A PROOF OF POLIGNAC'S CONJECTURE AND INFINITE TWIN PRIMES

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ABSTRACT. Polignac's Conjecture is widely considered to remain unproven since first proposed in 1849. We derive a generalised proof of the conjecture by regarding a prime of its indivisibility and probability of its occurrence. The specific case of the Twin Prime Conjecture is proven as an example.

INTRODUCTION AND PRELIMINARIES

Polignac's Conjecture states that *for every even natural number k, there are infinitely many consecutive prime pairs p and p′ such that p′ − p = k* [1]. In the case of $k = 2$, this is known as the Twin Prime Conjecture. In Lemma 1, we define a prime in terms of its indivisibility. We then define a prime's probability of occurring as a function of this property. In Lemma 2, we elaborate how this relates to, and proves the conjecture.

- $\mathbb{P}$ is the set of all prime numbers
- $\mathbb{P}(n)$ is the n^{th} prime
- ε is any arbitrarily small number

LEMMA 1

A prime can be defined as *a natural number wholly indivisible except by 1 and itself*. That is, a prime must produce a remainder > 0 for any division where the divisor is not 1 or itself.

$$p \bmod d > 0 : p \in \mathbb{P}, \mathrm{d} \neq 1, \mathrm{d} \neq \mathrm{p}$$

The probability that a natural odd *m* is prime can be expressed as the probability that for all divisions of *m* by all primes < *m*, no remainders of zero will occur.

For any natural numerator *a* and any natural denominator *b*, the probability of producing a non-zero remainder can be expressed as:

$$\Pr(a \bmod b > 0) = \frac{b-1}{b}$$

Therefore, the probability of any natural odd *m* being prime can be expressed as the product of probabilities of divisions producing a nonzero remainder:

$$\Pr(m \in \mathbb{P}) = \prod_{\mathrm{n}=1}^{|S|} \frac{S(n)-1}{S(n)} : S = \{n : n \in \mathbb{P}, \mathbb{P}(n) < m\}$$

EXAMPLE. Consider the probability of 7 being prime in terms of its whole divisibility:

$$m = 7$$

$$\Pr(m \in \mathbb{P}) = \prod_{\mathrm{n}=1}^{3} \frac{S(n)-1}{S(n)} \; : \mathrm{S} = \{2, 3, 5\}$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}$$
$$= 0.2\overline{6}$$

LEMMA 2

The probability of any odd natural *m* being prime in terms of its indivisibility converges toward zero as m → ∞:

$$\lim_{m \to \infty} \Pr(m \in \mathbb{P}) = 0$$

However, *m* can be arbitrarily large. Per the second Borel–Cantelli lemma [2], *if the sum of probabilities of events diverges to infinite, then the probability that infinitely many of them occur is 1*. That is, any event of arbitrarily small probability > 0 will certainly occur given an infinite sample:

$$\Pr\left(\limsup_{m \to \infty} E_n\right) = 1 : \ E_n = \langle \mathrm{n} : \ \Pr(m \in \mathbb{P}) \rangle_{n=1}^{\infty}$$

Given any natural odd *m* and any natural even *k* > *m*, the probability of *m* + *k* being prime is always > 0. As *m* and *k* can be arbitrarily large, no *m* + *k* is impossibly prime. As *m* can be arbitrarily large, there are an infinite quantity of each interval. Necessarily, Polignac's Conjecture must be true.

EXAMPLE. Consider the Twin Prime Conjecture where *m* is any arbitrarily large prime, and k = 2. The probability of *m* such that *m* + 2 is also prime:

$$k = 2$$
$$P\big((m+2) \in \mathbb{P}\big) = \prod_{\mathrm{n}=1}^{|S|} \frac{S(n) - 1}{S(n)} : S = \{\mathrm{n} : \mathrm{n} \in \mathbb{P}, \mathbb{P}(\mathrm{n}) < (\mathrm{m} + 2)\}$$
$$= \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \ldots \times \frac{\max S - 1}{\max S}$$
$$= \epsilon$$
$$\Pr\left(\limsup_{n \to \infty} \langle \epsilon_{\mathrm{n}} \rangle_{n=1}^{\infty}\right) = 1 : m \in \mathbb{N}, |\mathbb{N}| = \infty$$

DISCUSSION

This deduction inadvertently proves a stronger formulation of Polignac's Conjecture that *for every odd natural m and prime k, there are infinitely many even natural intervals between m and k*. That is, the conjecture remains true even if *m* is merely odd and not necessarily prime. It may be noteworthy that a novel proof of Euclid's Theorem is also given; the probability of any arbitrarily large number being prime never intersects zero.

REFERENCES

1. de Polignac, A. "Recherches nouvelles sur les nombres premiers" (1849)
2. Émile Borel, M. "Les probabilités dénombrables et leurs applications arithmétiques" (1909)

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