

$|R^{**}_{hi} - R_c| = l^*\cosh\theta_f/\tanh\theta_f$ (80c); $l^*W/c = l^*[\sinh\theta_f V_{WU}\hat{} + \cosh\theta_f U/c]$ (81);

The coord time differences along the vertical U/c axis are :

$cT_{fi} = l^*$ (82a); for the 1st hyperbola. $cT^*_{fi} = l^*\cosh\theta_f$ (82b); eq81 shows that this is the time needed and it occurs at the second trajectory, but at $T^*_{initial}$ the 3rd trajectory starts at R^{**}_{hi} on the horizontal axis. We have:

$(R^{**}_{hi} - R_c) - (R_{hi} - R_c) = [l^*\cosh\theta_f/\tanh\theta_f - l^*/\sinh\theta_f] V\hat{}_{WU} = l^*\sinh\theta_f V\hat{}_{WU}$ (83);

From which we get: $[R^*_{hf} - R_{hi}] = [l^* \sinh\theta_{WU} V\text{^}_{WU} + l^* \cosh\theta_{WU} U/c] = l^* W/c$ (84); This shows that the trajectory starts at R_{hi} and ends at R^*_{hf} but we cannot equate the start R_{hi} with R' and R^*_{hf} with R as was the case with the non acceleration field.To take into account that U sees W as in its past not its future, it seems that we should reverse the trajectories. In other word put –U/c for the initial tangent, -W/c for its final, $-V\text{^}_{WU}$ for the initial unit normal.$V\text{^}_{UW}$ for the final unit normal.Put $-R_{hi}$,$-R^*_{hi}$, $-R^{**}_{hi}$,$-R_{hf}$,$-R^*_{hf}$, etc. The times T_i ,T^*_I,T^{**}_I ,would be less negative than the T_f , T^*_f ,T^{**}_f ,etc.Whether this is correct will require more experience with the various trajectories.We are dealing with various reflections.We will not discuss the proper times or other aspects in order not to lengthen this article unduly.

8F The "conformal" 4 velocity U* = kU and $F^{uu'\ tot}_{ij}$.

$F^{uu'\ nonacc}_{ij} = F^{na}_{ij} = [q'/(E.U'/-c)^3][ExU'/c]_{ij}$;

$F^{uu'\ acc}_{ij} = (q'/l)[(W/c)x\ (\alpha^{uu'}{}_{w}/c^2)] = [q'/(E.U'/-c)^3][ExdE/cd\tau']_{ij}$ (85);

We want to put the total field in a similar form.

$F^{uu'\ tot}_{ij} = [q'/(E.U'/-c)^2][(W/c)xV_{UW}] = [q'/(E.U'/-c)^2][(W/c)xU/c]/\cosh\theta_{UW}$

$= [q'/(E.U'/-c)^2][(W/c)x(U/c)/\cosh\theta_{U'W}](\cosh\theta_{U'W}/\cosh\theta_{UW})$

$= [q'/(E.U'/-c)^3][ExkU/c]_{ij}$ (86);

$k = [\cosh\theta_{U'W}/\cosh\theta_{UW}]$ eq 77a is the conformal factor that was used in the definition of l*=kl, in the previous subsection when we discussed the hyperbolic trajectories. Set U* =kU (87); we get for $F^{uu'\ tot}_{ij} = F^{na}_{ij} + F^{uu'\ acc}_{ij}$

$[q'/(E.U'/-c)^3][ExU^*/c] = [q'/(E.U'/-c)^3][ExU'/c] + [q'/(E.U'/-c)^3][ExdE/cd\tau']$ (88);

We can set : $U^*/c = [U'/c + dE/dc\tau']$ (89);

Since $dE/cd\tau' = (U'/c)x(a^{uu'}/c^2)_{ij}E_j = [(E.a^{uu'}/c^2)U'/c + (a^{uu'}/c^2)(E.U'/-c)]$;

$U^*/c = U'/c + [(E.a^{uu'}/c^2)U'/c + (a^{uu'}/c^2)(E.U'/-c)]$ (90); $E.U^* = E.U'$ (91);

The purpose of this exercise is to show that we get exactly the same form with these new fields as we get with the real total and acceleration field as was mentioned before.Now we have the exact equations.We only need to substitute a 4 velocity U'_h for U, and put the real 4 acceleration a' of q' and the generalization of the real 3 acceleration a'(t') instead of $\alpha^{uu'}{}_{w}/c^2$. Deriving U'_h from the real total field, however is tricky and will be done in the next article.

8G dE/dcτ . It is not necessary to redo all the previous calculations.It suffices to use the relation dE/cdτ = (dE/cdτ')dτ'/dτ =(1/k)dE/cdτ' with k=E.U'/E.U $=\cosh\theta_{U'W}/\cosh\theta_{UW}$ .From eq8.85 :

$F^{uu'\,acc}{}_{ij} = [q'/(E.U'/-c)^3][ExdE/dc\tau'];$

$[q'/(E.U'/-c)^3][ExdE/cd\tau] = (1/k)F^{uu'\,acc}{}_{ij}$ (92); to be consistent we must take

$(1/k)F^{uu'na}{}_{ij} = (1/k)F^{na}{}_{ij} =[q'/(E.U'/-c)^3][Ex(1/k)U'/c]_{ij}$ (93);

$(1/k)F^{uu'tot}{}_{ij} = [q'/(E.U'/-c)^3][ExU/c]_{ij}$ (94); we have used eq8.86.

$(1/k)F^{uu'acc}{}_{ij} = (1/k) F^{uu'tot}{}_{ij} - (1/k)F^{uu'na}{}_{ij}$ (95);

$[q'/(E.U'/-c)^3][ExdE/cd\tau] = [q'/(E.U'/-c)^3][ExU/c]- [q'/(E.U'/-c)^3][Ex(1/k)U'/c]$ (96);

This is the desired formula.Remember that $dE/cd\tau =[V_{UW} - V_{U'W}]/c(1-V^2{}_{UW}/c^2)^{1/2}$ We can recalculate all the entities discussed in the previous subsection from that.There are a few more things worth noting.

$d\tau' =(1-v'^2/c^2)^{1/2}dt'$ and $d\tau= (1-v^2/c^2)^{1/2}dt$ can be generalized.

Let $d\tau'=(1-V^2{}_{U'W})^{1/2}dT_{u'w}$ (97a); $d\tau = (1-V^2{}_{UW}/c^2)^{1/2}dT_{uw}$ (97b);

$\cosh\theta_{U'W}dc\tau' = cdT_{u'w}$ (98a); $\cosh\theta_{UW}dc\tau = cdT_{uw}$ (98b);

$dE/\cosh\theta_{U'W}dc\tau' = dE/cdT_{u'w} = (V_{UW} - V_{U'W})/c$ (99a);

$dE/\cosh\theta_{UW}dc\tau = dE/cdT_{uw} = (V_{UW} - V_{U'W})/c$ (99b); $dT_{u'w} =dT_{uw}$ (100);

$d(W/c)/cdT_{u'w} = d(W/c)/cdT_{uw} =(V_{UW}-V_{U'W})/cl =(\alpha^{uu'}{}_{w}/c^2)/(1-V^2{}_{U'W}/c^2)$(101);

One can obtain the LW potential.

$dR'/dcT_{u'w} = (dR'/cd\tau')(d\tau'/dT_{u'w}) = (U'/c)(-c^2/U'.W)$ (101a);

$(q'/l)dR'/cdT_{u'w} = q'(U'/c)/(E.U'/-c) = -q'[U'/E.U'] = A_{LW}$ (101b);

$dR/cT_{uw} = (dR/cd\tau)(d\tau/dT_{uw}) = (U/c)(-c^2/U.W)$ (101c);

$(q/l)dR/cdT_{uw} = q(U/c)/(E.U/-c) = -q[U/E.U] = A_{LW}^u$ (101d);

We will not discuss the 2nd derivatives $d^2E/d\tau'^2, d^2E/d\tau^2, d^2E/d\tau d\tau'$ because the 4 accelerations of q and q' are entangled in a complicated way which makes it difficult to easily interpret them.A detailed exposition will be left to a future article.The same is true of the 2nd derivatives with respect to T_{uw} and $T_{u'w}$.

8H Quadratic expressions.

We will only briefly discuss the energy stress tensor of the total field $F^{uu'tot}_{ij}$.F^{na}_{ij} has already been discussed in details. What needs to be pointed out is some profound difference between the real total field , the real acceleration field and the formal fields and energy stress tensor.If there is no acceleration, the total field becomes the nonacceleration field and the the same is true of their respective energy stress tensor.No such thing accurs with the formal field , the total formal field does not merge into the nonacceleration field.The physical meaning of the formal total field is unclear, yet it closely resembles the real total field in a formal way.For the real total field ,its projection on U/c is of major importance but here it has only geometrical meaning.The importance of the formal fields seem to lie in their geometrical meaning.This may change in the future as they are better understood.We can write the total formal field as:

$F^{uu'tot}_{ij} = (q'/l*^2)[(W/c)x(V_{UW}/c)(1-V^2_{UW}/c^2)] = (-q'/l*^2)[(U/c)x(V_{WU}/c)(1-V^2_{WU}/c^2)$

$=(q'/l*^2)[\tanh\theta_{UW}/\cosh^2\theta_{UW}][(W/c)xV\char`^_{UW}]; l* = kl;$

$T^{uu'tot}_{ij} = (1/2)T^{uu'tot}[\delta_{ij} -2[(W_iW_j/-c^2) + V\char`^^i_{UW}V\char`^^j_{UW}]$

$= (1/2)T^{uu'tot}[\delta_{ij} -2[(U_iU_j/-c^2) + V\char`^^i_{WU}V\char`^^j_{WU}]$ (102);

$T^{uu'tot} = (q'^2/l*^4)(1/4)[(d^2\tanh\theta/d\theta^2)_{\theta=\theta uw}]^2$

$= (q'^2/l*^4)[\tanh\theta_{UW}/\cosh^2\theta_{UW}]^2$ (103);

We can project W onto the U,U' plane and obtain a 4 velocity which we could label $U^{uu',w}{}_o$ .By comparison U projected onto the U',W plane which we used for the Lorentz force and which we called $U_o = U^{u'w,u}{}_o$.We can project U' onto the U,W plane.We shall skip the details of these projections in this article.

9 Conclusion.

We have shown that it is possible to have a consistent electrodynamics with a timelike interval of interaction and that electromagnetism becomes deeply connected with the geodesics of the Beltrami Klein model of 3 dim hyperbolic geometry in velocity space.A Minkowski space-time foliation of hyperboloids of two sheets describes the hyperbolic geometry.The unit normal of the hyperboloids (which are 4 velocities) determine the initial and final points of the geodesics segments between any two 4 velocities,including the formal 4 velocity of interaction. Electromagnetism seems to be interested in the geodesic distances in the tangent space of the hyperboloids and in the accelerations of these geodesics (the acceleration term).The acceleration terms are the fields.The inverse of the fields also have meanings and are connected with a foliation of hyperboloids of one sheet described by a Rindler-like coordinate system, with the 4-velocities representing the initial and final tangents of virtual timelike hyperbolic motions.The interval of propagation or its projections connect one timelike hyperbolic motion to another and can be thought of having transmitted the influence from the source charge to the field point or the test charge.It must be emphasized that we are not dealing with emission of virtual particles, we are dealing with geodesic segments and the acceleration part of the geodesic segments.Can these be connected to emissions and annihilations of particles? This is not obvious at this time but see refs10,11.The existence of a timelike e.m. does not seem to have been considered by other authors.The results presented in this work and in a previous one (ref.1) indicate that there is a serious gap in our understanding of electromagnetism and that this will eventually transfer to our understanding of cosmology and particle physics.The geometric methods developed here are powerful and completely general. They could be used in hyperbolic geometry and even in relativistic scattering cross sections of binary systems. (refs10,11).

References.

1 John Suss Timelike Electrodynamics Vixra 11/27/2020

Total field eq 22. Non acceleration field eq 26. Acceleration field eq 26bis.

2 V.Fock The Theory of Space Time and Gravitation. Pergamon Press 1959

Relative velocity sect16,p37 eqs 16.07-16.13.sect.17.The Lobachevsky-Einstein velocity space.Eq 17.01,02 for the metric.Eq17.29,17.30 for the geodesic.Eq17.33 Einstein addition (subtraction) formula.

3 John G. Ratcliffe. Foundations of Hyperbolic Manifolds.Springer-Verlag 1994.

Chapter 3 Hyperbolic Geometry.Chapter 4 Inversive Geometry.

4 J.L. Synge. Relativity: the special theory. North Holland Publishing Co. 1965

Chapt.5 ,sect.5,p131 eq65 and fig10.Definition of projective relative velocities in term of 4 velocities.

5 Svante Janson. Riemannian Geometry: some examples including map projections.122p revised 3/15/20

Example 7.7 Klein ball (disk) model starting p80.Eq 7.86 Christoffel symbols.

Eq 7.92 geodesic acceleration eq. Eq 7.93 geodesic. Eq 7.82 metric tensor.

6 David Tong.Electromagnetism.University of Cambridge.Lent term 2015

Lienard Wiechert potential bottom of p152 no eq number.Total field eq6.54 p153.

7 L.D. Landau and E.M. Lifschitz.The Classical Theory of Fields.Revised 2nd edit Pergamon Press.Addison-Wesley Publishing Co. 1962

Relative velocity p17 problem1 .Transformation of relative cross sections p37 sect.12 eqs 12.1-12.5. Lienard Wiechert potential p185 sect.63 eq63.3

8 C.Moeller.The Theory of Relativity.Oxford U.Press 1962 ed.

Chapt.IV,sect.47 p121-123.Successive rest systems of a particle in arbitrary rectilinear motion and in constant circular motion.eq160.

Chapt.VIII,sect.97.Rigid frames of reference moving in the direction of the X-axis.p255-257.

Chapt.V,sect.60 p159.The e.m. energy-momentum tensor.eq.106.

Lienard Wiechert potential.Chapter V, sect.56,p149 eq51.Total field p150 eq58.see also p150 eq57.

9 Richard A. Mould. Basic Relativity. Springer-Verlag 1994.

Sect.8.4 Uniformly accelerated coordinates Fig 8.7

10 Anton A Baldin. The Lobachevsky space in relativistic nuclear physics. LHE, JINR, Dubna

Section 2 The relative velocity space. General characteristics of particle distributions. Eqs 5,6,7 .Section 3.Geometric characteristics of particle distributions in the rapidity space.Defines total,longitudinal and transverse rapidity eqs 8-11 Also fig7.

11 Mirco Cannoni.Lorentz Invariant Relative Velocity and Relativistic Binary Collisions. Arxiv 5 May 2016.

Eq18,19 relative velocity.Fig 1 scattering diagram for ultrarelativistic particles in Beltrami Klein interior disk.Eq 43,44 invariant relativistic flux.