

A Relational Analysis of Quantum Symmetry

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Abstract. Carlo Rovelli’s “relational interpretation” of quantum mechanics tells us that our understanding of quantum states is limited to their interactions with other quantum states. This implies that we have no understanding of the symmetry properties of a state vector except when considered in relation to at least one other state vector. The $SU(3)\times SU(2)\times U(1)$ symmetry of the Standard Model is derived from their interactions and therefore cannot necessarily be applied to a single state vector. Steven Weinberg showed that mixed density matrices can have symmetries that are not available to state vectors.

We explore the symmetry properties of finite symmetry groups for mixed density matrices. Just as mixed density matrices can define mixed states that require multiple state vectors, we find that the symmetries of mixed density matrices define “mixed symmetries” that are similar in structure to the symmetry of the Standard Model. We explore the point group symmetries and show one that gives the Standard Model symmetry. With this symmetry, we can generalize the Pauli spin matrices to a set that has irreducible representations matching the Standard Model plus a dark matter candidate.

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1. Introduction

Carlo Rovelli’s paper “Relational Quantum Mechanics” [1] uses a simple example to show that interactions between two quantum (or classical) systems are more complicated than the sum of the interactions inside the individual systems. I.e. since absolute positions cannot be determined, the number of degrees of freedom in a 1-d collection of positions of N particles is $N - 1$. Two such collections have twice as many degrees of freedom $2N - 2$, but when

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the collections interact an addition degree of freedom appears, the distance between the collections giving $2N - 1$. This shows that isolated systems are simpler than interacting systems. The consequence of this that we will be exploring is the conclusion that the symmetry of an interaction can be different (and generally more complicated) than the symmetry of the individual states.

In the usual quantum mechanics, a pure density matrix is mathematically created as a product of two state vectors, for example

$$\rho_{+y} = | + y \rangle \langle + y | \quad (1.1)$$

where “+ y ” indicates spin-1/2 in the + y direction. Since this is an interaction between two quantum states, that is, the bra and the ket, the relational interpretation suggests that we consider the symmetry of the density matrix as more fundamental than symmetry of the kets and bras. Does this make a difference? The bra is a 2 of $SU(2)$ while the ket is a $\bar{2}$, the density matrix consists of all $2 \times 2 = 4$ products of an element of the bra and an element of the ket. Therefore the density matrix transforms under $SU(2)$ as

$$2 \times \bar{2} = 3 + 1. \quad (1.2)$$

This symmetry decomposition is compatible with the fact that given a real unit vector $\vec{u} = (u_x, u_y, u_z)$, the pure density matrix corresponding to spin in the \vec{u} direction is

$$\rho_u = \frac{1}{2} \begin{pmatrix} +u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1.3)$$

that is, the pure density matrix is the sum of two components, the first of which transforms as a real 3-vector (u_x, u_y, u_z) and the second of which transforms as a scalar. This simple example shows that the symmetry of density matrices is not the same as the symmetry of state vectors. Steven Weinberg makes this last point in his 2014 paper “Quantum Mechanics Without State Vectors” [2] and also includes a list of quantum oddities that are eliminated in the density matrix formulation.¹

This spin-1/2 example calls into question the assumption that individual electrons have $SU(2)$ symmetry. Considered as density matrices, their symmetry is more naturally the symmetry of real unit vectors, $SO(3)$. We use $SU(2)$ when we create a mathematical fiction and consider what happens when we violate Rovelli’s relational interpretation by splitting interacting electron state vectors into individual state vectors. As far as spin-1/2 goes, this argument has no physical consequences. Density matrices and state vectors are both fully sufficient to model spin-1/2 particles. But we will meet interesting results when we apply the same reasoning to the Standard Model symmetry.

¹From Weinberg’s abstract: “This change in the description of physical states opens up a large variety of new ways that the density matrix may transform under various symmetries, different from the unitary transformations of ordinary quantum mechanics.”

Our concern is with a finite symmetry group G of size N with group members given by $g \in G$. With density matrices ρ given by their matrix² elements ρ_{mn} , the transformation defined by g is a set of complex numbers $K_{m'm,n'n}[g]$, and the transformation is

$$g(\rho)_{m'n'} = \sum_{mn} K_{m'm,n'n}[g] \rho_{mn}, \quad (1.4)$$

given as equation (3) in Weinberg.[2] We can see that Weinberg's notation includes the situation we will cover. Different from Weinberg, we will analyze the symmetry using group theory. For Abelian finite groups, density matrices and state vectors will have the same symmetries but it is advantageous to cover the non Abelian situation as a generalization of the Abelian case so we will begin with Abelian symmetries.

2. Density Matrix Symmetry, Abelian Case

In addition to symmetries different from state vectors, such as the electron spin symmetry being $SO(3)$ instead of $SU(2)$, density matrices live in the operator space and so allow some improvements in describing the relationship between symmetry and quantum states, particularly in the Abelian case. Note that $SU(2)$ and $SO(3)$ are non Abelian symmetries; for the Abelian symmetries, density matrices and state vectors have the same properties.

Mixed density matrices allow thermodynamics so we can unify an understanding of entropy, symmetry and quantum states that is more difficult in state vectors. These are also closely related to the Fourier transforms and the group character tables.

For simplicity, we will discuss cyclic Abelian symmetries; the same ideas apply to all finite Abelian symmetry groups. Let G be a cyclic Abelian group of size N which is generated by s so that $G = \{s^n | 0 \leq n < N\}$. These symmetry operators act on the state vectors by left multiplication:

$$|s^2\rangle = s s |s^0\rangle, \quad (2.1)$$

and the corresponding density matrix symmetry transformation:

$$\rho_2 = s s \rho_0 s^\dagger s^\dagger, \quad (2.2)$$

where $s^\dagger s = s s^\dagger = 1$. Choosing the basis where the state vector $|s^j\rangle$ has 1 in the j th position and the rest zero, we have s with 1s below the diagonal and at the top right corner, and zero everywhere else. I.e., for $N = 4$ we have

$$s = s^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.3)$$

The various powers of s have the diagonal 1s in other positions.

²For simplicity, we are not specifying the size of the matrices here.

The N symmetry operators are not enough to form a basis for the $N \times N$ matrices but it turns out that they are just enough to form a basis for the N pure density matrices. The easiest one is obtained by summing over all the powers of s and dividing by N . So for $N = 4$ we have

$$\rho_0 = (s^0 + s^1 + s^2 + s^3)/4 = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad (2.4)$$

For $0 < n < N$, the trace of s^n is zero while the trace of s^0 is N . So the above pure density matrix has $1/N = 1/4$ as the coefficient for s^0 . And the other s^j take this same coefficient.

Learned readers might notice that the character table for a finite symmetry group has a row of 1s for its top entry. In fact, at least for Abelian symmetries, each row of the character table, when divided by the size of the group N , defines the coefficients for a pure density matrix. Continuing our example, the cyclic group of size 4, the point group C_4 , has character table:

C_4	s^0	s^1	s^2	s^3
$4\rho_0$	+1	+1	+1	+1
$4\rho_1$	+1	+i	-1	-i
$4\rho_2$	+1	-1	+1	-1
$4\rho_3$	+1	-i	-1	+i

(2.5)

where we've modified the usual character table labels by replacing the irrep labels with the corresponding pure density matrix, and the group elements with powers of the generating element s^1 . These coefficients define the four pure density matrices associated with the C_4 symmetry, for example:

$$\rho_3 = (+1s^0 - is^1 - 1s^2 + is^3)/4 = \frac{1}{4} \begin{pmatrix} +1 & +i & -1 & -i \\ -i & +1 & +i & -1 \\ -1 & -i & +1 & +i \\ +i & -1 & -i & +1 \end{pmatrix}. \quad (2.6)$$

The point here is that the symmetry group's character table defines the pure density matrices.

We can convert the ρ_3 density matrix to a state vector by taking any of its non zero columns and normalizing. The choice of four columns gives us four different phases but these don't mean much in state vectors so we'll take the second column and we have:

$$|3\rangle = 0.5 \begin{pmatrix} +i \\ +1 \\ -i \\ -1 \end{pmatrix} \quad (2.7)$$

The non Abelian case will be somewhat different from this.

Learned readers might also notice that the $N \times N$ coefficients defining the irreps and the pure density matrices is also the matrix for the inverse of

the discrete Fourier transform:

$$F_4 = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -i & -1 & +i \\ +1 & -1 & +1 & -1 \\ +1 & +i & -1 & -i \end{pmatrix}, \quad (2.8)$$

with normalization chosen to normalize the resulting state vectors. The inverse Fourier transform is given by the complex transpose:

$$F_4^{-1} = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & +i & -1 & -i \\ +1 & -1 & +1 & -1 \\ +1 & -i & -1 & +i \end{pmatrix}. \quad (2.9)$$

If we act on our ket $|3\rangle$ with the Fourier transform F_4 we obtain:

$$F_4 |3\rangle = \frac{1}{2} \begin{pmatrix} +1 & +1 & +1 & +1 \\ +1 & -i & -1 & +i \\ +1 & -1 & +1 & -1 \\ +1 & +i & -1 & -i \end{pmatrix} \frac{1}{2} \begin{pmatrix} +i \\ +1 \\ -i \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ +i \end{pmatrix}. \quad (2.10)$$

Similarly, the bra $\langle 3|$ is transformed by F_4^{-1} to $(0, 0, 0, -i)$ and the density matrix ρ_3 is transformed to

$$F_4 \rho_3 F_4^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.11)$$

Thus the Fourier transform diagonalizes the pure density matrices. Note that a mixed density matrices will have at least two off diagonal entries non zero.

Finally, it's very important to see what the Fourier transform does to the symmetry group elements. Of course it leaves s^0 unchanged, so let's apply it to s^1 :

$$F_4 s^1 F_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +i \end{pmatrix}. \quad (2.12)$$

We see that the Fourier transform diagonalizes the symmetry group. Of course diagonalization is only possible for an Abelian symmetry group; we will be looking for a generalization of the Fourier transform which does as much diagonalization as possible.

Given a set of probabilities $0 \leq p_j \leq 1$ that sum to 1, we define the mixed density matrix defined with these probabilities as

$$\rho_p = \sum_j p_j \rho_j, \quad (2.13)$$

so that the pure density matrices are given when only one of the p_j are 1 and the rest are zero. The Boltzmann relation of thermodynamics tells us that

the probabilities p_j are related to energy by an exponential. This is in the context of a Hamiltonian H where we have

$$\rho_H(T) \propto \exp(-H/T), \quad (2.14)$$

for a temperature T and we've selected units where the Boltzmann constant k_B is unity. The proportionality constant is determined by the requirement that the trace of ρ is 1, that is, that the probabilities sum to 1. We can square to obtain

$$(\rho_H(T))^2 \propto (\exp(-H/T))^2 = \exp(-H/(T/2)), \quad (2.15)$$

and we see that squaring gives a result that is proportional to the density matrix at half the temperature. The process can be continued indefinitely. To avoid nasty surprises, we should divide by the trace after squaring to normalize the probabilities so they sum to 1.

Under the assumption that there was a single p_j that was larger than any other, the limit as the temperature goes to zero is:

$$\rho_{(H(T))^\infty} \propto \rho_H(0) = |j\rangle\langle j|. \quad (2.16)$$

Thus we can find the pure density matrices defined on a space by choosing random mixed density matrices and cooling them down to find their zero temperature limits.

The Von Neumann entropy is defined as

$$S(\rho) = -\text{Tr } \rho \log \rho = \sum p_k \log(p_k). \quad (2.17)$$

Degeneracies complicate the entropy of systems with non Abelian symmetry[3] but the Abelian case is straight forward. In the absence of degeneracies, the p_k are uniquely defined and the entropy is minimized to zero when exactly one of the p_k is one. These are the pure states.

3. Density Matrix Symmetry, non Abelian

Our problem is that of finding the pure density matrix quantum states from a non Abelian symmetry. Reviewing the situation for Abelian symmetry analysis of the previous section the pure states can be found by cooling random states to their zero temperature limit. The result of this are the various irreducible representations of the symmetry. These irreducible representations are defined by the character table of the group and the Fourier transformation is a transformation from the coordinate space to these irreducible representations. So in the non Abelian case, what we are looking for is a generalization of the Fourier transform.

As far as the generalized Fourier transform goes, in physics the continuous non Abelian symmetries are better understood than the finite non Abelian symmetries. So for pedagogical reasons it makes sense to describe the generalized Fourier transform for $\text{SO}(3)$. This will help cement the ideas we need for the finite symmetries. First let's discuss an Abelian continuous symmetry then $\text{SO}(3)$.

For quantum states defined on the unit circle, the symmetry is given by rotations of the circle by an angle θ . These are an Abelian symmetry group so the solution for the pure states on the circle is defined by a Fourier transform:

$$\hat{\psi}(k) = \int \psi(\theta) \exp(ik\theta) d\theta \quad (3.1)$$

This converts a wave function that depends on the coordinate θ into one that depends on wave number k . The pure states are the wave functions that are given by a single value of k .

For our first non Abelian symmetry, consider the quantum states defined on the unit sphere, that is, the points in 3-space that are a distance 1 from the origin. We will label these points with the usual two angles (θ, ϕ) . The symmetry is SO(3) and the irreducible representations of the symmetry are in the form $1 + 3 + 5 + 7 + 9 + \dots$. The “1” irrep is a discrete point while the other irreps are all degenerate.

To put SO(3) into the form of a generalized Fourier transform the irreps correspond to spherical harmonics $Y_l^m(\theta, \phi)$. Thus the spherical harmonics define a generalized Fourier transform which takes us from (θ, ϕ) coordinate space to (l, m) wave number space. Again any wave function that corresponds to a single value of (l, m) is a pure state. The first eigenfunction equation is

$$L^2 Y_l^m(\theta, \phi) = l(l+1) Y_l^m(\theta, \phi) \quad (3.2)$$

so that Y_l^m is an eigenfunction of the square of total orbital angular momentum L^2 . The corresponding equation for the m eigenvalue is

$$L_z Y_l^m(\theta, \phi) = m Y_l^m(\theta, \phi). \quad (3.3)$$

Thus we see that the m eigenvalue equation requires the specification of a direction z . Changing the direction will form linear superpositions of the Y_l^m with the same l but different m values. Therefore the spherical harmonics are degenerate over m . That is, we can make pure states with mixtures of the same l but different m . This corresponds with the fact that the SO(3) irreps $3, 5, 7, 9, \dots$ are degenerate with degeneracy given by the number of different m each corresponds to. That is, the 3 irrep is triply degenerate, etc.

So in order to solve the corresponding quantum state problem for a finite symmetry group we need to find the generalization of the Fourier transform for a non Abelian finite symmetry. We did this for the Abelian case by using the character table. This worked because the character tables for an Abelian symmetry are $N \times N$ tables when there are N elements in the group. This happened because each of the group elements was in its own “conjugacy class”. Two group elements a and b are in the same conjugacy class when there is another element c with $ac = cb$. In an Abelian symmetry, one cancels c from this equation and the result is that the conjugacy classes each have a single element.

The smallest non Abelian finite group has size 6; we will use it to show how to generalize the Fourier transform to non Abelian finite symmetries. This group is the permutation group on three elements, S_3 . It is the same as

the dihedral group of degree 3, D_3 . We will use permutation notation, so the group elements are:

$$S_3 = \{ [], [123], [132], [12], [13], [23] \}. \quad (3.4)$$

There are three conjugacy classes, $\{ [] \}$, $\{ [123], [132] \}$ and $\{ [12], [13], [23] \}$ so there are three irreps and the character table is 3×3 :

$$\begin{array}{c|ccc} S_3 & \{ [] \} & \{ [123], [132] \} & \{ [12], [13], [23] \} \\ \hline A_1 & 1 & +1 & +1 \\ A_2 & 1 & +1 & -1 \\ E & 2 & -1 & 0 \end{array}, \quad (3.5)$$

where we've borrowed the names for the irreps, A_1 , A_2 and E from those traditionally used for the D_3 point group. In the point group notation, E is used for irreps which are doubly degenerate, and we see that the character of the identity $[]$ is 2 so that it is indeed doubly degenerate.

To get a generalized Fourier transform, we need to expand the character table above, Equation (3.5) to 6×6 . It's obvious how to expand the columns to 6, we simply replace the conjugacy classes with the individual group elements. This gives an expanded character table:

$$\begin{array}{c|cccccc} S_3 & [] & [123] & [132] & [12] & [13] & [23] \\ \hline A_1 & 1 & +1 & +1 & +1 & +1 & +1 \\ A_2 & 1 & +1 & +1 & -1 & -1 & -1 \\ E & 2 & -1 & -1 & 0 & 0 & 0 \end{array}. \quad (3.6)$$

We need to supply three missing rows. Before continuing, it is useful to compute the products of these three irreps:

$$\begin{array}{c|ccc} * & A_1 & A_2 & E \\ \hline A_1 & 6A_1 & 0 & 0 \\ A_2 & 0 & 6A_2 & 0 \\ E & 0 & 0 & 3E \end{array} \quad (3.7)$$

So if we divide A_1 and A_2 by 6 and E by 3 they will be projection operators. In general, one gets the projection operators for a non Abelian finite group by dividing by the group size (in this case 6) and multiplying by the character of the identity (in this case, 1, 1 and 2).

The entries in a character table are the traces of the given representative of the symmetry. This is a clue that the missing rows will have zero trace and can be written as an $SU(2)$ doublet. Examining the table, we see that there is one missing row that must distinguish between the $[123]$ and $[132]$ elements. We can use any non zero multiple of $[123] - [132]$, but let's look ahead a little and pick a multiple convenient for $SU(2)$. Choosing the multiple as a complex number α and squaring we find

$$\begin{aligned} (\alpha[123] - \alpha[132])^2 &= \alpha^2([123]^2 - [123][132] - [132][123] + [132]^2), \\ &= \alpha^2(-2 + [132] + [123]) = -\alpha^2 E \end{aligned} \quad (3.8)$$

Now the Pauli spin matrix σ_x squares to 1 which is a projection operator. The corresponding projection operator here is $E/3$ so we can arrange for our

first new row to correspond to σ_x by choosing $\alpha = i/\sqrt{3}$. Let's rewrite the character table with this new row and also adjust the other irreps so that they are projection operators:

$$\begin{array}{c|cccccc}
 S_3 & [1] & [123] & [132] & [12] & [13] & [23] \\
 \hline
 A_1/6 & 1/6 & +1/6 & +1/6 & +1/6 & +1/6 & +1/6 \\
 A_2/6 & 1/6 & +1/6 & +1/6 & -1/6 & -1/6 & -1/6 \\
 E/3 & 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\
 \sigma_x & 0 & i/\sqrt{3} & -i/\sqrt{3} & 0 & 0 & 0
 \end{array} \quad (3.9)$$

The next two rows will arise from differences in [12], [13] and [23]. Choosing $\sigma_y = \alpha([12] - [13])$ and requiring that σ_y square to $E/3$ we find $\alpha = 1/\sqrt{3}$ so we can set $\sigma_y = ([12] - [13])/ \sqrt{3}$ and define σ_z by $\sigma_z = i\sigma_x\sigma_y$ to obtain

$$\begin{array}{c|cccccc}
 S_3/6 & [1] & [123] & [132] & [12] & [13] & [23] \\
 \hline
 A_1/6 & 1/6 & +1/6 & +1/6 & +1/6 & +1/6 & +1/6 \\
 A_2/6 & 1/6 & +1/6 & +1/6 & -1/6 & -1/6 & -1/6 \\
 E/3 & 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\
 \sigma_x & 0 & i/\sqrt{3} & -i/\sqrt{3} & 0 & 0 & 0 \\
 \sigma_y & 0 & 0 & 0 & 1/\sqrt{3} & -1/\sqrt{3} & 0 \\
 \sigma_z & 0 & 0 & 0 & 1/3 & 1/3 & -2/3
 \end{array} \quad (3.10)$$

This table defines the “generalized Fourier transform” for the D_3 finite symmetry group. It tells us that this space supports two singlets, $A_1/6$ and $A_2/6$, and an $SU(2)$ doublet with $E/6$ as the unit matrix. This is a “mixed symmetry”.

When the symmetry is Abelian, the number of elements in the symmetry group is the same as the number of positions in the coordinate space. That is, to get to any particular position there is a unique symmetry element so they can be put into correspondence. Because of that, we were able to define the Fourier transform so that it took the coordinate positions to irreducible representations just as it took the density matrices to irreducible representations and the symmetry from the density matrix and state vector calculations were identical. With the non Abelian case the numerical coincidence no longer applies so the “mixed symmetry” is purely a density matrix symmetry. In the case of D_3 , the size of the coordinate space is 6, and the generalized Fourier transform takes this to density matrices as

$$6 = 1^2 + 1^2 + 2^2. \quad (3.11)$$

If the group had been Abelian there would need to be six irreps and the Fourier transform would have diagonalized them with an equation of $6 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$. Other non Abelian finite symmetries will also have sizes that become sums of squares.

Similar calculations can be done for other non Abelian finite symmetry groups. In each case, we will obtain a generalized Fourier transform that will tell us the quantum states supported by a space with that symmetry. A more general problem (and the usual use of point group symmetries) is to use a point group symmetry to classify properties of a chemical with that

symmetry. That situation is more complicated in that each irrep may not appear or may appear multiple times. Since the method we've used here gives each irrep exactly once, it has been used as a convenient method of finding the possible irreps of a point group.[4]

This author came upon the idea of "mixed symmetry" only after extensive computer calculations. As mentioned in the Abelian section, we can find the particle content of an algebra by squaring (cooling) from random high temperature initial states. It was only after solving several non Abelian finite groups that the group theory described here became evident. The ideas could then be extended to the finite group of size 144 we propose for the Standard Model.

To show how the calculational method of finding density matrix particle content works, let's apply the method to D_3 , but use S_3 notation. The first step is to write software that allows multiplication in the complex S_3 algebra. The elements of this algebra are complex 6-vectors where the 6 entries are indexed by the 6 group elements $\{[], [123], [132], [12], [13], [23]\}$.

For example, suppose the 6 complex numbers are $2/3, -1/3, -1/3, 0, 0$ and 0 . These happen to be the coefficients for $E/3$ given above in the table of projection operators given above in Equation (3.10). We use the finite group elements as the basis for the vector space and write this element as:

$$P_E = (2[] - [123] - [132])/3. \quad (3.12)$$

We can check that this is a projection operator by squaring:

$$\begin{aligned} P_E^2 &= (4[]^2 + [123]^2 + [132]^2 - 2[][123] - 2[123][] \\ &\quad - 2[][132] - 2[132][] + [123][132] + [132][123])/9, \\ &= (4[] + [132] + [123] - 2[123] - 2[123] - 2[132] - 2[132] + [] + [])/9, \\ &= (6[] - 3[123] - 3[132])/9, \\ &= P_E \end{aligned} \quad (3.13)$$

So the first step is to verify that your computer algebra satisfies the projection operators. Also verify that they multiply each other to give zero and that the Pauli spin matrix analogs have the correct commutation relations and square to P_E .

In any of these group algebras, the high temperature limit is proportional to the identity. The proportionality constant arranges for the trace to be 1 and so is the group size N . In the D_3 case, the high temperature limit is $[/math]/6. To find the pure states, we will begin with states near the high temperature limit. To keep the trace 1, we can keep the coefficient of $[/math] as $1/6$ and choose small random numbers for the remaining 5 coefficients.$$

Another requirement for a density matrix is that it be Hermitian. For an element defined by $\sum a_g g$ where a_g are complex coefficients, the Hermitian requirement is that if $gh = 1$, then $a_g = a_h^\dagger$. For the D_3 group, this means that $A_{[]} , A_{[12]} , A_{[12]}$ and $A_{[23]}$ are real and that $A_{[123]}$ is the complex conjugate of $A_{[132]}$.

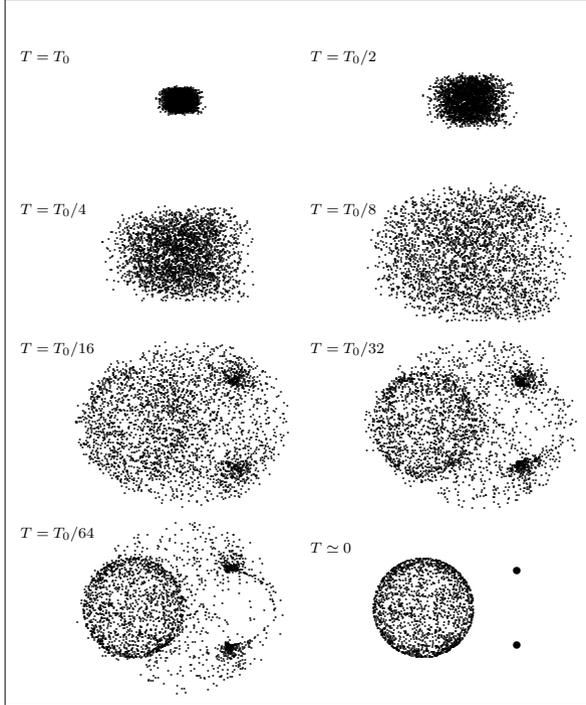


FIGURE 1. Beginning with 3000 random density matrices near the high temperature limit $T = T_0$, we square the density matrix six times to show the states beginning to converge to the two singletons A_1 and A_2 , and the $SU(2)$ doublet E . Of the six degrees of freedom in the D_3 algebra, we choose the x and y axes so as to spread A_1 , A_2 and E apart, and to show the Bloch sphere for E . Continuing the cooling process, the final image $T \simeq 0$ shows the cold temperature limit (pure states).

With these requirements, a typical Hermitian mixed density state near the high temperature limit is

$$\rho_{T_0} = []/6 + ((3 + 4i)[123] + (3 - 4i)[132] + 2[12] + 3[13] - 4[23])/100. \quad (3.14)$$

Next we cool this state by repeatedly squaring it and adjusting it so that its trace is again 1. This adjustment is done by dividing by the $[]$ coefficient times 6 so that the adjusted state will have a $[]$ coefficient of $1/6$. As we do this, the state eventually approaches stability. There are three possibilities. This limit will be either $A_1/6$, $A_2/6$, or will be on the $SU(2)$ Bloch sphere whose elements are defined by:

$$\rho_u = (E/3 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z)/2, \quad (3.15)$$

where (u_x, u_y, u_z) is a real unit 3-vector and $E/3$, σ_x , σ_y and σ_z are taken from Equation (3.10).

Beginning with a few thousand different random initial conditions, we get a picture of how mixed density matrices converge to the A_1 and A_2 singlets and the E $SU(2)$ doublet in Figure (1). To get the nice graph it helps to know the generalized Fourier transform. It's also possible, as the author did, to solve the quantum state problem for an algebra like this one without knowledge of the generalized Fourier transform. One runs a few thousand initial random high temperature states to their cold limit and looks for coincidences. Where there are a lot of identical final states, that is a singlet state. Doublet and higher states appear as clusters of states on the surface of a region. Doubly degenerate states will appear as Bloch spheres, that is, as the surfaces of 3-dimensional spheres. Higher $SU(N)$ will be larger dimensional.

It should be clear that the $SU(2)$ doublet in D_3 is a candidate for a particle with an internal $SU(2)$ symmetry. This provides an appropriate interface for a gauge boson with $SU(2)$ symmetry, and compatible with the relational principle, it arises from a finite symmetry which does not possess any $SU(2)$ symmetry. In the next section we will discuss the Standard Model with the object of fitting it into an appropriate finite symmetry.

When a finite symmetry has a triple degeneracy the associated irrep (which are labeled “ T ” in the literature), similar to the “ E ” case, will be the projection operator for the unit matrix of a 3×3 density matrix. There will be eight missing degrees of freedom all with zero trace and they can be written as the Gell-Mann matrices. Thus the T degeneracies define particles with internal $SU(3)$ triplet symmetry. Similarly a 4 dimensional degeneracy will define a particle with the $SU(4)$ fundamental representation, etc. Thus we can read off the density matrix particle content of a finite group by examining the characters of unity in their character table.

4. Standard Model as a Mixed Symmetry

The Standard Model symmetry has been measured to be $SU(3) \times SU(2) \times U(1)$ with only a small number of the infinite number of representations being observed. This is in extreme distinction to the case we expect with true Lie Group symmetries. For example, $SO(3)$ is the symmetry for orbital angular momentum. There are an infinite number of representations of size 1, 3, 5, ... and there does not appear to be a limit to how many of these we can observe in the lab.

Why then does the Standard Model restrict itself to only a few representations? Our proposal is that the Standard Model symmetry is not a Lie group symmetry appropriate to state vectors, but is instead a “mixed symmetry” that we can derive from a finite non Abelian group, in fact, a point group. Then the reason that the Standard Model has been fit into a Lie Group symmetry is sometimes expressed as “if the only tool you have is a hammer, all your problems will appear to you as nails.”

We will be treating the mass interaction separately here, so that we will be eventually concerned with modifications of the Pauli spin matrices and the Weyl equation rather than the gamma matrices and the Dirac equation. This is more compatible with the $SU(3) \times SU(2) \times U(1)$ representations assigned to the elementary fermions. And for the moment we will be concerned only with the fermions of the first generation. For that generation the representation assignments are:

	SU(3)	SU(2) + I_{Wz}	$Y_W/2$
ν_L	1	$2 + 1/2$	$-1/2$
ν_R	1	1	0
e_L^-	1	$2 - 1/2$	$-1/2$
e_R^-	1	1	-1
u_L	3	$2 + 1/2$	$+1/3$
d_L	3	$2 - 1/2$	$+1/2$
u_R	3	1	$+2/3$
d_R	3	1	$-1/3$

(4.1)

where the SU(2) entry $2 - 1/2$ for the left handed electron indicates its the $-1/2$ member of an SU(2) doublet and the $Y_W/2$ entries are half the U(1) weak hypercharge values. In order to account for massive neutrinos, we've added a right handed neutrino. For the SU(2) singlets, I_{Wz} is zero and the electric charge is $I_{Wz} + Y_W/2$.

The above table would be difficult to fit to the mixed symmetry of a finite group as the left-handed quarks are triplets under SU(3) and doublets under SU(2). Since the internal symmetries found in the previous section depend on the character of the identity, only the triplet or the doublet could be had in a single state, not both. This sort of problem has never stopped a theoretical physicist before; our solution will be to note that SU(2) is not a perfect symmetry as would be derived from a perfect point symmetry, but instead is approached only in the high temperature limit. So we can expect to see states arrive as pairs with some similarities between the SU(2) weak doublets but not as actual doublets. It's also likely that a complete list will include some dark matter states. These will be singlets under SU(3) and SU(2) and have zero $Y_W/2$.

With the SU(2) adjustment, we are looking for a point group symmetry which has four singlets and four triplet degeneracies. Only one such point group exists. It is the octahedral group of size 48. It consists of all the "90 degree" rotations in SO(3). That is, each of these rotations map the axes to \pm axes, half as proper the other half as improper rotations. In addition to four singlets and four triplets, there are also two doublets which we will

assign to the left and right handed dark matter. The character table is:

O_h	E	C_2	C_3	C_2	C_4	i	σ_h	S_6	S_4	σ_d
Hand:	L	L	L	L	L	R	R	R	R	R
Ev/Od:	E	E	E	O	O	E	E	E	O	O
Size:	1	3	8	6	6	1	3	8	6	6
A_{1g}	1	+1	+1	+1	+1	+1	+1	+1	+1	+1
A_{2g}	1	+1	-1	-1	+1	+1	-1	+1	+1	-1
E_g	2	-1	0	0	+2	+2	0	-1	+2	0
T_{1g}	3	0	-1	+1	-1	+3	+1	0	-1	-1
T_{2g}	3	0	+1	-1	-1	+3	-1	0	-1	+1
A_{1u}	1	+1	+1	+1	+1	-1	-1	-1	-1	-1
A_{2u}	1	+1	-1	-1	+1	-1	+1	-1	-1	+1
E_u	2	-1	0	0	+2	-2	0	+1	-2	0
T_{1u}	3	0	-1	+1	-1	-3	-1	0	+1	+1
T_{2u}	3	0	+1	-1	-1	-3	+1	0	+1	-1

In the above, of the ten columns, the five on the left are proper rotations (i.e. left handed) while the five on the right are improper or right handed. Each of these five are split into a set of 3 even and two odd, as labeled. The “Size” is the number of elements in that conjugacy class.

There are various ways of assigning the irreps to the known elementary fermions. Dark matter is clearly E_g and E_u , and its differences with the other states makes it clear why it is dark. Dark matter has an SU(2) internal symmetry which suggests an analogy to the SU(3) internal symmetry of quarks so that we can call them “duarks”. There should be a “dark gluon” or “duon” that will follow an SU(2) triplet symmetry. Dark quark colors will correspond to spin-up and spin-down in spin-1/2 SU(2) but since they are dark we propose calling them “doom” and “gloom”, also appropriate for the present Covid19 pandemic. In analogy with quarks, a duark and antiquark may combine to form a “deson”. Since there will be no decay by electroweak processes, they are likely to be stable. Two duarks of different dark colors may combine to form a “daryon”. [5]

The Standard Model has three generations. To account for these in density matrix form we need to have three times as many group elements. There is only one group with three elements and it is Abelian so increasing our symmetry group by it will simply triple the number of states. This increases the size of our group from the octahedral group O_h , to the tripled octahedral group O_h^3 of size $N = 48 \times 3 = 144$.

The universe is not made out of symmetry, instead whatever the universe is made out of happens to possess certain symmetries. So an attractive problem is to find a physical object that happens to possess the symmetry we’ve described by a mixed symmetry. For example, orbital angular momentum defines a pattern of irreducible representations of sizes 1, 3, 5, 7, ... and from this we can suppose that it arises from an SO(3) symmetry. We then look for a physical model with SO(3) symmetry and conclude that orbital angular momentum might be described by a system with SO(3) symmetry.

This suggests looking for powers of a real vector r and after some trials we find the Schroedinger equation for the hydrogen atom. We will now illustrate a similar process with a finite symmetry group.

The Standard Model arises from a finite group of size 144 which will be too large for a reasonable example. So instead, let us write down a modification of the Pauli spin matrices that will possess D_3 symmetry. What we are looking for are equations that can be rewritten so that there are two singlets A_1 and A_2 , and one doublet E . Of course we could simply write down that result but it would not at all be convincing; it's what was done with the Standard Model already, in its arbitrary choice of irreducible representations of $SU(3) \times SU(2) \times U(1)$. So instead, the underlying physical system will have D_3 symmetry and we will use the D_3 generalized Fourier transform to rewrite its equations into A_1, A_2, E form.

The relational interpretation of quantum mechanics explains the EPR paradox as a consequence of what observers know about a quantum system and not a matter of a state that is physically entangled. This suggests that to understand the nature of quantum states we do not need to spend effort on understanding how quantum mechanics treats states with two or more electrons. Instead, we need to understand single particle quantum mechanics; the important issue is the difference between an electron and a quark, not the difference between one electron and two electrons. Besides, two electron states are already well understood. So we will be concerned with quantum states where each different fermion appears zero or once. The reader can think of this as the "quantum information theory" version of particle physics.

The wave equation for left or right handed fermions is the Weyl equation:

$$\sigma^\mu \partial_\mu \psi = (I_2 \partial_t \pm \sigma_x \partial_x \pm \sigma_y \partial_y \pm \sigma_z \partial_z) \psi, \quad (4.3)$$

with the plus signs taken for the left handed particles while minus signs are for the right handed. To convert this into an equation for N different fermions, we could add a superscript k to the wave function ψ , where $k = 1, 2, \dots, N$ defines N different fermions. In order for a single Weyl equation to support different fermions, we will also have to apply the same superscript to the Pauli spin matrices σ^μ . Having done this, we can arrange for the N fermions to have whatever symmetry relationship we'd like but this is not compatible with the principle of linear superposition and irreducible representations of symmetry.

To actually derive a mixed symmetry we need to begin with a more complicated set of equations. We can do this by arranging for the Pauli spin matrices to couple with the wave functions ψ . So let's use g as a superscript for the Pauli spin matrices and h as a superscript for the wave function:

$$\sigma^{\mu g} \partial_\mu \psi^h \quad (4.4)$$

Since g and h can each run over the N elements of the finite group, this gives us N^2 terms. To arrange for a coupling with the finite group we can sum these using the finite group product. That is, two of these terms, one with g and h , the other with g' and h' , will be added together if $gh = g'h'$.

For example, since $[\] [23] = [23] = [123][13]$, we'll be adding those terms together as in:

$$\sigma^{\mu[\]} \partial_{\mu} \psi^{[23]} + \sigma^{\mu[123]} \partial_{\mu} \psi^{[13]}. \quad (4.5)$$

So there are $6^2 = 36$ such terms and we'll be combining them into six equations. Each of these six equations involves all six ψ^g , that is, they are coupled. In case this has not been sufficiently clear, let's write out the first of these 6 coupled equations for the D_3 symmetry, the one that gives an overall $[\]$ group element:

$$0 = \sigma^{\mu[\]} \partial_{\mu} \psi^{[\]} + \sigma^{\mu[123]} \partial_{\mu} \psi^{[132]} + \sigma^{\mu[132]} \partial_{\mu} \psi^{[123]} + \sigma^{\mu[12]} \partial_{\mu} \psi^{[12]} + \sigma^{\mu[13]} \partial_{\mu} \psi^{[13]} + \sigma^{\mu[23]} \partial_{\mu} \psi^{[23]}. \quad (4.6)$$

That's a long line. To make it shorter, let's leave off the μ and ∂ . We will add these back in when we're done. So the 6 coupled equations are:

$$\begin{aligned} 0 &= \sigma^{[\]} \psi^{[\]} + \sigma^{[123]} \psi^{[132]} + \sigma^{[132]} \psi^{[123]} + \sigma^{[12]} \psi^{[12]} + \sigma^{[13]} \psi^{[13]} + \sigma^{[23]} \psi^{[23]}, \\ 0 &= \sigma^{[\]} \psi^{[123]} + \sigma^{[123]} \psi^{[\]} + \sigma^{[132]} \psi^{[132]} + \sigma^{[12]} \psi^{[23]} + \sigma^{[13]} \psi^{[12]} + \sigma^{[23]} \psi^{[13]}, \\ 0 &= \sigma^{[\]} \psi^{[132]} + \sigma^{[123]} \psi^{[123]} + \sigma^{[132]} \psi^{[\]} + \sigma^{[12]} \psi^{[13]} + \sigma^{[13]} \psi^{[23]} + \sigma^{[23]} \psi^{[12]}, \\ 0 &= \sigma^{[\]} \psi^{[12]} + \sigma^{[123]} \psi^{[23]} + \sigma^{[132]} \psi^{[13]} + \sigma^{[12]} \psi^{[\]} + \sigma^{[13]} \psi^{[123]} + \sigma^{[23]} \psi^{[132]}, \\ 0 &= \sigma^{[\]} \psi^{[13]} + \sigma^{[123]} \psi^{[12]} + \sigma^{[132]} \psi^{[23]} + \sigma^{[12]} \psi^{[132]} + \sigma^{[13]} \psi^{[\]} + \sigma^{[23]} \psi^{[123]}, \\ 0 &= \sigma^{[\]} \psi^{[23]} + \sigma^{[123]} \psi^{[13]} + \sigma^{[132]} \psi^{[12]} + \sigma^{[12]} \psi^{[123]} + \sigma^{[13]} \psi^{[132]} + \sigma^{[23]} \psi^{[\]}. \end{aligned} \quad (4.7)$$

Our task is to uncouple these equations using the generalized Fourier transform.

The A_1 and A_2 irreps of Equation (3.9) are the same except they differ in the signs of the $[12]$, $[13]$ and $[23]$ components. So we can combine the two into the same calculation with the plus signs accounting for A_1 and the minus signs for A_2 . Leaving off the overall factor of $1/6$, the 6 equations are transformed into:

$$\begin{aligned} 0 &= \sigma^{[\]} \psi^{[\]} + \sigma^{[123]} \psi^{[132]} + \sigma^{[132]} \psi^{[123]} + \sigma^{[12]} \psi^{[12]} + \sigma^{[13]} \psi^{[13]} + \sigma^{[23]} \psi^{[23]} \\ &+ \sigma^{[\]} \psi^{[123]} + \sigma^{[123]} \psi^{[\]} + \sigma^{[132]} \psi^{[132]} + \sigma^{[12]} \psi^{[23]} + \sigma^{[13]} \psi^{[12]} + \sigma^{[23]} \psi^{[13]} \\ &+ \sigma^{[\]} \psi^{[132]} + \sigma^{[123]} \psi^{[123]} + \sigma^{[132]} \psi^{[\]} + \sigma^{[12]} \psi^{[13]} + \sigma^{[13]} \psi^{[23]} + \sigma^{[23]} \psi^{[12]} \\ &\pm \sigma^{[\]} \psi^{[12]} \pm \sigma^{[123]} \psi^{[23]} \pm \sigma^{[132]} \psi^{[13]} \pm \sigma^{[12]} \psi^{[\]} \pm \sigma^{[13]} \psi^{[123]} \pm \sigma^{[23]} \psi^{[132]} \\ &\pm \sigma^{[\]} \psi^{[13]} \pm \sigma^{[123]} \psi^{[12]} \pm \sigma^{[132]} \psi^{[23]} \pm \sigma^{[12]} \psi^{[132]} \pm \sigma^{[13]} \psi^{[\]} \pm \sigma^{[23]} \psi^{[123]} \\ &\pm \sigma^{[\]} \psi^{[23]} \pm \sigma^{[123]} \psi^{[13]} \pm \sigma^{[132]} \psi^{[12]} \pm \sigma^{[12]} \psi^{[123]} \pm \sigma^{[13]} \psi^{[132]} \pm \sigma^{[23]} \psi^{[\]}. \end{aligned} \quad (4.8)$$

This doesn't appear to be much of a simplification but it factors. Bringing the μ and ∂ back we have the Weyl equations for the A_1 and A_2 irreps:

$$0 = (\sigma^{\mu[\]} + \sigma^{\mu[123]} + \sigma^{\mu[132]} \pm \sigma^{\mu[12]} \pm \sigma^{\mu[13]} \pm \sigma^{\mu[23]}) \partial_{\mu} (\psi^{[\]} + \psi^{[123]} + \psi^{[132]} \pm \psi^{[12]} \pm \psi^{[13]} \pm \psi^{[23]}) \quad (4.9)$$

There are four more generalized Fourier components, that for E , σ_x , σ_y and σ_z . The process is similar and we recommend it to the reader as an exercise. The pure $SU(2)$ density matrix states are then found by choosing a real unit 3-vector (u_x, u_y, u_z) and computing $(E/3 + u_x \sigma_x + u_y \sigma_y + u_z \sigma_z)/2$. The result has an internal $SU(2)$ symmetry similar to the color $SU(3)$ symmetry of the quarks. That is, under the relational interpretation, color $SU(3)$ is a

symmetry of quark interactions so all we need for a single quark is for it to have the right “socket” for the interaction, that is, something to couple to.

5. Speculations

This paper has discussed the symmetry of the fundamental fermion states and has not covered their gauge interactions. We presume that papers on the subject will follow, not necessarily by this author. But if this theory is a better way of understanding the Standard Model symmetry, what does it say about the nature of space and time?

A point group symmetry depends on space only, so the first conclusion is that special relativity is only a low temperature, long wavelength approximation. The octahedral group implies cubic axes so that in addition to losing special relativity we even lose the apparent $SO(3)$ symmetry of space alone. There are already some clues in the literature that this might be the case, for example Iwo Bialynicki-Birula showed how to put the Weyl equation on a cubic lattice with quantum cellular automata.[6]

One of the major problems with modeling fermions on a lattice is “fermion doubling” where one finds twice as many fermions for each dimension. In the case of the Pauli spin matrices and the Weyl equation there are three doublings so the number of fermions increases by a factor of 8. These doublings are caused by spatial transformations that invert the x , y and z axes and as such are included in the octahedral point group. Thus we can suppose that this paper gives a solution to the fermion doubling problem when one makes it even worse by also allowing right angle rotations.

If space-time really is a cubic lattice, in addition to effects at very small distances there might also be effects in the long distance limit, or even in black holes. No firm evidence exists at the moment.

6. Conclusion

Of course the author has received assistance from many physicists. Of particular note are his advisors in the physics program at Washington State University, Michael Forbes, Sukanta Bose and Fred Gittes. Their interest and encouragement was critical to maintaining the long effort required here. And that long effort was too long by two or three years as WSU has a limitation on how long one can take to write a thesis. The author thanks his advisors for continuing to assist after that time ran out. The author hopes that the university will find a way of making an exception and that this paper will be considered as a partial fulfilment of the PhD degree in physics at WSU.

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