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COLLATZ CONJECTURE - THE PROOF

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ABSTRACT. In this paper, we prove the Collatz conjecture. The proof consists of two parts. The first, shows that if an integer can be iterated through the Collatz conjecture to one, it is the equivalent of the condition that it can be presented as a certain equation. In the second part, we prove that for every initial integer, this equation can be found. To achieve this, we propose a procedure that can be iterated, and we prove that by doing this we arrive at this equation. We also prove that initial integer can be presented in an infinite number of ways in the form of needed equation. All analysis is done using binary representation of numbers.

1. Introduction

The Collatz conjecture is a well known mathematical problem. It claims that for every positive integer I_0 if iterating

$$I_{n+1} = \begin{cases} \frac{1}{2} \cdot I_n & for,\ I_n \ \ even \\ \\ 3 \cdot I_n + 1 & for,\ I_n \ \ odd \end{cases} \tag{1.1}$$

ultimately we get 1.

The purpose of this paper is to prove that the Collatz conjecture is true. The proof consists of two parts:

Theorem 1.1. *If the Collatz conjecture is true for a positive integer I_0, it is the equivalent of the condition that a positive integer n and a sequence of integers $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ exists, for which*

$$3^n I_0 \ = \ 2^{m_n} - 2^{m_{n-1}} 3^0 - 2^{m_{n-2}} 3^1 - \ \cdots \ - \ 2^{m_1} 3^{n-2} \ - \ 2^{m_0} 3^{n-1}. \tag{1.2}$$

Theorem 1.2. *For every positive integer I_0, such a positive integer n and a sequence of integers $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ can be found. Therefore, (by Theorem 1.1) the Collatz conjecture is true.*

2. Remarks and Definitions

To understand how the Collatz conjecture works and make it more accessible, we have to iterate integers in their binary representations. This paper explains when binary numbers are even or odd, how they are affected by different operations

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and examines how they iterate through the Collatz formula. The definitions and remarks introduced below are used over the course of this paper.

Remark 2.1. An integer is odd when in binary representation its least significant bit is 1. An integer is even when in binary representation its least significant bit is 0.

Remark 2.2. Every even positive integer can be reduced to the odd positive integer by recursively dividing it by 2 until the result is odd.

When I_{even} is an even positive integer, I_{odd} is an odd positive integer and p is the number of divisions by 2 required for I_{even} to became the odd integer I_{odd}, then

$$\frac{I_{even}}{2^p} \; = \; I_{odd}. \tag{2.1}$$

Example 2.3. Reduction of an even integer to an odd integer in binary representation.

Let I_{even} be an even positive integer

$$I_{even} = 20 = 10100_b.$$

Then

$$\begin{aligned} \frac{I_{even}}{2^p} &= \frac{20}{2^2} \\ &= \frac{10\mathbf{100}_b}{\mathbf{100}_b} \\ &= 10\mathbf{1}_b \\ &= 5 = I_{odd}. \end{aligned}$$

We see that an even positive integer I_{even} can be reduced to an odd positive integer I_{odd}. In this case 20 is reduced to 5.

Remark 2.4. By multiplying an odd positive integer by 3 and adding 1, we get a result which is always even

$$3 \cdot I_{odd} + 1 = I_{even}. \tag{2.2}$$

Example 2.5. Example in binary representation.
Let I_{odd} be an odd positive integer

$$I_{odd} = 7 = 111_b.$$

Then

$$\begin{aligned} 3I_{odd} + 1 &= 21 + 1 \\ &= 10101_b + 1 \\ &= 1011\mathbf{0}_b \\ &= 22 = I_{even}. \end{aligned}$$

We see that by multiplying an odd positive integer I_{odd} by 3 and increasing by 1, we get an even positive integer I_{even}.

Definition 2.6. For any positive integer I, let $lsb(I)$ be **the least significant nonzero bit** in the binary representation of I.

Example 2.7. Binary numbers with their least significant nonzero bits in bold:

$$lsb(10110101\mathbf{1}000_b) = \mathbf{1000}_b,$$
$$lsb(100101\mathbf{1}0_b) = \mathbf{10}_b,$$
$$lsb(10110101\mathbf{1}00_b) = \mathbf{100}_b,$$
$$lsb(110011\mathbf{1}_b) = \mathbf{1}_b,$$
$$lsb(110111\mathbf{1}000_b) = \mathbf{1000}_b.$$

Remark 2.8. For every odd positive integer I_{odd}

$$lsb(I_{odd}) = 2^0 = 1. \tag{2.3}$$

Example 2.9. We find $lsb(I_{odd})$ for an odd positive integer I_{odd}.
For $I_{odd} = 25$ we have

$$lsb(25) = lsb(1100\mathbf{1}_b) \;=\; 2^0 = 1.$$

Remark 2.10. For every even positive integer I_{even}

$$lsb(I_{even}) = 2^p, \tag{2.4}$$

where p is a positive integer, and then

$$\frac{I_{even}}{2^p} \;=\; I_{odd}, \tag{2.5}$$

therefore

$$I_{even} = 2^p I_{odd}. \tag{2.6}$$

Example 2.11. We find $lsb(I_{even})$ for an even positive integer I_{even}.
For $I_{even} = 28$ we have

$$lsb(28) = lsb(11\mathbf{1}00_b) \;= 2^2$$

and thus

$$\begin{aligned}\frac{I_{even}}{2^p} &= \frac{28}{lsb(28)}\\ &= \frac{28}{2^2}\\ &= \frac{11\mathbf{100}_b}{\mathbf{100}_b}\\ &= \;7 = I_{odd}.\end{aligned}$$

When we divide 28 by $lsb(28)$ it gives us an odd positive integer 7.

Definition 2.12. Let O denote a **base odd integer** of I and be defined as

$$O = \frac{I}{lsb(I)}, \tag{2.7}$$

where I can be an even or odd positive integer.

Example 2.13. Finding a base odd integer.
We check the case for an odd integer

$$I = 9 = 1001_b,$$
$$lsb(I) = lsb(100\mathbf{1}_b) = \mathbf{1}_b,$$

$$\begin{aligned} O &= \frac{I}{lsb(I)} \\ &= \frac{100\mathbf{1}_b}{\mathbf{1}_b} \\ &= 1001_b \\ &= 9. \end{aligned}$$

We conclude that for odd integers

$$O = I. \tag{2.8}$$

Notice that when I is an odd positive integer, its base odd integer O is equal to I.
Now we check the case for an even integer

$$I = 20 = 10100_b,$$
$$lsb(I) = lsb(10\mathbf{100}_b) = \mathbf{100}_b,$$

$$\begin{aligned} O &= \frac{I}{lsb(I)} \\ &= \frac{10\mathbf{100}_b}{\mathbf{100}_b} \\ &= 101_b \\ &= 5. \end{aligned}$$

To find the base odd integer O for an even integer I, we divide integer I by 2 until we get an odd result. We do this by dividing I by its least significant nonzero bit $lsb(I)$.

3. Proof of Theorem 1.1

Proof. For any positive integer I_0, we find its base odd integer using (2.7) and it is

$$O_0 = \frac{I_0}{lsb(I_0)}. \tag{3.1}$$

Value of $lsb(I_0)$ is in the form of 2^p, where $p \geq 0$ and $p = 0$ when I_0 is odd, thus

$$O_0 = \frac{I_0}{2^p}, \tag{3.2}$$

where $p \geq 0$.

We iterate this odd positive integer O_0 through the Collatz conjecture. We have

$$\frac{3\,\dfrac{\overset{\displaystyle 3^{\frac{3^{\frac{3^{\frac{3O_0+1}{2^{p_0}}}+1}{2^{p_1}}}+1}{2^{p_2}}}+1}{\cdots}}{2^{p_{n-2}}}+1}{2^{p_{n-1}}}=1, \tag{3.3}$$

and O_n is odd for every n, so $(3O_n + 1)$ is always even, therefore

$$p_0, p_1, p_2, ..., p_{n-2}, p_{n-1} \geq 1. \tag{3.4}$$

Equation (3.3) can be also presented like this

$$\left(\left(\left(\left(\left((3O_0+1)\frac{3}{2^{p_0}}+1\right)\frac{3}{2^{p_1}}+1\right)\frac{3}{2^{p_2}}+1\right)...\right)\frac{3}{2^{p_{n-2}}}+1\right)\frac{1}{2^{p_{n-1}}}=1. \tag{3.5}$$

By performing simple algebraic transformations we get

$$\begin{aligned} 3^n O_0 \;=\; & (2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}) - (2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0})\,3^0 - \\ & -(2^{p_{n-3}}...\,2^{p_1}2^{p_0})\,3^1 - \cdots - 2^{p_1}2^{p_o}3^{n-3} \;-\; 2^{p_o}3^{n-2} \;-\; 3^{n-1}. \end{aligned} \tag{3.6}$$

Now, we can substitute O_0 from (3.2)

$$\begin{aligned} 3^n \tfrac{I_0}{2^p} \;=\; & (2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}) - (2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0})\,3^0 - \;... \\ & -(2^{p_{n-3}}...\,2^{p_1}2^{p_0})\,3^1 - \cdots - 2^{p_1}2^{p_o}3^{n-3} \;-\; 2^{p_o}3^{n-2} \;-\; 3^{n-1}, \end{aligned}$$

and multiply both sides by 2^p

$$\begin{aligned} 3^n I_0 \;=\; & (2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p) - (2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p)\,3^0 - \;... \\ & -(2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p)\,3^1 - \cdots - 2^{p_1}2^{p_o}2^p3^{n-3} \;-\; 2^{p_o}2^p3^{n-2} \;-\; 2^p3^{n-1}. \end{aligned}$$

We substitute the following:

$$\begin{aligned} 2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_n}, \\ 2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_{n-1}}, \\ 2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_{n-2}}, \\ &... \\ 2^{p_1}2^{p_o}2^p &= 2^{m_2}, \\ 2^{p_o}2^p &= 2^{m_1}, \\ 2^p &= 2^{m_0}, \end{aligned} \tag{3.7}$$

where all $p_0, p_1, p_2, ..., p_{n-2}, p_{n-1} \geq 1$ and $p \geq 0$.

We finally have

$$3^n I_0 \;=\; 2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \;\cdots\; - \;2^{m_1}3^{n-2} \;-\; 2^{m_0}3^{n-1}, \tag{3.8}$$

where $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ and m_0 can eventually be 0, when I_0 is odd.

□

We prove in opposite direction.

Proof. We start from integer I_0 that fulfils equation

$$3^n I_0 \ = \ 2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2} \ - \ 2^{m_0}3^{n-1},$$

where n is a positive integer and $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ are integers.

We divide both sides by 3^n. We have

$$I_0 \ = \ \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2} \ - \ 2^{m_0}3^{n-1}}{3^n}.$$

Now we iterate through the Collatz equation, combines multiple divisions by 2 into single division by 2^p. In each iteration we receive odd integer from even integer or even integer from odd integer.

We divide I_0 by 2^{m_0}, where $m_0 \geq 0$ to receive odd integer. If I_0 is already odd then $m_0 = 0$, so $2^0 = 1$ and division by 1 does not affect the result. If I_0 is even, $m_0 > 0$ and m_0 is the number that represents how many times I_0 has to be divided by 2 to become odd. We have

$$I_1 \ = \ \frac{I_0}{\mathbf{2^{m_0}}} = \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2}}{3^n\mathbf{2^{m_0}}} - \frac{3^{n-1}}{3^n}$$

which is an odd integer. For odd integer

$$I_1 \ = \ \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2}}{3^n 2^{m_0}} - \frac{1}{3}$$

we multiply by 3

$$I_1 \cdot \mathbf{3} \ = \ \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2}}{\mathbf{3^{n-1}}2^{m_0}} - \frac{\mathbf{3}}{3}$$

and add 1

$$I_2 \ = \ I_1 \cdot 3 + \mathbf{1} = \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2}}{3^{n-1}2^{m_0}}.$$

We put the last term to separate quotient

$$I_2 \ = \ \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_2}3^{n-3}}{3^{n-1}2^{m_0}} - \frac{2^{m_1}3^{n-2}}{2^{m_0}3^{n-1}}.$$

We know that I_2 is even (from Remark 2.4), so it can be divided by 2^{p_0}, where $p_0 > 0$, to get an odd integer. We know that $m_1 > m_0$, so to make the right side of equation odd, we need $m_1 = p_0 + m_0$, which gives $2^{p_0} = \frac{2^{m_1}}{2^{m_0}}$ and then divide I_2 by 2^{p_0}

$$I_3 \ = \ \frac{I_2}{\mathbf{2^{p_0}}} = \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_2}3^{n-3}}{3^{n-1}\mathbf{2^{m_1}}} - \frac{1}{3}.$$

Now that I_3 is odd, we multiply it by 3 again

$$I_3 \cdot \mathbf{3} \ = \ \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_2}3^{n-3}}{\mathbf{3^{n-2}}2^{m_1}} - \frac{\mathbf{3}}{3}$$

and add 1

$$I_4 \ = \ I_3 \cdot 3 + \mathbf{1} = \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_3}3^{n-4} - \ 2^{m_2}3^{n-3}}{3^{n-2}2^{m_1}}.$$

We put the last term to separate quotient

$$I_4 \;=\; \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \;\cdots\; - \;2^{m_3}3^{n-4}}{3^{n-2}2^{m_1}} - \frac{2^{m_2}3^{n-3}}{2^{m_1}3^{n-2}}.$$

We know that I_4 is even, so it can be divided by 2^{p_1}, where $p_1 > 0$, to become an odd integer. We know that $m_2 > m_1$, so to make the right side of equation odd, we need $m_2 = p_1 + m_1$, which gives $2^{p_1} = \frac{2^{m_2}}{2^{m_1}}$ and then divide I_4 by 2^{p_1}

$$I_5 \;=\; \frac{I_4}{2^{p_1}} = \frac{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \;\cdots\; - \;2^{m_3}3^{n-4}}{3^{n-2}2^{m_2}} - \frac{3^{n-3}}{3^{n-2}}.$$

We can continue this process till

$$I_{k-4} \;=\; \frac{2^{m_n} - 2^{m_{n-1}}3^0}{3^2 2^{m_{n-2}}} - \frac{3^1}{3^2}.$$

We know I_{k-4} is odd, we multiply it by 3

$$I_{k-4} \cdot 3 \;=\; \frac{2^{m_n} - 2^{m_{n-1}}3^0}{\mathbf{3^1} 2^{m_{n-2}}} - \frac{\mathbf{3^2}}{3^2}$$

and add 1

$$I_{k-3} = I_{k-4} \cdot 3 + 1 \;=\; \frac{2^{m_n} - 2^{m_{n-1}}3^0}{3^1 2^{m_{n-2}}}.$$

We put the last term to separate quotient

$$I_{k-3} \;=\; \frac{2^{m_n}}{3^1 2^{m_{n-2}}} - \frac{2^{m_{n-1}}3^0}{2^{m_{n-2}}3^1}.$$

We know that I_{k-3} is even, so it can be divided by $2^{p_{n-2}}$, where $p_{n-2} > 0$, to become an odd integer. We know that $m_{n-1} > m_{n-2}$, so to make the right side of equation odd, we need $m_{n-1} = p_{n-2} + m_{n-2}$, which gives $2^{p_{n-2}} = \frac{2^{m_{n-1}}}{2^{m_{n-2}}}$ and then divide I_{k-3} by $2^{p_{n-2}}$

$$I_{k-2} = \frac{I_{k-3}}{2^{p_{n-2}}} \;=\; \frac{2^{m_n}}{3^1 \mathbf{2^{m_{n-1}}}} - \frac{3^0}{3^1}.$$

We know I_{k-2} is odd, we multiply it by 3

$$I_{k-2} \cdot 3 = \frac{2^{m_n}}{\mathbf{3^0} 2^{m_{n-1}}} - \frac{\mathbf{3^1}}{3^1}$$

and add 1

$$I_{k-1} \;=\; I_{k-2} \cdot 3 + 1 = \frac{2^{m_n}}{3^0 2^{m_{n-1}}}.$$

We know that I_{k-1} is even, so it can be divided by $2^{p_{n-1}}$, where $p_{n-1} > 0$, to become an odd integer. We know that $m_n > m_{n-1}$, so to make the right side of equation odd, we need $m_n = p_{n-1} + m_{n-1}$, which gives $2^{p_{n-1}} = \frac{2^{m_n}}{2^{m_{n-1}}}$ and then divide I_{k-1} by $2^{p_{n-1}}$

$$I_k = \frac{I_{k-1}}{2^{p_{n-1}}} \;=\; \frac{2^{m_n}}{\mathbf{2^{m_n}}} = 1.$$

□

Notice that for any initial positive integer I_0 that fulfils equation

$$3^n I_0 \;=\; 2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \;\cdots\; - \;2^{m_1}3^{n-2} \;-\; 2^{m_0}3^{n-1},$$

where n is a positive integer and $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ are integers, we have a sequence of integers

$$I_0, I_1, I_2, I_3, ...I_{k-3}, I_{k-2}, I_{k-1}, I_k$$

and $I_k = 1$.

4. Procedure

We consider the following procedure.

Procedure 1.

Step 1. Take any positive integer I_0 and define

$$A_0 = 2^p, \text{where } p \in \mathbb{Z}^+ \; and \; A_0 > I_0, \tag{4.1}$$

$$B_0 = A_0 - I_0, \tag{4.2}$$

$$C_0 = 0. \tag{4.3}$$

We have

$$\boldsymbol{3^0 I_0 = A_0 - B_0 - C_0}. \tag{4.4}$$

Step 2. Multiply both sides of the equation by 3 using the following transformations

$$\boldsymbol{3^n I_0 \;=\; 3 \cdot 3^{n-1} I_0}, \tag{4.5}$$

$$\boldsymbol{A_n \;=\; 4 \cdot A_{n-1}}, \tag{4.6}$$

$$\boldsymbol{B_n \;=\; 3 \cdot B_{n-1} + A_{n-1} - lsb(B_{n-1})}, \tag{4.7}$$

$$\boldsymbol{C_n \;=\; 3 \cdot C_{n-1} + lsb(B_{n-1})}. \tag{4.8}$$

We have general formula for n^{th} iteration

$$\boldsymbol{3^n I_0 \;=\; A_n \;-\; B_n -\; C_n}. \tag{4.9}$$

Step 3. Iterate Step 2 forever.

Remark 4.1. Notice that

$$B_n > 0, \; for \; all \; n, \tag{4.10}$$

$$C_n > 0, \; for \; n > 0. \tag{4.11}$$

From (4.9) we have

$$A_n = 3^n I_0 + B_n + C_n \tag{4.12}$$

therefore

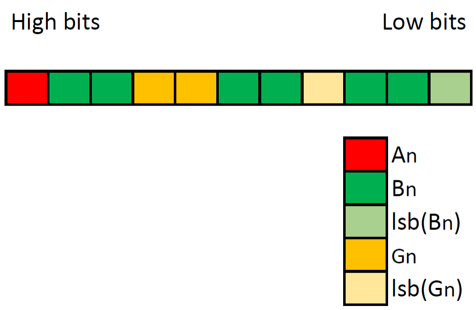
$$A_n > B_n, \; for \; all \; n. \tag{4.13}$$

We know from (4.1) and (4.6) that A_n is always in the form of a single bit therefore

$$A_n > every \; bit \; in \; B_n, \; for \; all \; n. \tag{4.14}$$

We define G_n as a sum of all bits between $lsb(B_n)$ and A_n that are not part of B_n.

FIGURE 1. Definition of G_n.



Lemma 4.2. *When iterating Procedure 1, for any initial positive integer* I_0

$$\boldsymbol{B_{n+1} = 4B_n + G_n}. \tag{4.15}$$

Proof. We know that

$$G_n > lsb(B_n), \tag{4.16}$$

when there are some gaps between bits in B_n, or

$$G_n = 0,\ G_n < lsb(B_n) \tag{4.17}$$

if there are no gaps between bits in B_n and all bits are next to each other.

From Figure 1 we see that

$$A_n = B_n + G_n + lsb(B_n). \tag{4.18}$$

From (4.7) we have

$$B_{n+1} = 3B_n + A_n - lsb(B_n), \tag{4.19}$$

we substitute A_n from (4.18)

$$B_{n+1} = 3B_n + B_n + G_n + lsb(B_n) - lsb(B_n) \tag{4.20}$$

therefore

$$B_{n+1} = 4B_n + G_n. \tag{4.21}$$

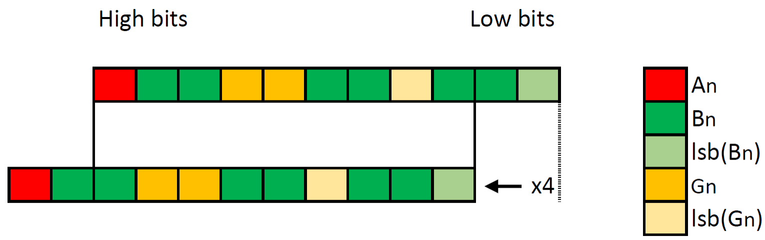
□

In the following examples we check how G_n changes when we iterate from n to $n+1$. We explore all possible scenarios:
when $lsb(G_n) > 4lsb(B_n)$ in Example 4.4,
when $lsb(G_n) = 4lsb(B_n)$ in Example 4.5,
when $lsb(G_n) = 2lsb(B_n)$ in Example 4.6.

Remark 4.3. Notice how G_{n+1} depends on the position of $lsb(G_n)$.

When we multiply number by 4, we shift the binary representation of such number by two positions towards higher bits.

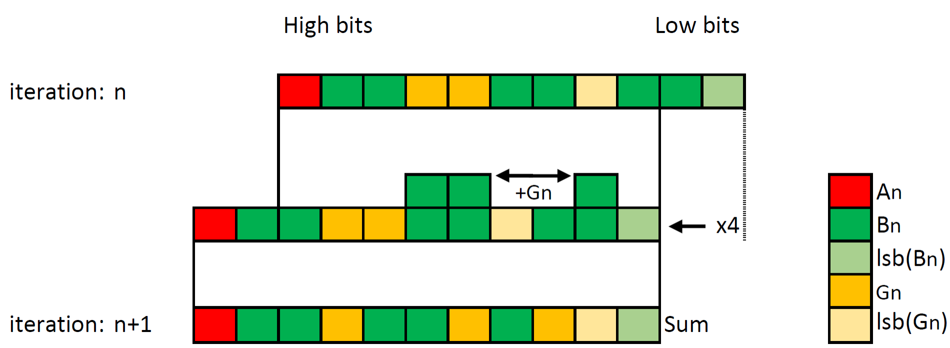
FIGURE 2. Multiplication by 4 in binary notation - all bits shifted by 2 positions.



We multiply (shift) all bits in B_n as well as all bits in G_n.

Example 4.4 (when $lsb(G_n) > 4lsb(B_n)$). In (4.21) when we change from iteration n to $n+1$, we shift all of the bits in B_n, but we also add G_n to have B_{n+1}.

FIGURE 3. Binary operations for $B_{n+1} = 4B_n + G_n$.



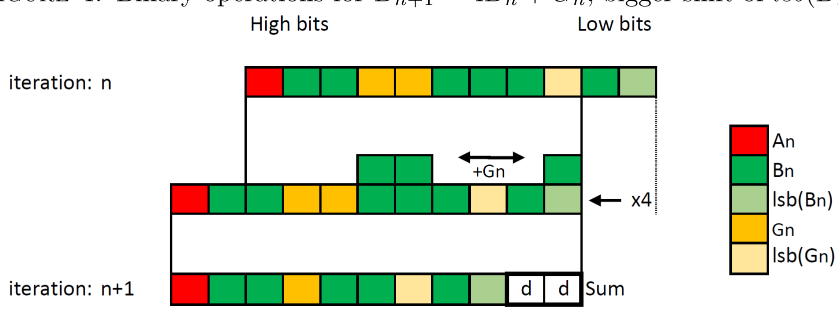
Notice how one G_n was added to $4B_n$ therefore when iterating from n to $n+1$ in this example we have

$$\begin{aligned} G_{n+1} &= 4G_n - G_n \\ &= 3G_n, \end{aligned} \tag{4.22}$$

(compare to Figure 2).

Example 4.5 (**when $lsb(G_n) = 4lsb(B_n)$**)**.** In another case, after the operation $B_{n+1} = 4B_n + G_n$, $lsb(B_n)$ can be shifted by more than two positions.

FIGURE 4. Binary operations for $B_{n+1} = 4B_n + G_n$, bigger shift of $lsb(B_n)$.



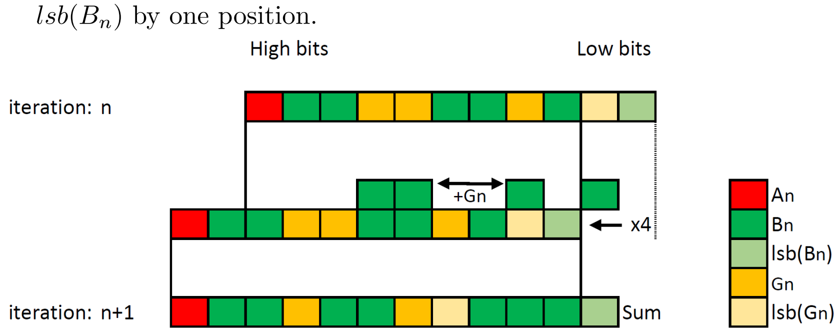
By comparing with Example 4.4, notice a bigger shift of $lsb(B_n)$ (by 4 positions). Some bits transfered from G_n were left behind $lsb(B_{n+1})$ and G_{n+1} is lower than $3G_n$

$$\begin{aligned} G_{n+1} &= 4G_n - G_n - d \\ &= 3G_n - d, \end{aligned} \tag{4.23}$$

where d represents all bits that are smaller than $lsb(B_{n+1})$.

Example 4.6 (**when $lsb(G_n) = 2lsb(B_n)$**)**.** It is also possible that after the operation $B_{n+1} = 4B_n + G_n$, $lsb(B_n)$ can be shifted by only one bit.

FIGURE 5. Binary operations for $B_{n+1} = 4B_n + G_n$, shift of $lsb(B_n)$ by one position.



In such case

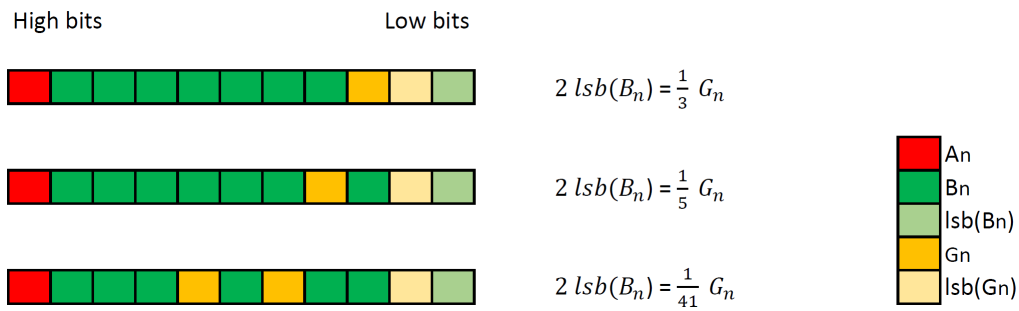
$$\begin{aligned} G_{n+1} &= 4G_n - G_n + 2lsb(B_n) \\ &= 3G_n + 2lsb(B_n) \end{aligned} \tag{4.24}$$

Notice that almost always we have

$$2lsb(B_n) \le \frac{1}{3}G_n \tag{4.25}$$

(see Figure 6),

FIGURE 6. Comparison of $2lsb(B_n)$ to G_n in different cases when $lsb(G_n) = 2lsb(B_n)$.

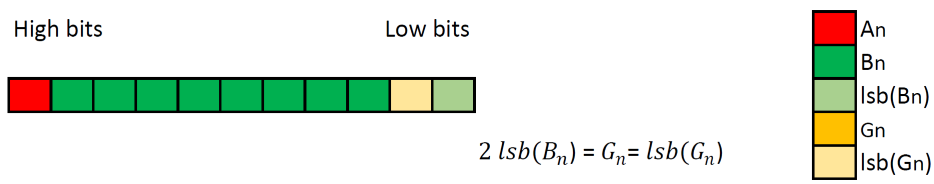


with only one exception when

$$2lsb(B_n) = G_n \tag{4.26}$$

(see Figure 7).

FIGURE 7. Special case when $2lsb(B_n) = G_n$.



In this special case when

$$2lsb(B_n) = G_n \tag{4.27}$$

we have

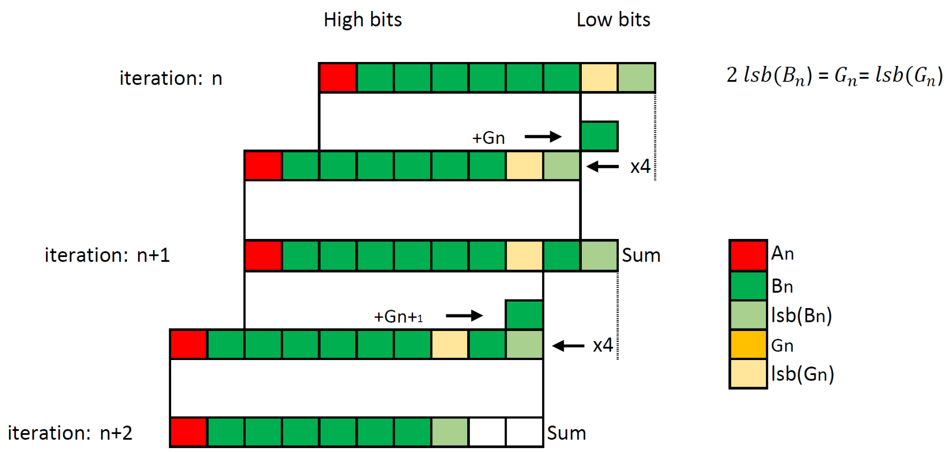
$$G_{n+1} = 4G_n \tag{4.28}$$

and

$$G_{n+2} = 0 \tag{4.29}$$

in next two iterations (compare Figure 8).

FIGURE 8. Two iterations after $2lsb(B_n) = G_n$ we have $G_{n+2} = 0$.



Remark 4.7. From all above examples, which present all possible scenarios for change of G_n, we can conclude that

$$G_{n+1} = a \cdot G_n, \ \textit{where } a \leq 3\frac{1}{3}, \tag{4.30}$$

or

$$G_{n+1} = 4G_n, \tag{4.31}$$

but then $G_{n+2} = 0$.

Lemma 4.8. *When iterating Procedure 1, for any initial positive integer I_0 such iteration number k exists that starting from this iteration and for all the following iterations, when $n \geq k$,*

$$\boldsymbol{A_n = B_n + lsb(B_n)}. \tag{4.32}$$

Proof. We iterate Procedure 1 for any initial positive integer I_0. We know from Lemma 4.2 that

$$B_{n+1} = 4B_n + G_n. \tag{4.33}$$

We divide both sides by B_n

$$\frac{B_{n+1}}{B_n} = \frac{4B_n}{B_n} + \frac{G_n}{B_n} \tag{4.34}$$

thus

$$\frac{B_{n+1}}{B_n} = 4 + \frac{G_n}{B_n}. \tag{4.35}$$

We evaluate $\frac{G_n}{B_n}$ for $n+1$. From (4.33) we know the change of B_n and from Remark 4.7 we know the change of G_n therefore

$$\frac{G_{n+1}}{B_{n+1}} = \frac{a \cdot G_n}{4B_n + G_n}, \ \textit{where } a \leq 3\frac{1}{3}, \tag{4.36}$$

or $a = 4$, but then $G_{n+2} = 0$.

In (4.36) we see that the numerator grows slower (since it is always multiplied by $a \leq 3\frac{1}{3}$) than the denominator (which is always multiplied by 4^+) therefore

$$\frac{G_{n+1}}{B_{n+1}} = \frac{G_n}{B_n} \cdot x, \ where \ x < 1, \tag{4.37}$$

thus

$$\lim_{n \to \infty} \frac{G_n}{B_n} = 0. \tag{4.38}$$

From (4.35) and (4.38) we have

$$\lim_{n \to \infty} \frac{B_{n+1}}{B_n} = 4, \tag{4.39}$$

which is

$$\lim_{n \to \infty} B_{n+1} = 4B_n. \tag{4.40}$$

From (4.7) we have

$$B_{n+1} = 3B_n + A_n - lsb(B_n), \tag{4.41}$$

we substitute into (4.40)

$$\lim_{n \to \infty} (3B_n + A_n - lsb(B_n)) = 4B_n \tag{4.42}$$

therefore

$$\lim_{n \to \infty} (A_n - lsb(B_n)) = B_n. \tag{4.43}$$

This means that for any small value of d, such iteration k exists that starting from this iteration and for all the following iterations n, where $n \geq k$

$$|A_n - lsb(B_n) - B_n| < d \tag{4.44}$$

therefore

$$-d < A_n - lsb(B_n) - B_n < d. \tag{4.45}$$

For $d = 1$ we have

$$-1 < A_n - lsb(B_n) - B_n < 1. \tag{4.46}$$

We know from (4.1) and (4.6) that A_n and $lsb(B_n)$ (by definition) are single bits and B_n is an positive integer therefore

$$A_n = B_n + lsb(B_n) \tag{4.47}$$

and there are no gaps between the bits in B_n, thus

$$G_n = 0. \tag{4.48}$$

We conclude that such iteration k exists that starting from this iteration and for all the following iterations n, where $n \geq k$

$$A_n = B_n + lsb(B_n). \tag{4.49}$$

□

5. Proof of Theorem 1.2

Proof. We start Procedure 1 for any positive integer I_0. From Lemma 4.8 we know that such iteration number k exists that for all next iterations when $n \geq k$

$$lsb(B_n) = A_n - B_n. \tag{5.1}$$

From general formula on n^{th} iteration (4.9) we have

$$3^n I_0 \ = \ A_n \ - \ B_n - \ C_n. \tag{5.2}$$

For iterations where $n \geq k$, we substitute for $A_n - B_n$. We have

$$3^n I_0 \ = \ lsb(B_n) - \ C_n. \tag{5.3}$$

Notice that $lsb(B_n)$ is a single bit in the form of

$$lsb(B_n) = 2^{m_n}, \ where \ m_n \in \mathbb{Z}^+ \tag{5.4}$$

and C_n is created by iterating Procedure 1, based on the formula

$$C_n = 3C_{n-1} + lsb(B_{n-1}). \tag{5.5}$$

With each iteration, C_n is multiplied by 3 (all bits are multiplied by 3) and new higher bit is added, so it can be presented as

$$C_n = 3^{n-1} \cdot 2^{m_0} + 3^{n-2} \cdot 2^{m_1} + ... + 3^1 \cdot 2^{m_{n-2}} + 3^0 \cdot 2^{m_{n-1}}, \tag{5.6}$$

where

$$m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0.$$

We substitute in (5.3)

$$\begin{aligned} 3^n I_0 \ &= \ 2^{m_n} \ - \ (3^{n-1} \cdot 2^{m_0} + 3^{n-2} \cdot 2^{m_1} + ... + 3^1 \cdot 2^{m_{n-2}} + 3^0 \cdot 2^{m_{n-1}}) \\ &= \ 2^{m_n} \ - \ 3^{n-1} \cdot 2^{m_0} - 3^{n-2} \cdot 2^{m_1} - ... - 3^1 \cdot 2^{m_{n-2}} - 3^0 \cdot 2^{m_{n-1}} \end{aligned} \tag{5.7}$$

and we sort terms to get

$$3^n I_0 \ = \ 2^{m_n} - 2^{m_{n-1}} 3^0 - 2^{m_{n-2}} 3^1 - \ \cdots \ - \ 2^{m_1} 3^{n-2} \ - \ 2^{m_0} 3^{n-1},$$

where all m's form a sequence of integers that

$$m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0.$$

We conclude that for every initial positive integer I_0, when iterating Procedure 1, such positive integer k exists that for every positive integer $n \geq k$ a sequence of integers

$$m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$$

exists, for which

$$3^n I_0 \ = \ 2^{m_n} - 2^{m_{n-1}} 3^0 - 2^{m_{n-2}} 3^1 - \ \cdots \ - \ 2^{m_1} 3^{n-2} \ - \ 2^{m_0} 3^{n-1}.$$

□

6. Extension of Theorem 1.2

Theorem 6.1. *For every initial positive integer I_0, an infinite number of equations exists that satisfies Theorem 1.2, therefore, it can be extended in an infinite number of ways to form the following expression*

$$\boldsymbol{I_0} = \frac{\mathbf{2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \cdots - 2^{m_1}3^{n-2} - 2^{m_0}3^{n-1}}}{\mathbf{3^n}}, \tag{6.1}$$

where n is a positive integer and all m's form a sequence of integers that

$$m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0.$$

Proof. The proof of Theorem 1.2 confirms that. □

7. Examples

Presented below are various examples of positive integers, confirming the Theorems proven above.

$$3^6 \cdot \mathbf{9} = 2^{13} - 2^9 3^0 - 2^6 3^1 - 2^4 3^2 - 2^3 3^3 - 2^2 3^4 - 2^0 3^5 \tag{7.1}$$

$$3^7 \cdot \mathbf{9} = 2^{15} - 2^{13} 3^0 - 2^9 3^1 - 2^6 3^2 - 2^4 3^3 - 2^3 3^4 - 2^2 3^5 - 2^0 3^6 \tag{7.2}$$

$$3^8 \cdot \mathbf{9} = 2^{17} - 2^{15} 3^0 - 2^{13} 3^1 - 2^9 3^2 - 2^6 3^3 - 2^4 3^4 - 2^3 3^5 - 2^2 3^6 - 2^0 3^7 \tag{7.3}$$

$$\begin{aligned} 3^{12} \cdot \mathbf{6541} = 2^{32} - 2^{28} 3^0 - 2^{25} 3^1 - 2^{23} 3^2 - 2^{22} 3^3 - 2^{21} 3^4 - 2^{17} 3^5 \\ -2^{15} 3^6 - 2^{13} 3^7 - 2^{10} 3^8 - 2^9 3^9 - 2^3 3^{10} - 2^0 3^{11} \end{aligned} \tag{7.4}$$

$$3^7 \cdot \mathbf{435} = 2^{20} - 2^{16} 3^0 - 2^{11} 3^1 - 2^{10} 3^2 - 2^9 3^3 - 2^4 3^4 - 2^1 3^5 - 2^0 3^6 \tag{7.5}$$

$$\begin{aligned} 3^{41} \mathbf{27} = 2^{70} - 2^{66} 3^0 - 2^{61} 3^1 - 2^{60} 3^2 - 2^{59} 3^3 - 2^{56} 3^4 - 2^{52} 3^5 \\ -2^{50} 3^6 - 2^{48} 3^7 - 2^{44} 3^8 - 2^{43} 3^9 - 2^{42} 3^{10} - 2^{41} 3^{11} - 2^{38} 3^{12} \\ -2^{37} 3^{13} - 2^{36} 3^{14} - 2^{35} 3^{15} - 2^{34} 3^{16} - 2^{33} 3^{17} - 2^{31} 3^{18} - 2^{30} 3^{19} \\ -2^{28} 3^{20} - 2^{27} 3^{21} - 2^{26} 3^{22} - 2^{23} 3^{23} - 2^{21} 3^{24} - 2^{20} 3^{25} - 2^{19} 3^{26} \\ -2^{18} 3^{27} - 2^{16} 3^{28} - 2^{15} 3^{29} - 2^{14} 3^{30} - 2^{12} 3^{31} - 2^{11} 3^{32} - 2^9 3^{33} \\ -2^7 3^{34} - 2^6 3^{35} - 2^5 3^{36} - 2^4 3^{37} - 2^3 3^{38} - 2^1 3^{39} - 2^0 3^{40} \end{aligned} \tag{7.6}$$

$$\begin{aligned} 3^{34} \cdot \mathbf{121} = 2^{61} - 2^{57} 3^0 - 2^{52} 3^1 - 2^{51} 3^2 - 2^{50} 3^3 - 2^{47} 3^4 - 2^{43} 3^5 \\ -2^{41} 3^6 - 2^{39} 3^7 - 2^{35} 3^8 - 2^{34} 3^9 - 2^{33} 3^{10} - 2^{32} 3^{11} - 2^{29} 3^{12} \\ -2^{28} 3^{13} - 2^{27} 3^{14} - 2^{26} 3^{15} - 2^{25} 3^{16} - 2^{24} 3^{17} - 2^{22} 3^{18} - 2^{21} 3^{19} \\ -2^{19} 3^{20} - 2^{18} 3^{21} - 2^{17} 3^{22} - 2^{14} 3^{23} - 2^{12} 3^{24} - 2^{11} 3^{25} - 2^{10} 3^{26} \\ -2^9 3^{27} - 2^7 3^{28} - 2^6 3^{29} - 2^5 3^{30} - 2^3 3^{31} - 2^2 3^{32} - 2^0 3^{33} \end{aligned} \tag{7.7}$$

(7.8)

$$\begin{aligned}
3^{174}\cdot \mathbf{8388607} = 2^{299} &- 2^{295}3^{0} - 2^{290}3^{1} - 2^{289}3^{2} - 2^{288}3^{3} - 2^{285}3^{4} \\
&-2^{281}3^{5} - 2^{279}3^{6} - 2^{277}3^{7} - 2^{273}3^{8} - 2^{272}3^{9} - 2^{271}3^{10} - 2^{270}3^{11} - 2^{267}3^{12} \\
&-2^{266}3^{13} - 2^{265}3^{14} - 2^{264}3^{15} - 2^{263}3^{16} - 2^{262}3^{17} - 2^{260}3^{18} - 2^{259}3^{19} - 2^{257}3^{20} \\
&-2^{256}3^{21} - 2^{255}3^{22} - 2^{252}3^{23} - 2^{250}3^{24} - 2^{249}3^{25} - 2^{248}3^{26} - 2^{247}3^{27} - 2^{245}3^{28} \\
&-2^{244}3^{29} - 2^{243}3^{30} - 2^{241}3^{31} - 2^{240}3^{32} - 2^{236}3^{33} - 2^{235}3^{34} - 2^{234}3^{35} - 2^{233}3^{36} \\
&-2^{232}3^{37} - 2^{229}3^{38} - 2^{227}3^{39} - 2^{225}3^{40} - 2^{224}3^{41} - 2^{223}3^{42} - 2^{221}3^{43} - 2^{219}3^{44} \\
&-2^{218}3^{45} - 2^{214}3^{46} - 2^{213}3^{47} - 2^{207}3^{48} - 2^{206}3^{49} - 2^{204}3^{50} - 2^{201}3^{51} - 2^{200}3^{52} \\
&-2^{198}3^{53} - 2^{197}3^{54} - 2^{196}3^{55} - 2^{195}3^{56} - 2^{193}3^{57} - 2^{190}3^{58} - 2^{187}3^{59} - 2^{185}3^{60} \\
&-2^{184}3^{61} - 2^{183}3^{62} - 2^{180}3^{63} - 2^{179}3^{64} - 2^{178}3^{65} - 2^{173}3^{66} - 2^{172}3^{67} - 2^{171}3^{68} \\
&-2^{170}3^{69} - 2^{169}3^{70} - 2^{168}3^{71} - 2^{166}3^{72} - 2^{165}3^{73} - 2^{163}3^{74} - 2^{162}3^{75} - 2^{160}3^{76} \\
&-2^{158}3^{77} - 2^{157}3^{78} - 2^{151}3^{79} - 2^{150}3^{80} - 2^{148}3^{81} - 2^{147}3^{82} - 2^{146}3^{83} - 2^{145}3^{84} \\
&-2^{143}3^{85} - 2^{139}3^{86} - 2^{138}3^{87} - 2^{131}3^{88} - 2^{130}3^{89} - 2^{128}3^{90} - 2^{126}3^{91} - 2^{123}3^{92} \\
&-2^{122}3^{93} - 2^{121}3^{94} - 2^{120}3^{95} - 2^{119}3^{96} - 2^{118}3^{97} - 2^{117}3^{98} - 2^{116}3^{99} - 2^{114}3^{100} \\
&-2^{113}3^{101} - 2^{112}3^{102} - 2^{108}3^{103} - 2^{107}3^{104} - 2^{105}3^{105} - 2^{102}3^{106} - 2^{101}3^{107} \\
&-2^{100}3^{108} - 2^{99}3^{109} - 2^{98}3^{110} - 2^{94}3^{111} - 2^{93}3^{112} - 2^{91}3^{113} - 2^{90}3^{114} - 2^{89}3^{115} \\
&-2^{87}3^{116} - 2^{86}3^{117} - 2^{84}3^{118} - 2^{83}3^{119} - 2^{81}3^{120} - 2^{80}3^{121} - 2^{78}3^{122} - 2^{74}3^{123} \\
&-2^{72}3^{124} - 2^{71}3^{125} - 2^{69}3^{126} - 2^{67}3^{127} - 2^{66}3^{128} - 2^{65}3^{129} - 2^{61}3^{130} - 2^{60}3^{131} \\
&-2^{59}3^{132} - 2^{58}3^{133} - 2^{57}3^{134} - 2^{56}3^{135} - 2^{54}3^{136} - 2^{53}3^{137} - 2^{52}3^{138} - 2^{49}3^{139} \\
&-2^{46}3^{140} - 2^{42}3^{141} - 2^{40}3^{142} - 2^{39}3^{143} - 2^{36}3^{144} - 2^{34}3^{145} - 2^{32}3^{146} - 2^{30}3^{147} \\
&-2^{29}3^{148} - 2^{28}3^{149} - 2^{24}3^{150} - 2^{22}3^{151} - 2^{21}3^{152} - 2^{20}3^{153} - 2^{19}3^{154} - 2^{18}3^{155} \\
&-2^{17}3^{156} - 2^{16}3^{157} - 2^{15}3^{158} - 2^{14}3^{159} - 2^{13}3^{160} - 2^{12}3^{161} - 2^{11}3^{162} - 2^{10}3^{163} \\
&-2^{9}3^{164} - 2^{8}3^{165} - 2^{7}3^{166} - 2^{6}3^{167} - 2^{5}3^{168} - 2^{4}3^{169} - 2^{3}3^{170} - 2^{2}3^{171} - 2^{1}3^{172} \\
&-2^{0}3^{173}
\end{aligned}$$

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