Generalized Quantum Evidence Theory on Interference Effect ⋆

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Abstract. In this paper, CET is generalized to quantum framework of Hilbert space in an open world, called generalized quantum evidence theory (GQET). Differ with classical GET, interference effects are involved in GQET. Especially, when a GQBBA turns into a classical GBBA, interference effects disappear, so that GQB and GQP functions of GQET degenerate to classical GBel and GPl functions of classical GET, respectively.

Keywords: Generalized quantum evidence theory · Generalized quantum mass function · Generalized quantum belief function · Generalized quantum Plausibility function · Complex evidence theory · Complex mass function · Uncertainty reasoning · Interference effect · Quantum decision.

1 Generalized quantum evidence theory

In this section, CET [2, 3] is generalized to quantum framework of Hilbert space in an open world, called generalized quantum evidence theory (GQET).

Definition 1. *(Quantum FOD). Let $|\Phi\rangle$ be a quantum FOD (QFOD), making of a set of mutually exclusive and collectively non-empty events, each of which is expressed as an orthonormal basis $|\phi_g\rangle$ in a Hilbert space:*

$$|\Phi\rangle = \{|\phi_1\rangle, \ldots, |\phi_g\rangle, \ldots, |\phi_n\rangle\}. \tag{1}$$

Definition 2. *(Quantum proposition). The power set of $|\Phi\rangle$ is denoted as:*

$$2^{|\Phi\rangle} = \{|\emptyset\rangle, \{|\phi_1\rangle\}, \{|\phi_2\rangle\}, \ldots, \{|\phi_n\rangle\}, \{|\phi_1\phi_2\rangle\}, \ldots, \{|\phi_1\phi_2\ldots\phi_g\rangle\}, \ldots, |\Phi\rangle\}, \tag{2}$$

in which $|\emptyset\rangle$ denotes an unknown event.

Eq. (2) can be simply represented as:

$$2^{|\Phi\rangle} = \{|\emptyset\rangle, |\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle, |\phi_{12}\rangle, \ldots, |\phi_{12\ldots g}\rangle, \ldots, |\phi_{12\ldots n}\rangle\}. \tag{3}$$

$|\psi_j\rangle$ is defined as a quantum proposition, when $|\psi_j\rangle \in 2^{|\Phi\rangle}$.

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Definition 3. *(Generalized quantum mass function). A generalized quantum mass function (GQMF) $\mathbb{Q}_{\mathbb{M}}$ in QFOD $|\Phi\rangle$, also called a GQBBA, mapping from $2^{|\Phi\rangle}$ to $\mathbb{C}$, is defined by*

$$\mathbb{Q}_{\mathbb{M}}: \quad 2^{|\Phi\rangle} \to \mathbb{C}, \tag{4}$$

and satisfies:

$$\begin{aligned} &\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle) = \mathbf{m}(|\psi_j\rangle)e^{i\theta(|\psi_j\rangle)}, \quad |\psi_j\rangle \in 2^{|\Phi\rangle}, \\ &\sum_{|\psi_j\rangle \in 2^{|\Phi\rangle}} \mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle) = 1, \end{aligned} \tag{5}$$

in which $i = \sqrt{-1}$, $\mathbf{m}(|\psi_j\rangle) \in [0,1]$ denotes the magnitude of $\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$, and $\theta(|\psi_j\rangle)$ denotes a phase term.

$\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)$ is also represented as:

$$\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle) = x_j + y_j i, \quad |\psi_j\rangle \in 2^{|\Phi\rangle}, \tag{6}$$

and its magnitude is expressed as:

$$|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)| = \mathbf{m}(|\psi_j\rangle) = \sqrt{x_j^2 + y_j^2}, \tag{7}$$

where $\sqrt{x_j^2 + y_j^2} \in [0,1]$.

Definition 4. *(Quantum focal element). If $|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|$ or $\mathbf{m}(|\psi_j\rangle) > 0$, $|\psi_j\rangle$ is defined as a quantum focal element. The value of $|\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle)|$ or $\mathbf{m}(|\psi_j\rangle)$ represents the degree to which QBBA supports $|\psi_j\rangle$.*

Definition 5. *(Quantum evidential combination rule). Let $\mathbb{Q}_{\mathbb{M}h}$ and $\mathbb{Q}_{\mathbb{M}k}$ be two independently GQBBAs with propositions $|\psi_p\rangle$ and $|\psi_q\rangle$ in QFOD $|\Phi\rangle$, respectively. Quantum evidential combination rule (QECR), denoted as $\mathbb{Q}_{\mathbb{M}} = \mathbb{Q}_{\mathbb{M}h} \oplus \mathbb{Q}_{\mathbb{M}k}$ is defined by:*

$$\mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle) = \frac{\sum\limits_{|\psi_p\rangle \cap |\psi_q\rangle = \psi_j} \mathbb{Q}_{\mathbb{M}h}(|\psi_p\rangle)\mathbb{Q}_{\mathbb{M}k}(|\psi_q\rangle)}{1 - \mathbb{K}}, \quad |\psi_j\rangle \in 2^{|\Phi\rangle}, \tag{8}$$

with

$$\mathbb{K}_{\mathbb{Q}} = \sum_{|\psi_p\rangle \cap |\psi_q\rangle = |\emptyset\rangle \| |\psi_p\rangle \cup |\psi_q\rangle \neq |\emptyset\rangle} \mathbb{Q}_{\mathbb{M}h}(|\psi_p\rangle)\mathbb{Q}_{\mathbb{M}k}(|\psi_q\rangle), \tag{9}$$

in which $\mathbb{K}_{\mathbb{Q}}$ is the conflict coefficient between $\mathbb{Q}_{\mathbb{M}h}$ and $\mathbb{Q}_{\mathbb{M}k}$.

Definition 6. *(Generalized quantum belief function) Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBBA with proposition $|\psi_j\rangle \in 2^{|\Phi\rangle}$. The generalized quantum belief (GQB) function, denoted as GQBel is defined by:*

$$\mathrm{GQBel}(|\psi_j\rangle) = \begin{cases} \sum\limits_{|\phi_p\rangle \subseteq |\psi_j\rangle} \mathbb{Q}_{\mathbb{M}}(|\psi_p\rangle), & |\psi_j\rangle \neq \emptyset, \\ \mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle), & |\psi_j\rangle = \emptyset. \end{cases} \tag{10}$$

Let $|\phi_p\rangle, |\phi_q\rangle \subseteq |\psi_j\rangle$ ($|\psi_j\rangle \neq \emptyset$) with s items. The amplitude of $\mathrm{GQBel}(|\psi_j\rangle)$ is calculated as:

$$\begin{aligned}|\mathrm{GQBel}(|\psi_j\rangle)| &= \sum_{|\phi_p\rangle\subseteq|\psi_j\rangle} |\mathbb{Q}_{\mathbb{M}}(|\psi_p\rangle)| + 2\sum_{p=1}^{s-1}\sum_{q=p+1}^{s} |\mathbb{Q}_{\mathbb{M}}(|\psi_p\rangle)||\mathbb{Q}_{\mathbb{M}}(|\psi_q\rangle)| \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)] \\ &= \sum_{|\phi_p\rangle\subseteq|\psi_j\rangle} \mathbf{m}(|\psi_p\rangle) + 2\sum_{p=1}^{s-1}\sum_{q=p+1}^{s} \mathbf{m}(|\psi_p\rangle)\mathbf{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)].\end{aligned} \tag{11}$$

When $\cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)] = 0$, Eq. (11) converges to classical GBel [1], such that:

$$|\mathrm{GQBel}(|\psi_j\rangle)| = \sum_{|\phi_p\rangle\subseteq|\psi_j\rangle||\psi_j\rangle\neq\emptyset} \mathbf{m}(|\psi_p\rangle). \tag{12}$$

Definition 7. *(Interference effect in GQB function) Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBBA with propositions $|\psi_p\rangle, |\phi_q\rangle \subseteq |\psi_j\rangle$ ($|\psi_j\rangle \neq \emptyset$) with s items. The interference effect for $|\psi_j\rangle$ in GQB function is defined by:*

$$\mathrm{Int}_{\mathrm{GQB}}(|\psi_j\rangle) = 2\sum_{p=1}^{s-1}\sum_{q=p+1}^{s} \mathbf{m}(|\psi_p\rangle)\mathbf{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)], \tag{13}$$

where $\mathbf{m}$ *and* θ *are magnitude and phase terms of* $\mathbb{Q}_{\mathbb{M}}$*.*

When GQBBA only consists of singletons, the amplitude of $\mathrm{GQBel}(|\psi_{12\ldots n}\rangle)$ is calculated as:

$$|\mathrm{GQBel}(|\phi_{12\ldots n}\rangle)| = \sum_{|\phi_p\rangle\subseteq|\phi_{12\ldots n}\rangle} \mathbf{m}(|\phi_p\rangle) + \mathrm{Int}_{\mathrm{GQB}}(|\phi_{12\ldots n}\rangle); \tag{14}$$

and $\mathrm{Int}_{\mathrm{GQB}}(|\phi_{12\ldots n}\rangle)$ is calculated as:

$$\mathrm{Int}_{\mathrm{GQB}}(|\phi_{12\ldots n}\rangle) = 2\sum_{p=1}^{n-1}\sum_{q=p+1}^{n} \mathbf{m}(|\psi_p\rangle)\mathbf{m}(|\psi_q\rangle) \cos[\theta(|\phi_p\rangle) - \theta(|\phi_q\rangle)]. \tag{15}$$

Definition 8. *(Generalized quantum Plausibility function) Let $\mathbb{Q}_{\mathbb{M}}$ be a GQBBA with proposition $|\psi_j\rangle \in 2^{|\Phi\rangle}$. The generalized quantum Plausibility (GQP) function, denoted as GQPl is defined by:*

$$\mathrm{GQPl}(|\psi_j\rangle) = \begin{cases} \sum_{|\phi_u\rangle\cap|\psi_j\rangle\neq\emptyset} \mathbb{Q}_{\mathbb{M}}(|\psi_u\rangle), & |\psi_j\rangle \neq \emptyset, \\ \mathbb{Q}_{\mathbb{M}}(|\psi_j\rangle), & |\psi_j\rangle = \emptyset. \end{cases} \tag{16}$$

Let $|\phi_u\rangle, |\phi_v\rangle \cap |\psi_j\rangle \neq \emptyset$ ($|\psi_j\rangle \neq \emptyset$) with t items. The amplitude of $\mathrm{GQPl}(|\psi_j\rangle)$ is calculated as:

$$\begin{aligned}|\mathrm{GQPl}(|\psi_j\rangle)| &= \sum_{|\phi_u\rangle\cap|\psi_j\rangle\neq\emptyset} |\mathbb{Q}_{\mathbb{M}}(|\psi_u\rangle)| + 2\sum_{u=1}^{t-1}\sum_{v=u+1}^{t} |\mathbb{Q}_{\mathbb{M}}(|\psi_u\rangle)||\mathbb{Q}_{\mathbb{M}}(|\psi_v\rangle)| \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)] \\ &= \sum_{|\phi_u\rangle\cap|\psi_j\rangle\neq\emptyset} \mathbf{m}(|\psi_u\rangle) + 2\sum_{u=1}^{t-1}\sum_{v=u+1}^{t} \mathbf{m}(|\psi_u\rangle)\mathbf{m}(|\psi_v\rangle) \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)].\end{aligned} \tag{17}$$

When $\cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)] = 0$, Eq. (17) converges to classical GPl [1], such that:

$$|\mathrm{GQPl}(|\psi_j\rangle)| = \sum_{|\phi_u\rangle \cap |\psi_j\rangle \neq \emptyset \| \psi_j\rangle \neq \emptyset} \mathbf{m}(|\psi_u\rangle). \tag{18}$$

Definition 9. *(Interference effect in GQP function) Let* $\mathbb{Q}_{\mathbb{M}}$ *be a GQBBA with propositions* $|\phi_u\rangle, |\phi_v\rangle \cap |\psi_j\rangle \neq \emptyset$ $(|\psi_j\rangle \neq \emptyset)$ *with* t *items. The interference effect for* $|\psi_j\rangle$ *in GQP function is defined by:*

$$\mathrm{Int}_{\mathrm{GQP}}(|\psi_j\rangle) = 2\sum_{u=1}^{t-1} \sum_{v=u+1}^{t} \mathbf{m}(|\psi_u\rangle)\mathbf{m}(|\psi_v\rangle) \cos[\theta(|\phi_u\rangle) - \theta(|\phi_v\rangle)]. \tag{19}$$

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