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COLLATZ CONJECTURE - THE PROOF

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1. Introduction

The Collatz conjecture is a well known mathematical problem. It claims that for every positive integer I_o if iterating

$$I_{n+1} = \begin{cases} \frac{1}{2} \cdot I_n & for,\ I_n \ \ even \\ \\ 3 \cdot I_n + 1 & for,\ I_n \ \ odd \end{cases} \tag{1.1}$$

ultimately we get 1.

The purpose of this paper is to prove that the Collatz conjecture is true. The proof consists of two parts:

Theorem 1.1. *If the Collatz conjecture is true for a positive integer I_0, it is equivalent of the condition that a positive integer n and a sequence of integers $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ exists, for which*

$$3^n I_0 \ = \ 2^{m_n} - 2^{m_{n-1}}3^0 - 2^{m_{n-2}}3^1 - \ \cdots \ - \ 2^{m_1}3^{n-2} \ - \ 2^{m_0}3^{n-1}. \tag{1.2}$$

Theorem 1.2. *For every positive integer I_0, such a positive integer n and a sequence of integers $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ can be found. Therefore, (by Theorem 1.1) the Collatz conjecture is true.*

2. Remarks and Definitions

To understand how the Collatz conjecture works and make it more accessible, we have to iterate integers in their binary representations. This paper explains when binary numbers are even or odd, how they are affected by different operations and examines how they iterate through the Collatz formula. The definitions and remarks introduced below are used over the course of this paper.

Remark 2.1. An integer is odd when in binary representation its least significant bit is 1. An integer is even when in binary representation its least significant bit is 0.

Remark 2.2. Every even positive integer can be reduced to the odd positive integer by recursively dividing it by 2 until the result is odd.

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When I_{even} is an even positive integer, I_{odd} is an odd positive integer and p is the number of divisions by 2 required for I_{even} to became the odd integer I_{odd}, then

$$\frac{I_{even}}{2^p} = I_{odd}. \tag{2.1}$$

Example 2.3. Reduction of an even integer to an odd integer in binary representation.

Let I_{even} be an even positive integer

$$I_{even} = 20 = 10100_b.$$

Then

$$\begin{aligned} \frac{I_{even}}{2^p} &= \frac{20}{2^2} \\ &= \frac{10\mathbf{100}_b}{\mathbf{100}_b} \\ &= 10\mathbf{1}_b \\ &= 5 = I_{odd}. \end{aligned}$$

We see that an even positive integer I_{even} can be reduced to an odd positive integer I_{odd}. In this case 20 is reduced to 5.

Remark 2.4. By multiplying an odd positive integer by 3 and adding 1, we get a result which is always even

$$3 \cdot I_{odd} + 1 = I_{even}. \tag{2.2}$$

Example 2.5. Example in binary representation.
Let I_{odd} be an odd positive integer

$$I_{odd} = 7 = 111_b.$$

Then

$$\begin{aligned} 3I_{odd} + 1 &= 21 + 1 \\ &= 10101_b + 1 \\ &= 1011\mathbf{0}_b \\ &= 22 = I_{even}. \end{aligned}$$

We see that by multiplying an odd positive integer I_{odd} by 3 and increasing by 1, we get an even positive integer I_{even}.

Definition 2.6. For any positive integer I, let $lsb(I)$ be **the least significant nonzero bit** in the binary representation of I.

Example 2.7. Binary numbers with their least significant nonzero bits in bold:

$$\begin{aligned} lsb(10110101\mathbf{1}000_b) &= \mathbf{1000}_b, \\ lsb(1001011\mathbf{1}0_b) &= \mathbf{10}_b, \\ lsb(10110101\mathbf{1}00_b) &= \mathbf{100}_b, \\ lsb(110011\mathbf{1}_b) &= \mathbf{1}_b, \\ lsb(110111\mathbf{1}000_b) &= \mathbf{1000}_b. \end{aligned}$$

Remark 2.8. For every odd positive integer I_{odd}

$$lsb(I_{odd}) = 2^0 = 1. \tag{2.3}$$

Example 2.9. We find $lsb(I_{odd})$ for an odd positive integer I_{odd}.
For $I_{odd} = 25$ we have

$$lsb(25) = lsb(1100\mathbf{1}_b) \ = \ 2^0 = 1.$$

Remark 2.10. For every even positive integer I_{even}

$$lsb(I_{even}) = 2^p, \tag{2.4}$$

where p is a positive integer, and then

$$\frac{I_{even}}{2^p} \ = \ I_{odd}, \tag{2.5}$$

therefore

$$I_{even} = 2^p I_{odd}. \tag{2.6}$$

Example 2.11. We find $lsb(I_{even})$ for an even positive integer I_{even}.
For $I_{even} = 28$ we have

$$lsb(28) = lsb(11\mathbf{1}00_b) \ = 2^2$$

and thus

$$\begin{aligned} \frac{I_{even}}{2^p} &= \frac{28}{lsb(28)} \\ &= \frac{28}{2^2} \\ &= \frac{11\mathbf{100}_b}{\mathbf{100}_b} \\ &= \ 7 = I_{odd}. \end{aligned}$$

When we divide 28 by $lsb(28)$ it gives us an odd positive integer 7.

Definition 2.12. For any positive integer I, let $msb(I)$ be **the most significant bit** in a binary representation of I.

Example 2.13. Binary numbers with their most significant bits in bold:

$$\begin{aligned} msb(\mathbf{1}01101011000_b) &= \mathbf{100000000000}_b, \\ msb(\mathbf{1}0010110_b) &= \mathbf{10000000}_b, \\ msb(\mathbf{1}0110101100_b) &= \mathbf{10000000000}_b, \\ msb(\mathbf{1}100111_b) &= \mathbf{1000000}_b, \\ msb(\mathbf{1}101111000_b) &= \mathbf{1000000000}_b. \end{aligned}$$

Definition 2.14. For any positive integer I, let $N(I)$ be **the number of consecutive nonzero bits attached to $lsb(I)$** in the binary representation of I.

Example 2.15. Binary numbers with consecutive nonzero bits attached to lsb in bold:

$$\begin{aligned} N(1011010\mathbf{11}000_b) &= 1, \\ N(10010\mathbf{11}0_b) &= 1, \\ N(10110\mathbf{111}100_b) &= 3, \\ N(1100\mathbf{11}_b) &= 1, \\ N(\mathbf{11111}1000_b) &= 5. \end{aligned}$$

Definition 2.16. Let O denote a **base odd integer** of I and be defined as

$$O = \frac{I}{lsb(I)}, \tag{2.7}$$

where I can be an even or odd positive integer.

Example 2.17. Finding a base odd integer.
We check the case for an odd integer

$$\begin{aligned} I &= 9 = 1001_b, \\ lsb(I) &= lsb(100\mathbf{1}_b) = \mathbf{1}_b, \end{aligned}$$

$$\begin{aligned} O &= \frac{I}{lsb(I)} \\ &= \frac{100\mathbf{1}_b}{\mathbf{1}_b} \\ &= 1001_b \\ &= 9. \end{aligned}$$

We conclude that for odd integers

$$O = I. \tag{2.8}$$

Notice that when I is an odd positive integer, its base odd integer O is equal to I.
Now we check the case for an even integer

$$\begin{aligned} I &= 20 = 10100_b, \\ lsb(I) &= lsb(10\mathbf{100}_b) = \mathbf{100}_b, \end{aligned}$$

$$\begin{aligned} O &= \frac{I}{lsb(I)} \\ &= \frac{10\mathbf{100}_b}{\mathbf{100}_b} \\ &= 101_b \\ &= 5. \end{aligned}$$

To find the base odd integer O for an even integer I, we divide integer I by 2 until we get an odd result. We do this by dividing I by its least significant nonzero bit $lsb(I)$.

3. Simplification of The Collatz conjecture

Using the above remarks and definitions, standard form of the Collatz conjecture (1.1) can be substantially simplified. Despite each of the following simplifications iterating integers in slightly different way, all of them are fully aligned with original definition and therefore can be used to prove the Collatz conjecture.

Example 3.1. Iteration of the Collatz conjecture (1.1) starting from $I_0 = 11$.

Table 1. Original Collatz iterations starting from $I_0 = 11$.

n	I_n	$(I_n)_b$	$even/odd$	p_n	$(2^{p_n})_b$
0	11	1011	o		
1	34	100010	e	1	10
2	17	10001	o		
3	52	110100	e	2	100
4	26	11010	e		
5	13	1101	o		
6	40	101000	e	3	1000
7	20	10100	e		
8	10	1010	e		
9	5	101	o		
10	16	10000	e	4	10000
11	8	1000	e		
12	4	100	e		
13	2	10	e		
14	1	1	o		

In binary notation, division by 2 is simply a shift of the whole number by one position(bit) to the right. In Table 1, we see it for every even integer. Instead of multiple divisions by 2, it can be shortened to one operation. We divide by 2^{p_n}, where p_n is a positive integer and represents a number of consecutive zeros at the end of a binary number. Notice that 2^{p_n} is the least significant nonzero bit of an even integer, defined earlier in Definition 2.6. Merging all single divisions by 2 into one division by 2^{p_n}, we can simplify iterations of the Collatz conjecture to iterations presented in Table 2.

Formally, this simplification of Collatz conjecture can be define as

$$I_{n+1} = \begin{cases} \frac{I_n}{2^{p_n}} & for,\ I_n \ \ even, \\ \\ 3 \ \cdot I_n + 1 & for,\ I_n \ \ odd, \end{cases} \tag{3.1}$$

where $2^{p_n} = lsb(I_n)$ is the least significant nonzero bit of I_n.

Symbol I_n is kept as a representation of elements in the series, even if some elements are omitted in comparison to the original Collatz conjecture proposition (1.1).

TABLE 2. Collatz iterations with divisions by 2^{p_n}.

n	I_n	$(I_n)_b$	*even/odd*	p_n	$(2^{p_n})_b$
0	11	1011	*o*		
1	34	100010	*e*	1	10
2	17	10001	*o*		
3	52	110100	*e*	2	100
4	13	1101	*o*		
5	40	101000	*e*	3	1000
6	5	101	*o*		
7	16	10000	*e*	4	10000
8	1	1	*o*		

Since now each even integer is producing odd integer and each odd integer is producing even integer, we can consolidate both operations into one. This time, we process only odd positive integers, so we substitute I_n with O_n using definition (2.16). We define this simplification of the Collatz conjecture as

$$O_{n+1} = \frac{3 \ \cdot O_n + 1}{2^{p_n}}, \tag{3.2}$$

where $2^{p_n} = lsb(3 \cdot O_n + 1)$ is the least significant nonzero bit of $(3 \cdot O_n + 1)$.

Notice that $(3 \cdot O_n + 1)$ is always even, so $2^{p_n} \geq 2$ for every n. This simplification of Collatz conjecture results in iterations of odd integers only. To start from an even integer, we simply reduce it to an odd integer, by dividing it by 2 as many times as needed to achieve an odd result.

TABLE 3. Collatz iterations simplified to odd integers only.

n	O_n	$(O_n)_b$	e/o	$3O_n + 1$	$(3O_n + 1)_b$	p_n	$(2^{p_n})_b$
0	11	1011	*o*	34	100010	1	10
1	17	10001	*o*	52	110100	2	100
2	13	1101	*o*	40	101000	3	1000
3	5	101	*o*	16	10000	4	10000
4	1	1	*o*				

There is one more simplification we can do.

The process introduced below differs from the original Collatz proposition, however, it produces the same results. To distinguish it from the above explanations, symbol A_n is used as an element of the iterations.

Starting from any positive integer A_0, we do not need to constantly divide by 2^{p_n}. To keep this process aligned with the orginal Collatz conjecture, instead of always adding 1, we have to add the least significant nonzero bit of A_n. By this, we allow A_n to increase, ultimately reaching, instead of 1, integer in the form of 2^p, where p is a positive integer.

TABLE 4. Improved Collatz conjecture - iterations without divisions.

n	A_n	$(A_n)_b$	$3A_n$	$(3A_n)_b$	p_n	$(2^{p_n})_b$	$3A_n+2^{p_n}$	$(3A_n+2^{p_n})_b$	O_n
0	11	1011	33	100001	0	1	34	**10001**0	**17**
1	34	100010	102	1100110	1	10	104	**1101**000	**13**
2	104	1101000	312	100111000	3	1000	320	**101**000000	**5**
3	320	101000000	960	1111000000	6	1000000	1024	**1**0000000000	**1**
4	1024	10000000000							

Notice that corresponding odd integers are still present in such iterations in column O_n in Table 4. They are also visible in column $(3A_n + 2^{p_n})_b$ in bold, but for each iteration they are multiplied by constantly increasing powers of 2.

Formal definition of this improved Collatz conjecture is presented below.

Definition 3.2. For any positive integer A_0 if iterating

$$A_{n+1} = 3A_n + lsb(A_n), \tag{3.3}$$

where $lsb(A_n)$ is the least significant nonzero bit of A_n, ultimately we get $A_n = 2^p$, where p is positive integer.

This way we have two equivalent methods of iterating the Collatz conjecture. The first one, proposed in (3.2), is a simplified version of (1.1) that only skips all even numbers and, as original, finally reaches 1. The second one, without any divisions by 2, proposed in (3.3), ultimately reaches 2^p, where p is a positive integer. In this case, the result in binary representation is just 1 followed by the sequence of zeros. **Each of these two methods have exactly the same number of steps as they are strictly connected**.

Example 3.3. In Table 5, we see a comparison of iterations through both methods side by side; without divisions (3.3) as A_n and with divisions (3.2) as O_n, starting from 11.

TABLE 5. Equivalence of Collatz iterations without divisions A_n and with divisions O_n starting from 11.

n	A_n	$(A_n)_b$	O_n	$(O_n)_b$
0	11	**1011**	11	1011
1	34	**10001**0	17	10001
2	104	**1101**000	13	1101
3	320	**101**000000	5	101
4	1024	**1**0000000000	1	1

Example 3.4. In Table 6, we see a comparison of iterations through both methods side by side; without divisions (3.3) as A_n and with divisions (3.2) as O_n, starting from 57.

TABLE 6. Equivalence of Collatz iterations without divisions A_n and with divisions O_n starting from 57.

n	A_n	$(A_n)_b$	O_n	$(O_n)_b$
0	57	**111001**	57	111001
1	172	**101011**00	43	101011
2	520	**1000001**000	65	1000001
3	1568	**110001**00000	49	110001
4	4736	**100101**0000000	37	100101
5	14336	**111**00000000000	7	111
6	45056	**1011**000000000000	11	1011
7	139264	**10001**0000000000000	17	10001
8	425984	**1101**000000000000000	13	1101
9	1310720	**101**000000000000000000	5	101
10	4194304	**1**0000000000000000000000	1	1

Example 3.5. Relations between A_n, O_n, $lsb(A_n)$ and $msb(A_n)$ are shown in the example below:

$$\overbrace{\overbrace{\mathbf{100101}}^{O_n}0000000}^{A_n} \qquad \uparrow \ \underbrace{\ \ldots\ }_{msb(A_n)} \quad \underbrace{10000000}_{lsb(A_n)}$$

$\boldsymbol{A_n}$ is the entire integer, all bits in binary notation,
$\boldsymbol{O_n}$ is the odd base of A_n, which are only bits between first and last nonzero bits,
$\boldsymbol{lsb(A_n)}$ is the least significant nonzero bit of A_n in the form of 2^p,
$\boldsymbol{msb(A_n)}$ is the most significant bit of A_n in the form of 2^q,
where p, q are positive integers.

4. ELABORATION ON IMPROVED COLLATZ CONJECTURE

Considering iterations of A_n through the improved Collatz conjecture proposed in (3.3) a very interesting feature can be seen. The least significant nonzero bit $lsb(A_n)$ is almost always just a small fraction of A_n. Therefore, the most significant bit $msb(A_n)$ tends to grow with coefficient on average close to 3 with each iteration. Using the improved Collatz conjecture

$$A_{n+1} = 3A_n + lsb(A_n),$$

we usually get

$$\frac{lsb(A_n)}{A_n} \approx 0, \tag{4.1}$$

therefore, we can say that on average

$$msb(A_{n+1}) \approx 3 \cdot msb(A_n). \tag{4.2}$$

Small deviations from this rule can be observed, when interactions with other bits of lower significance occur (especially when O_n is small), which can temporarily make this coefficient slightly higher.

On the other hand, the least significant bit $lsb(A_n)$, being a part of A_n, is each time multiplied by 3 and additionally increased by adding $lsb(A_n)$. Therefore, the least significant bit of A_n tends to grow with coefficient on average close to 4 with each iteration.

When iterating

$$A_{n+1} = 3A_n + lsb(A_n) \tag{4.3}$$

on average, we have

$$\begin{aligned} lsb(A_{n+1}) &\approx 3 \cdot lsb(A_n) + lsb(A_n) \\ &\approx 4 \cdot lsb(A_n). \end{aligned} \tag{4.4}$$

A deviation from this rule can occur through interactions with other bits of A_n. The coefficient can be temporary much higher than 4, when a sequence of bits in the form of "...10101010101" appears at the end of O_n which is a part of A_n (see Figure 2 for A_0). In this case, we can observe a rapid shortening of A_n. This coefficient can also be temporarily smaller, when a sequence of consecutive 1's appears at the end of O_n. In this case, this coefficient is temporarily equal 2, until number of 1's is reduced one by one in the following iterations (compare Table 9).

Even if both described dependencies can be temporarily disturbed, eventually in a large number of iterations they become very evident. As a result of their interactions, the distance between the most significant bit $msb(A_n)$ and the least significant nonzero bit $lsb(A_n)$ gets shortened.

Notice that a difference in lengths between $msb(A_n)$ and $lsb(A_n)$ represents the length of O_n in bits.

We see

$$msb(A_n) - lsb(A_n) \to 0, \text{ as } n \to \infty, \tag{4.5}$$

and

$$msb(A_n) \ / \ lsb(A_n) \to 1, \text{ as } n \to \infty. \tag{4.6}$$

Example 4.1. Comparison of growth trends between the most significant bit $msb(A_n)$ and the least significant nonzero bit $lsb(A_n)$.

FIGURE 1. Comparison of growth trends between $msb(A_n)$ and $lsb(A_n)$ starting from 57.

n	A_n	$(A_n)_b$	O_n	$(O_n)_b$
0	57	**111001**	57	111001
1	172	**101011**00	43	101011
2	520	**1000001**000	65	1000001
3	1568	**110001**00000	49	110001
4	4736	**100101**0000000	37	100101
5	14336	**111**00000000000	7	111
6	45056	**1011**000000000000	11	1011
7	139264	**10001**0000000000000	17	10001
8	425984	**1101**000000000000000	13	1101
9	1310720	**101**000000000000000000	5	101
10	4194304	**1**0000000000000000000000	1	1

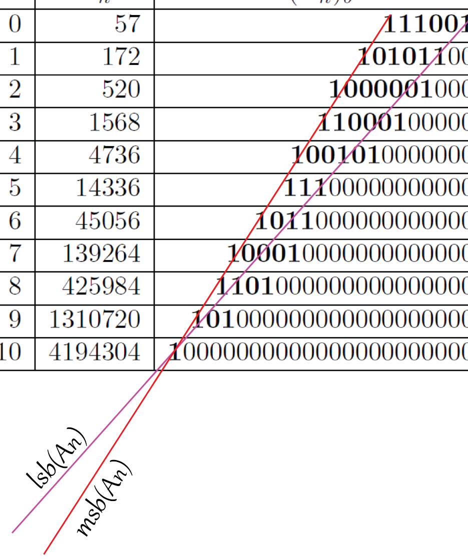
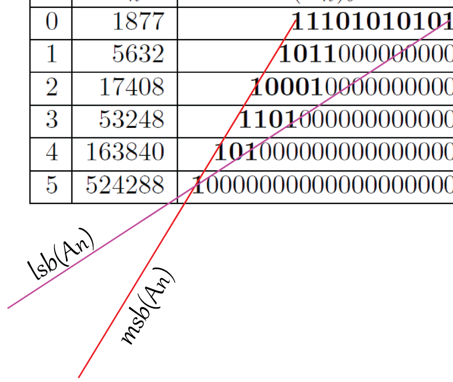


FIGURE 2. Comparison of growth trends between $msb(A_n)$ and $lsb(A_n)$. Special case when A_n contains a sequence of bits "...1010101".

n	A_n	$(A_n)_b$	O_n	$(O_n)_b$
0	1877	**11101010101**	1877	11101010101
1	5632	**1011**000000000	11	1011
2	17408	**10001**0000000000	17	10001
3	53248	**1101**000000000000	13	1101
4	163840	**101**000000000000000	5	101
5	524288	**1**0000000000000000000	1	1



When initial integer A_0 is very big, on average

$$\frac{msb(A_{n+1})}{msb(A_n)} = 3 \tag{4.7}$$

and on average

$$\frac{lsb(A_{n+1})}{lsb(A_n)} = 4, \tag{4.8}$$

we can propose a formula to estimate the number of iterations required to reach $O_n = 1$, which means $A_n = 2^p$, where p is a positive integer.
When using binary numbers, we know that each position represents a power of 2. Multiplication by 3 extends the length of a number by

$$\log_2(3) = 1.584963. \tag{4.9}$$

By continuous multiplication of a binary number by 3, its length increases on average by 1.584963 bits(positions) per operation.

We check how fast the least significant bit $lsb(A_n)$ increases its length, we have

$$\log_2(4) = 2. \tag{4.10}$$

We see that by continuous multiplication of the least significant nonzero bit by 4, its length increases on average by 2 bits(positions) per operation. We calculate how fast $lsb(A_n)$ approaches $msb(A_n)$.

We have

$$2 - 1.584963 = 0.415037, \tag{4.11}$$

thus $lsb(A_n)$ is on average 0.415037 bits(positions) closer to $msb(A_n)$ per iteration. Note that a number of needed iterations can be bigger, when at the end of A_0 we have a sequence of consecutive 1's "...1111111", or it can be dramatically smaller, when at the end we have a sequence of alternating 0 and 1 "...01010101".

Example 4.2. Starting from A_0, which is 20000 bits long, we can predict how many times we have to iterate, through the improved version of the Collatz conjecture (3.3), until we finally reach $O_n = 1$ (which means $A_n = 2^p$, where p is a positive integer). To approximate a number of iterations, we have to divide the length of A_0 in bits by 0.415037, in this case

$$\frac{20000}{0.415037} \approx \mathbf{48188}. \tag{4.12}$$

Exact number of required operations depends on detailed structure of bits in a particular initial integer. However, for big initial integers that do not end with consecutive 1's or alternating sequences of 0 and 1, exact number of iterations should be very close to an estimated one. In practice, starting from A_0, which was created as randomly generated 20000 bits, the exact number of operations needed to reach 1 was **48043**, which is only around 0.3% different from the estimated one.

On Figure 3, we see how length of O_n, in number of bits, decreases when iterating initial integer A_0 consisting of 20000 random bits.

Length of O_n, is the difference in bits between the length of $msb(A_n)$ and the length of $lsb(A_n)$ and decreases with almost perfect accuracy (see Figure 3). However, when we look closer at first 1000 iterations on Figure 4, we see local fluctuations. It is even more visible on Figure 5, where only first 100 iterations are presented.

Above elaboration, together with analysis of ending sequences of 1's, "...111111" described in Section 7 of this work, can be enough to proof the Collatz conjecture, however it is not used for this purpose in this work. It is only presented for better understanding how integers are processed iterating through the Collatz formula and what we can observe when analyzing their binary representations.

FIGURE 3. Decrease of O_n length for 20000 bits long initial A_0

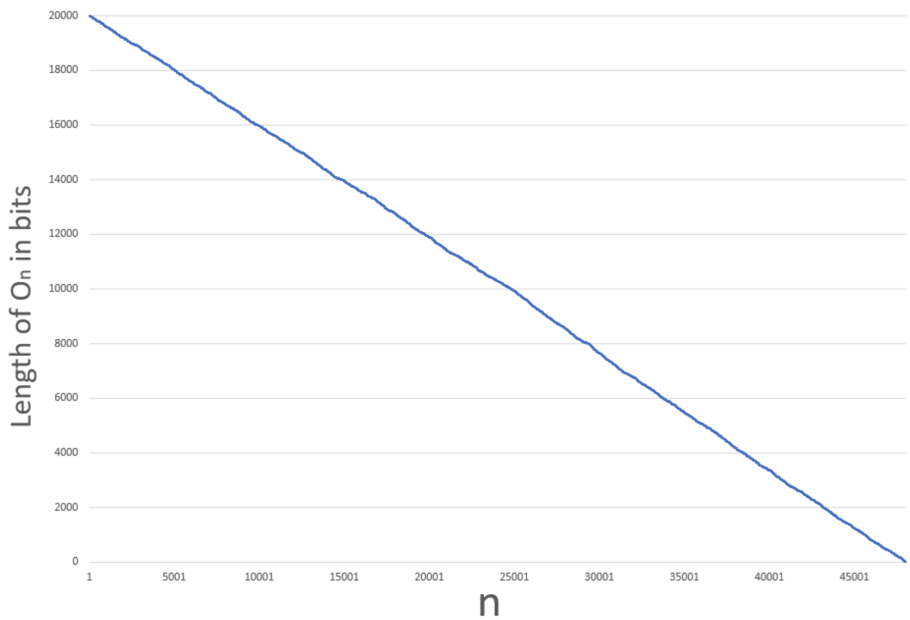


FIGURE 4. Decrease of O_n length for 20000 bits long initial A_0 *(first 1000 iterations)*.

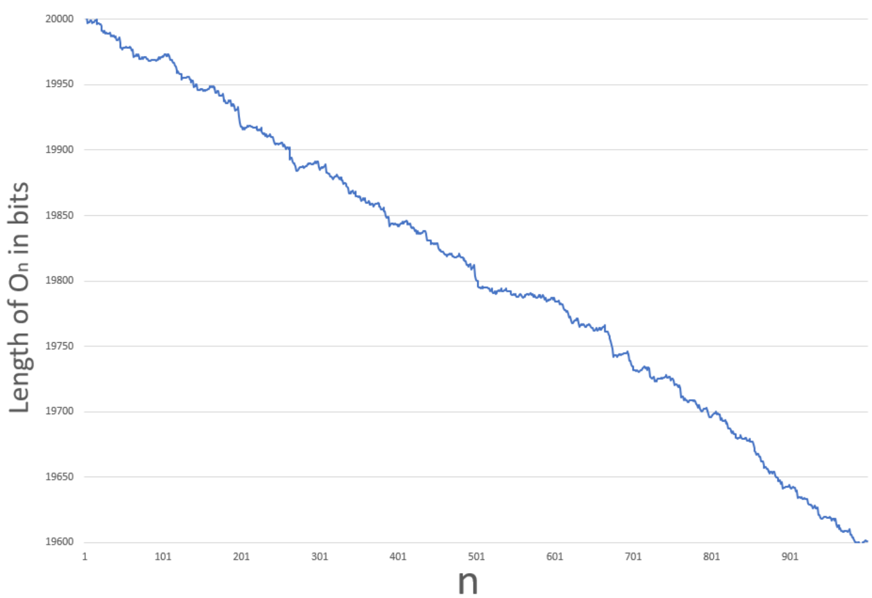
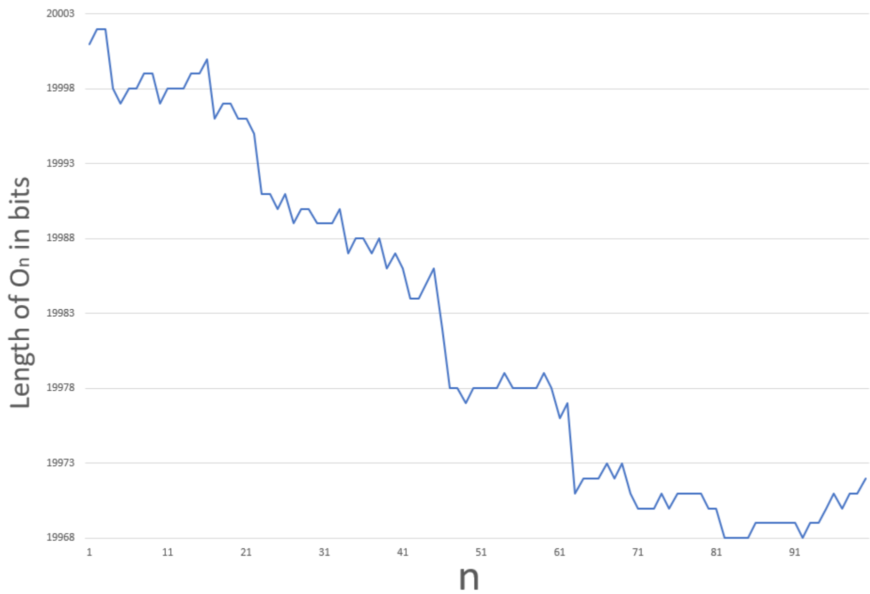


Figure 5. Decrease of O_n length for 20000 bits long initial A_0 *(first 100 iterations).*



5. Proof of Theorem 1.1

Proof. For any positive integer I_0, we find its base odd integer using (2.7) and it is

$$O_0 = \frac{I_0}{lsb(I_0)}. \tag{5.1}$$

Value of $lsb(I_0)$ is in the form of 2^p, where $p \geq 0$ and $p = 0$ when I_0 is odd, thus

$$O_0 = \frac{I_0}{2^p}, \tag{5.2}$$

where $p \geq 0$.

We iterate this odd positive integer O_0 through simplified Collatz conjecture presented in equation (3.2). We have

$$\frac{3\frac{3\frac{3\frac{3\frac{3O_0+1}{2^{p_0}}+1}{2^{p_1}}+1}{2^{p_2}}+1}{\cdots}}{2^{p_{n-2}}}+1}{2^{p_{n-1}}} = 1, \tag{5.3}$$

and O_n is odd for every n, so $(3O_n + 1)$ is always even, therefore

$$p_0, p_1, p_2, ..., p_{n-2}, p_{n-1} \geq 1. \tag{5.4}$$

Equation (5.3) can be also presented like this

$$\left(\left(\left(\left(\left((3O_0+1)\frac{3}{2^{p_0}}+1\right)\frac{3}{2^{p_1}}+1\right)\frac{3}{2^{p_2}}+1\right)...\right)\frac{3}{2^{p_{n-2}}}+1\right)\frac{1}{2^{p_{n-1}}}=1. \tag{5.5}$$

By performing simple algebraic transformations we get

$$\begin{aligned} 3^n O_0 \;=\; & \left(2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)-\left(2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)3^0- \\ & -\left(2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)3^1-\cdots-2^{p_1}2^{p_o}3^{n-3}\;-\;2^{p_o}3^{n-2}\;-\;3^{n-1}. \end{aligned} \tag{5.6}$$

Now, we can substitute O_0 from (5.2)

$$\begin{aligned} 3^n\tfrac{I_0}{2^p} \;=\; & \left(2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)-\left(2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)3^0-\;... \\ & -\left(2^{p_{n-3}}...\,2^{p_1}2^{p_0}\right)3^1-\cdots-2^{p_1}2^{p_o}3^{n-3}\;-\;2^{p_o}3^{n-2}\;-\;3^{n-1}, \end{aligned}$$

and multiply both sides by 2^p

$$\begin{aligned} 3^n I_0 \;=\; & \left(2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p\right)-\left(2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p\right)3^0-\;... \\ & -\left(2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p\right)3^1-\cdots-2^{p_1}2^{p_o}2^p3^{n-3}\;-\;2^{p_o}2^p3^{n-2}\;-\;2^p3^{n-1}. \end{aligned}$$

We substitute the following:

$$\begin{aligned} 2^{p_{n-1}}2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_n}, \\ 2^{p_{n-2}}2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_{n-1}}, \\ 2^{p_{n-3}}...\,2^{p_1}2^{p_0}2^p &= 2^{m_{n-2}}, \\ & ... \\ 2^{p_1}2^{p_o}2^p &= 2^{m_2}, \\ 2^{p_o}2^p &= 2^{m_1}, \\ 2^p &= 2^{m_0}, \end{aligned} \tag{5.7}$$

where all $p_0, p_1, p_2, ..., p_{n-2}, p_{n-1} \geq 1$ and $p \geq 0$.

We finally have

$$3^n I_0 \;=\; 2^{m_n}-2^{m_{n-1}}3^0-2^{m_{n-2}}3^1-\;\cdots\;-\;2^{m_1}3^{n-2}\;-\;2^{m_0}3^{n-1}, \tag{5.8}$$

where $m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0$ and m_0 can eventually be 0, when I_0 is odd.

□

6. Procedure

We consider the following procedure.

Procedure 1.

Step 1. Take any positive integer A_0 and define $B_0 = 0$, $C_0 = lsb(A_0)$ (the least significant nonzero bit of A_0 in its binary representation). Let our initial equation be

$$A_0 \;=\; A_0 \;-\; B_0\;. \tag{6.1}$$

Step 2. Multiply both sides of equation by 3

$$3A_0 \ = \ 3A_0 \ - \ 3B_0. \tag{6.2}$$

Step 3. On the right side of equation we add C_0 to A and B sections

$$3A_0 \ = \ (3A_0 + C_0) \ - \ (3B_0 \ + C_0). \tag{6.3}$$

This is still a valid equation because $B_0 = 0$ and C_0 elements cancel each other.

Step 4. We name A and B sections as A_1 and B_1, so we have

$$3^1 A_0 \ = \ A_1 \ - \ B_1, \tag{6.4}$$

where

$$A_1 = 3A_0 + C_0, \tag{6.5}$$

$$B_1 = 3B_0 + C_0, \tag{6.6}$$

and

$$C_0 = lsb(A_0). \tag{6.7}$$

Step 5. By repeating steps 2 to 4, we get universal equations for iteration n

$$\boldsymbol{3^n A_0 \ = \ A_n \ - \ B_n}, \tag{6.8}$$

where

$$\boldsymbol{A_n \ = \ 3A_{n-1} + C_{n-1}}, \tag{6.9}$$

$$\boldsymbol{B_n \ = \ 3B_{n-1} + C_{n-1}}, \tag{6.10}$$

$$\boldsymbol{C_{n-1} = \ lsb(A_{n-1})}. \tag{6.11}$$

By definition, C_{n-1} is the least significant nonzero bit of A_{n-1}, so

$$0 < C_{n-1} \leq A_{n-1}, \tag{6.12}$$

thus

$$3A_{n-1} < A_n \leq 4A_{n-1}. \tag{6.13}$$

Notice that for every iteration n

$$A_n \ > \ B_n, \tag{6.14}$$

$$A_n \ > \ 3A_{n-1}, \tag{6.15}$$

$$B_n \ > \ 3B_{n-1}, \tag{6.16}$$

$$C_n \ \geq \ 2C_{n-1}, \tag{6.17}$$

we can continue this procedure **forever**.

Remark 6.1. When increasing index from n to $n+1$ and calculating

$$A_{n+1} = 3A_n + C_n$$

we have

$$lsb(A_n) = C_n,$$

also

$$lsb(3A_n) = C_n,$$

but

$$lsb(3A_n + C_n) = C_{n+1},$$

therefore

$$C_{n+1} \geq \ 2C_n.$$

General rule for increase of C_n when iterating Procedure 1 is

$$C_{n+1} = 2^p \cdot C_n, \tag{6.18}$$

where p is a positive integer.

Note that minimal possible change of C_n as n increases is

$$C_{n+1} = 2 \cdot C_n. \tag{6.19}$$

Example 6.2. Minimal change of C_n.
We check how C_n increases starting from 3. For the initial value $A_n = 3$ we have

$$C_n = lsb(A_n) = lsb(3) = lsb(1\mathbf{1}_b) = 2^0 = 1.$$

We see that in the next iteration it is

$$C_{n+1} = lsb(3A_n + C_n) = lsb(10) = lsb(10\mathbf{1}0_b) = 2^1 = 2.$$

In this case the change of C_n is minimal

$$C_{n+1} = \mathbf{2^1} C_n.$$

Example 6.3. Non-minimal change of C_n.
Now we check how C_n increases starting from 5. For the initial value $A_n = 5$ we have

$$C_n = lsb(A_n) = lsb(5) = lsb(10\mathbf{1}_b) = 2^0 = 1.$$

We see that in the next iteration it is

$$C_{n+1} = lsb(3A_n + C_n) = lsb(16) = lsb(\mathbf{1}0000_b) = 2^4 = 16.$$

In this case the change of C_n is **non-**minimal

$$C_{n+1} = \mathbf{2^4} C_n.$$

7. Series of the consecutive minimal changes of C_n

Remark 7.1. When iterating through Procedure 1, we see that the number of consecutive minimal changes of C_n (when $C_{n+1} = 2C_n$) is limited by the number of consecutive nonzero bits attached to $lsb(A_n)$ in binary representation of A_n. This dependency can be formulated as follows.

The number of consecutive minimal changes of C_n is equal to $N(A_n)$, the number of consecutive nonzero bits attached to $lsb(A_n)$ in A_n (see Definition 2.14).

In case when there are no nonzero bits attached to $lsb(A_n)$, we have $C_{n+1} > 2C_n$ (see Table 7 for A_3 and A_4, where we have corresponding 4 and 4 in the column C_{n+1}/C_n). The number of consecutive minimal changes of C_n in example below (number of 2's in C_{n+1}/C_n) is equal to $N(A_0)$.

Example 7.2. In this case $N(A_0) = 3$, thus the number of consecutive minimal changes of C_n is 3.

Table 7. Relation between $N(A_n)$, the number of consecutive nonzero bits attached to $lsb(A_n)$, and the number of consecutive minimal changes of C_n.

n	A_n	$N(A_n)$	$3A_n$	C_n	C_{n+1}/C_n
0	101111_b	**3**	10001101_b	1_b	**2**
1	10001110_b	2	110101010_b	10_b	**2**
2	110101100_b	1	10100000100_b	100_b	**2**
3	10100001000_b	**0**	111100011000_b	1000_b	**4**
4	111100100000_b	**0**	10110101100000_b	100000_b	**4**
5	10110110000000_b	1	1000100010000000_b	10000000_b	...
...	...	...	...	...	...

Example 7.3. In this case $N(A_0) = 2$, thus the number of consecutive minimal changes of C_n is 2.

Table 8. Relation between how many times $\frac{C_{n+1}}{C_n} = 2$ and $N(A_n)$.

n	A_n	$N(A_n)$	$3A_n$	C_n	C_{n+1}/C_n
0	1110_b	**2**	101010_b	10_b	**2**
1	101100_b	1	10000100_b	100_b	**2**
2	10001000_b	**0**	110011000_b	1000_b	**4**
3	110100000_b	**0**	10011100000_b	100000_b	**8**
4	10100000000_b	1	111100000000_b	100000000_b	...
...	...	...	...	...	...

Remark 7.4. Notice that as we iterate and $n \to \infty$, each time $N(A_n)$ decreases to 0, gets new value and decreases to 0 again and so on ... (compare Example 7.6).

Remark 7.5. Notice that for every iteration n, for which $N(A_n) = 0$, we have $C_{n+1}/C_n > 2$.

Example 7.6. We check initial iterations of number 27 (11011_b) using the Procedure 1. In Table 9, we see how $N(A_n)$ decreases to 0. When $N(A_n)$ is greater than 0, C_n has minimal change ($C_{n+1} = 2C_n$). When $N(A_n)$ reaches 0, change of C_n is not minimal ($C_{n+1} > 2C_n$).

TABLE 9. $N(A_n)$ decreases to 0 in series, as $n \to \infty$.

n	A_n	$N(A_n)$	C_{n+1}/C_n
0	110**11**	1	2
1	1010010	0	4
2	**1111**1000	4	2
3	10**111**10000	3	2
4	1000**11**100000	2	2
5	110101**1**1000000	1	2
6	101000010000000	0	4
7	1111001000000000	0	4
8	10110**1**100000000000	1	2
9	10001001000000000000	0	4
10	1100**11**100000000000000	2	2
11	100110**1**1000000000000000	1	2
12	1110100100000000000000000	0	4
13	1010**111**10000000000000000000	3	2
14	100000**11**1000000000000000000000	2	2
15	1100010**1**1000000000000000000000	1	2
16	10010100010000000000000000000000	0	...
...	...	...	...

Remark 7.7. As explained above, minimal change of C_n requires a series of 1's at the end of O_n, which decreases with each iteration. To achieve constantly minimal change of C_n we need an infinitely long series of 1's. For every integer, such series is always limited, therefore inevitably after sequence of minimal changes of C_n,where $C_{n+1} = 2C_n$, we have bigger change, where $C_{n+1} > 2C_n$.

8. SERIES OF B_n

We check how B_n grows, assuming $C_0 = 1$ and **constantly minimal change of C_n** which is

$$C_{n+1} = 2C_n, \tag{8.1}$$

for every n.

TABLE 10. Change of B_n with $n \to \infty$, when $C_{n+1} = 2C_n$.

n	B_n	C_n	$3B_n + C_n$
0	0	2^0	2^0
1	2^0	2^1	$3^1 2^0 + 2^1$
2	$3^1 2^0 + 2^1$	2^2	$3^2 2^0 + 3^1 2^1 + 2^2$
3	$3^2 2^0 + 3^1 2^1 + 2^2$	2^3	$3^3 2^0 + 3^2 2^1 + 3^1 2^2 + 2^3$
4	$3^3 2^0 + 3^2 2^1 + 3^1 2^2 + 2^3$	2^4	$3^4 2^0 + 3^3 2^1 + 3^2 2^2 + 3^1 2^3 + 2^4$
5	$3^4 2^0 + 3^3 2^1 + 3^2 2^2 + 3^1 2^3 + 2^4$	2^5	$3^5 2^0 + 3^4 2^1 + 3^3 2^2 + 3^2 2^3 + 3^1 2^4 + 2^5$

From Table 10, we get the following formula

$$\boldsymbol{B_n = 3^{n-1} \cdot 2^0 + 3^{n-2} \cdot 2^1 + 3^{n-3} \cdot 2^2 + ... + 3^1 \cdot 2^{n-2} + 3^0 \cdot 2^{n-1}}, \tag{8.2}$$

assuming a minimal change of C_n

$$\boldsymbol{C_{n+1} = 2C_n}, \tag{8.3}$$

and

$$\boldsymbol{C_0 = 2^0 = 1}. \tag{8.4}$$

As explained earlier (Remark 7.7), as $n \to \infty$, C_{n+1} can not always equal $2C_n$, so we check how B_n grows when bigger change of C_n occurs every so often.

TABLE 11. Change of B_n, as $n \to \infty$, when C_{n+1} does **not** always equal $2C_n$.

n	B_n	C_n	C_{n+1}/C_n
0	0	2^0	2^1
1	2^0	2^1	2^1
2	$3^1 2^0 + 2^1$	2^2	$\mathbf{2^2}$
3	$3^2 2^0 + 3^1 2^1 + 2^2$	$\mathbf{2^4}$	2^1
4	$3^3 2^0 + 3^2 2^1 + 3^1 2^2 + \mathbf{2^4}$	2^5	$\mathbf{2^3}$
5	$3^4 2^0 + 3^3 2^1 + 3^2 2^2 + 3^1 \mathbf{2^4} + 2^5$	$\mathbf{2^8}$	$\mathbf{2^2}$
6	$3^5 2^0 + 3^4 2^1 + 3^3 2^2 + 3^2 \mathbf{2^4} + 3^1 2^5 + \mathbf{2^8}$	$\mathbf{2^{10}}$	2^1
7	$3^6 2^0 + 3^5 2^1 + 3^4 2^2 + 3^3 \mathbf{2^4} + 3^2 2^5 + 3^1 \mathbf{2^8} + \mathbf{2^{10}}$	2^{11}	
8	$3^7 2^0 + 3^6 2^1 + 3^5 2^2 + 3^4 \mathbf{2^4} + 3^3 2^5 + 3^2 \mathbf{2^8} + 3^1 \mathbf{2^{10}} + 2^{11}$		

In Table 11, we see the influence of non minimal changes of C_n on B_n. In all cases, without any artificial assumptions about minimal changes of C_n, formula for B_n is

$$\boldsymbol{B_n = 3^{n-1} \cdot 2^{m_0} + 3^{n-2} \cdot 2^{m_1} + 3^{n-3} \cdot 2^{m_2} + ... + 3^1 \cdot 2^{m_{n-2}} + 3^0 \cdot 2^{m_{n-1}}} \tag{8.5}$$

and differences between consecutive m's **can be greater than 1**. This can be presented as a condition for m's to be integers and

$$m_{n-1} > m_{n-2} > ... > m_1 > m_0 \geq 0. \tag{8.6}$$

9. Equation for 3^n

Theorem 9.1. *For every positive integer n, 3 to the power of n can be expanded into a sequence*

$$\mathbf{3^n = 3^{n-1}\cdot 2^0 + 3^{n-2}\cdot 2^1 + 3^{n-3}\cdot 2^2 + ... + 3^1\cdot 2^{n-2} + 3^0\cdot 2^{n-1} + 2^n}, \tag{9.1}$$

which can also be presented as

$$\mathbf{3^n = \sum_{k=0}^{n-1} \left(3^{n-1-k} 2^k\right) + 2^n}. \tag{9.2}$$

Remark 9.2. Notice that in (9.1), all differences between consecutive powers of 2 are **strictly equal to 1**.

Proof. For $n = 1$, we have

$$\begin{aligned} 3^1 &= 3^0\cdot 2^0 + 2^1 \\ &= 1 + 2 \\ &= 3^1. \end{aligned}$$

To see a bit more complicated case, for $n = 2$, we have

$$\begin{aligned} 3^2 &= 3^1\cdot 2^0 + 3^0\cdot 2^1 + 2^2 \\ &= 3 + 2 + 4 \\ &= 9 \\ &= 3^2. \end{aligned}$$

Assuming it is true for n, we check if it is true for $n + 1$. We start from

$$3^n = 3^{n-1}\cdot 2^0 + 3^{n-2}\cdot 2^1 + 3^{n-3}\cdot 2^2 + ... + 3^1\cdot 2^{n-2} + 3^0\cdot 2^{n-1} + 2^n \qquad |\ (\cdot 3).$$

After multiplication on both sides by 3

$$\begin{aligned} 3^{n+1} &= 3^n\cdot 2^0 + 3^{n-1}\cdot 2^1 + 3^{n-2}\cdot 2^2 + ... + 3^2\cdot 2^{n-2} + 3^1\cdot 2^{n-1} + 3\cdot 2^n \\ &= 3^n\cdot 2^0 + 3^{n-1}\cdot 2^1 + 3^{n-2}\cdot 2^2 + ... + 3^2\cdot 2^{n-2} + 3^1\cdot 2^{n-1} + (2+1)\cdot 2^n \\ &= 3^n\cdot 2^0 + 3^{n-1}\cdot 2^1 + 3^{n-2}\cdot 2^2 + ... + 3^2\cdot 2^{n-2} + 3^1\cdot 2^{n-1} + \left(2+3^0\right)\cdot 2^n \\ &= 3^n\cdot 2^0 + 3^{n-1}\cdot 2^1 + 3^{n-2}\cdot 2^2 + ... + 3^2\cdot 2^{n-2} + 3^1\cdot 2^{n-1} + 3^0\cdot 2^n \ + \ 2^{n+1}. \end{aligned}$$

Now we substitute $n + 1 = w$

$$3^w = 3^{w-1}\cdot 2^0 + 3^{w-2}\cdot 2^1 + 3^{w-3}\cdot 2^2 + ... + 3^1\cdot 2^{w-2} + 3^0\cdot 2^{w-1} + 2^w, \tag{9.3}$$

which proves that formula (9.1) is correct. □

10. Limit of $\frac{A_n}{B_n}$

Lemma 10.1. *When iterating Procedure 1 for any initial positive integer A_0 we have*

$$\frac{A_{n+1}}{B_{n+1}} < \frac{A_n}{B_n} \tag{10.1}$$

for all $n > 0$.

Proof. From (6.9) and (6.10) we have

$$A_{n+1} = A_n \left(3 + \frac{C_n}{A_n}\right) \tag{10.2}$$

and

$$B_{n+1} = B_n \left(3 + \frac{C_n}{B_n}\right). \tag{10.3}$$

From (6.14)

$$A_n > B_n$$

for all $n > 0$, so

$$\left(3 + \frac{C_n}{A_n}\right) < \left(3 + \frac{C_n}{B_n}\right),$$

therefore

$$\frac{\left(3 + \frac{C_n}{A_n}\right)}{\left(3 + \frac{C_n}{B_n}\right)} < 1.$$

Dividing (10.2) by (10.3)

$$\frac{A_{n+1}}{B_{n+1}} = \frac{A_n}{B_n} \cdot \frac{\left(3 + \frac{C_n}{A_n}\right)}{\left(3 + \frac{C_n}{B_n}\right)} \tag{10.4}$$

and finally

$$\frac{A_{n+1}}{B_{n+1}} < \frac{A_n}{B_n}, \tag{10.5}$$

for all $n > 0$. □

Lemma 10.2. *When iterating Procedure 1 for any initial positive integer A_0 we have*

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1. \tag{10.6}$$

Notice that we always have

$$C_n = 2^{p_n} \cdot C_{n-1} \tag{10.7}$$

where p_n is positive integer and

$$C_0 = 2^{p_0}, p_0 \geq 0. \tag{10.8}$$

Proof. We analyse

$$\lim_{n\to\infty} \frac{A_n}{B_n}.$$

From (6.8)

$$A_n = 3^n A_0 + B_n,$$

so

$$
\begin{aligned}
\lim_{n\to\infty}\frac{A_n}{B_n} &= \lim_{n\to\infty}\frac{3^nA_0+B_n}{B_n}\\
&= \lim_{n\to\infty}\left(\frac{B_n}{B_n}+\frac{3^nA_0}{B_n}\right)\\
&= \lim_{n\to\infty}\left(1+\frac{3^nA_0}{B_n}\right).
\end{aligned}
\tag{10.9}
$$

Substituting B_n from (8.2) and 3^n from (9.1) we have

$$\lim_{n\to\infty}\left(1+\frac{3^nA_0}{B_n}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{(3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...\ +3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}+\mathbf{2^n})A_0}{3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...+3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}}\right).$$

Notice that the above substitution for B_n is correct only under condition that C_n equals $2C_{n-1}$ for every n and $C_0=1$, see (8.3) and (8.4).

One can check that

$$3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...\ +3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}$$

grows much faster than $\mathbf{2^n}$ as $n\to\infty$, so

$$\lim_{n\to\infty}\left(\frac{3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...\ +3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}+2^n}{3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...+3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}}\right)=1,$$

which makes:

$$\lim_{n\to\infty}\left(1+\frac{(3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...\ +3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}+2^n)A_0}{3^{n-1}\cdot 2^0+3^{n-2}\cdot 2^1+3^{n-3}\cdot 2^2+...+3^1\cdot 2^{n-2}+3^0\cdot 2^{n-1}}\right)$$

$$=1+A_0.$$

Finally, we have

$$\lim_{n\to\infty}\frac{A_n}{B_n}=1+A_0, \tag{10.10}$$

when

$$\boldsymbol{C_n=2C_{n-1}} \tag{10.11}$$

for every n and

$$C_0=1. \tag{10.12}$$

For any initial A_0, the condition (10.11) is **impossible** to be fulfilled, when $n\to\infty$ (see Remark 7.4, Remark 7.1 and Remark 7.7). After certain number of repetitions (see Remark 7.5) $C_n>2C_{n-1}$ always occurs. Whenever $C_n>2C_{n-1}$ occurs, value of B_n in equation (10.9) grows gradually, faster than the value of 3^n. We use the equation presented in (8.5)

$$B_n = 3^{n-1}\cdot 2^{m_0} + 3^{n-2}\cdot 2^{m_1} + 3^{n-3}\cdot 2^{m_2} + ... + 3^1\cdot 2^{m_{n-2}} + 3^0\cdot 2^{m_{n-1}} \tag{10.13}$$

and

$$m_{n-1} > m_{n-2} > ... > m_2 > m_1 > m_0 \geq 0.$$

In above representation of B_n, differences between consecutive powers of 2 can be greater than 1, while in formula for 3^n, the difference between consecutive powers of 2 is always equal to 1, as $n \to \infty$ (compare with Remark 9.2). Therefore, we have

$$\boldsymbol{3^n \ll B_n,} \text{ as } \boldsymbol{n \to \infty}, \tag{10.14}$$

which is

$$\begin{aligned}&(3^{n-1}\cdot 2^0 + 3^{n-2}\cdot 2^1 + 3^{n-3}\cdot 2^2 + ... + 3^1\cdot 2^{n-2} + 3^0\cdot 2^{n-1} + 2^n)\\ &\quad \ll (3^{n-1}\cdot 2^{m_0} + 3^{n-2}\cdot 2^{m_1} + 3^{n-3}\cdot 2^{m_2} + ... + 3^1\cdot 2^{m_{n-2}} + 3^0\cdot 2^{m_{n-1}}),\end{aligned}$$

thus

$$\begin{aligned}&\lim_{n\to\infty}\left(\frac{3^n A_0}{B_n}\right)\\ &\quad = \lim_{n\to\infty}\left(\frac{3^{n-1}\cdot 2^0 + 3^{n-2}\cdot 2^1 + 3^{n-3}\cdot 2^2 + ... \ + 3^1\cdot 2^{n-2} + 3^0\cdot 2^{n-1} + 2^n}{3^{n-1}\cdot 2^{m_0} + 3^{n-2}\cdot 2^{m_1} + 3^{n-3}\cdot 2^{m_2} + ... + 3^1\cdot 2^{m_{n-2}} + 3^0\cdot 2^{m_{n-1}}}\right)\\ &\quad = 0.\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{n\to\infty}\frac{A_n}{B_n} &= \lim_{n\to\infty}\left(1 + \frac{3^n A_0}{B_n}\right)\\ &= \lim_{n\to\infty}(1+0)\\ &= 1\end{aligned} \tag{10.15}$$

when

$$C_n = 2^{p_n}\cdot C_{n-1}, \tag{10.16}$$

where p_n is an positive integer and

$$C_0 = 2^{p_0}, p_0 \ \geq 0. \tag{10.17}$$

□

Lemma 10.3. *Starting from any positive integer, when iterating Procedure 1, such iteration number k exists, that for all following iterations n*

$$\frac{A_n}{B_n} < 4 \tag{10.18}$$

and then

$$\boldsymbol{A_n = 2^p}, \textit{where } p \in \mathbb{Z}^+. \tag{10.19}$$

Proof. When starting Procedure 1 from any positive integer A_0, the condition for A_n to be in a form of 2^p (where p is positive integer) is such that the least significant nonzero bit of A_n is

$$lsb(A_n) = C_n = A_n. \tag{10.20}$$

From (6.9) and (6.10) we have

$$A_{n+1} = 3A_n + C_n \tag{10.21}$$

and

$$B_{n+1} = 3B_n + C_n. \tag{10.22}$$

We substitute $C_n = A_n$, so

$$\begin{aligned} A_{n+1} &= 3A_n + A_n \\ &= 4A_n, \end{aligned} \tag{10.23}$$

$$B_{n+1} = 3B_n + A_n. \tag{10.24}$$

We extract A_n from (10.24)

$$A_n = B_{n+1} - 3B_n$$

and substitute in (10.23)

$$\begin{aligned} A_{n+1} &= 4A_n \\ &= 4(B_{n+1} - 3Bn) \\ &= 4B_{n+1} - 12B_n. \end{aligned}$$

We divide both sides by B_{n+1}, to get

$$\frac{A_{n+1}}{B_{n+1}} = 4 - \frac{12B_n}{B_{n+1}}. \tag{10.25}$$

Now, we substitue from (6.8)

$$A_n = 3^n A_0 + B_n$$

in (10.24), so

$$\begin{aligned} B_{n+1} &= 3B_n + A_n \\ &= 3B_n + 3^n A_0 + B_n \\ &= 4B_n + 3^n A_0. \end{aligned}$$

Finally, we substitute B_{n+1} in (10.25)

$$\frac{A_{n+1}}{B_{n+1}} = 4 - \frac{12B_n}{4B_n + 3^n A_0}. \tag{10.26}$$

When $n \to \infty$, from (10.14) we have

$$3^n \ll B_n, \tag{10.27}$$

so

$$\frac{12B_n}{4B_n + 3^n A_0} \to \frac{12B_n}{4B_n} \to 3, \tag{10.28}$$

which gives

$$\frac{A_{n+1}}{B_{n+1}} \to 4 - 3 \to 1 \tag{10.29}$$

from (10.26) and it produces the same result, which is already proven in Lemma 10.2. On the other hand, when $3^n A_0$ is still comparable or bigger than $4B_n$, we have

$$\frac{12B_n}{4B_n + 3^n A_0} > 0, \tag{10.30}$$

which means that

$$\frac{A_{n+1}}{B_{n+1}} = 4 - 0^+ < 4. \tag{10.31}$$

We see that condition for A_n to be in a form of 2^p, leads us to an ultimate condition

$$\frac{A_{n+1}}{B_{n+1}} < 4. \tag{10.32}$$

From Lemma 10.1, we get that $\frac{A_n}{B_n}$ is continuously decreasing, so we formulate the final conclusion.

When iterating Procedure 1, as $n \to \infty$, at certain iteration k, we have $lsb(A_k) = A_k$. For all next iterations, where $n > k$

$$\frac{A_n}{B_n} < 4 \tag{10.33}$$

and also

$$A_n = 2^p, \tag{10.34}$$

where p is a positive integer.

□

11. Proof of Theorem 1.2

Proof. We start with any positive integer I_0, let $A_0 = I_0$. We start Procedure 1. Iterating this procedure, as $n \to \infty$ we have

$$3^n A_0 \;=\; A_n \;-\; B_n \tag{11.1}$$

from (6.8) also

$$A_n \;>\; B_n \tag{11.2}$$

from (6.14) and

$$\lim_{n\to\infty} \frac{A_n}{B_n} = 1 \tag{11.3}$$

from Lemma 10.2.

In binary notation, such situation occurs only when:
A_n is a single bit in the form of 2^{m_n}, where $m_n \in \mathbb{Z}^+$ and
B_n is the sum of almost all bits 2^{p_n}, where $0 \le p_n < m_n$, as follows

$$(A_n)_b = 100000000000000...,$$
$$(B_n)_b = \ 11111111111111....$$

From Lemma 10.3, we know that when iterating Procedure 1, such k exists, that for all following iterations n, where $n > k$

$$\frac{A_n}{B_n} < 4 \tag{11.4}$$

and then A_n is in the form of

$$A_n = 2^p \tag{11.5}$$

where p is a positive integer.

Therefore, for all $n > k$ we substitute in (11.1), $A_n = 2^{m_n}$ and B_n from (10.13), we finally have

$$\begin{aligned} 3^n A_0 &= 2^{m_n} - (3^{n-1} \cdot 2^{m_0} + 3^{n-2} \cdot 2^{m_1} + ... + 3^1 \cdot 2^{m_{n-2}} + 3^0 \cdot 2^{m_{n-1}}) \\ &= 2^{m_n} - 3^{n-1} \cdot 2^{m_0} - 3^{n-2} \cdot 2^{m_1} - ... - 3^1 \cdot 2^{m_{n-2}} - 3^0 \cdot 2^{m_{n-1}}. \end{aligned} \tag{11.6}$$

Now we sort elements and substitute $A_0 = I_0$ to conclude.

For any initial I_0, such positive integer k exists that for every positive integer $n > k$ sequence of integers

$$m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \ge 0$$

exists, for which

$$3^n I_0 = 2^{m_n} - 2^{m_{n-1}} 3^0 - 2^{m_{n-2}} 3^1 - \cdots - 2^{m_1} 3^{n-2} - 2^{m_0} 3^{n-1}.$$

□

12. Extension of Theorem 1.2

Theorem 12.1. *For every initial positive integer I_0, an infinite number of equations exists that satisfies Theorem 1.2, therefore, it can be extended in an infinite number of ways to form the following expression*
(12.1)

$$\boldsymbol{I_0 = \frac{2^{m_n} - 2^{m_{n-1}} 3^0 - 2^{m_{n-2}} 3^1 - \cdots - 2^{m_1} 3^{n-2} - 2^{m_0} 3^{n-1}}{3^n}},$$

where n is a positive integer and all m's form a sequence of integers that

$$m_n > m_{n-1} > m_{n-2} > ... > m_1 > m_0 \ge 0.$$

Proof. The proof of Theorem 1.2 confirms that. □

13. Examples

Presented below are various examples of positive integers, confirming the Theorems proven above.

$$3^6\cdot \mathbf{9} = 2^{13} - 2^9 3^0 - 2^6 3^1 - 2^4 3^2 - 2^3 3^3 - 2^2 3^4 - 2^0 3^5 \tag{13.1}$$

$$3^7\cdot \mathbf{9} = 2^{15} - 2^{13} 3^0 - 2^9 3^1 - 2^6 3^2 - 2^4 3^3 - 2^3 3^4 - 2^2 3^5 - 2^0 3^6 \tag{13.2}$$

$$3^8\cdot \mathbf{9} = 2^{17} - 2^{15} 3^0 - 2^{13} 3^1 - 2^9 3^2 - 2^6 3^3 - 2^4 3^4 - 2^3 3^5 - 2^2 3^6 - 2^0 3^7 \tag{13.3}$$

$$\begin{aligned} 3^{12}\cdot \mathbf{6541} = 2^{32} - 2^{28} 3^0 - 2^{25} 3^1 - 2^{23} 3^2 - 2^{22} 3^3 - 2^{21} 3^4 - 2^{17} 3^5 \\ -2^{15} 3^6 - 2^{13} 3^7 - 2^{10} 3^8 - 2^9 3^9 - 2^3 3^{10} - 2^0 3^{11} \end{aligned} \tag{13.4}$$

$$3^7\cdot \mathbf{435} = 2^{20} - 2^{16} 3^0 - 2^{11} 3^1 - 2^{10} 3^2 - 2^9 3^3 - 2^4 3^4 - 2^1 3^5 - 2^0 3^6 \tag{13.5}$$

$$\begin{aligned} 3^{41}\mathbf{27} = 2^{70} - 2^{66} 3^0 - 2^{61} 3^1 - 2^{60} 3^2 - 2^{59} 3^3 - 2^{56} 3^4 - 2^{52} 3^5 \\ -2^{50} 3^6 - 2^{48} 3^7 - 2^{44} 3^8 - 2^{43} 3^9 - 2^{42} 3^{10} - 2^{41} 3^{11} - 2^{38} 3^{12} \\ -2^{37} 3^{13} - 2^{36} 3^{14} - 2^{35} 3^{15} - 2^{34} 3^{16} - 2^{33} 3^{17} - 2^{31} 3^{18} - 2^{30} 3^{19} \\ -2^{28} 3^{20} - 2^{27} 3^{21} - 2^{26} 3^{22} - 2^{23} 3^{23} - 2^{21} 3^{24} - 2^{20} 3^{25} - 2^{19} 3^{26} \\ -2^{18} 3^{27} - 2^{16} 3^{28} - 2^{15} 3^{29} - 2^{14} 3^{30} - 2^{12} 3^{31} - 2^{11} 3^{32} - 2^9 3^{33} \\ -2^7 3^{34} - 2^6 3^{35} - 2^5 3^{36} - 2^4 3^{37} - 2^3 3^{38} - 2^1 3^{39} - 2^0 3^{40} \end{aligned} \tag{13.6}$$

$$\begin{aligned} 3^{34}\cdot \mathbf{121} = 2^{61} - 2^{57} 3^0 - 2^{52} 3^1 - 2^{51} 3^2 - 2^{50} 3^3 - 2^{47} 3^4 - 2^{43} 3^5 \\ -2^{41} 3^6 - 2^{39} 3^7 - 2^{35} 3^8 - 2^{34} 3^9 - 2^{33} 3^{10} - 2^{32} 3^{11} - 2^{29} 3^{12} \\ -2^{28} 3^{13} - 2^{27} 3^{14} - 2^{26} 3^{15} - 2^{25} 3^{16} - 2^{24} 3^{17} - 2^{22} 3^{18} - 2^{21} 3^{19} \\ -2^{19} 3^{20} - 2^{18} 3^{21} - 2^{17} 3^{22} - 2^{14} 3^{23} - 2^{12} 3^{24} - 2^{11} 3^{25} - 2^{10} 3^{26} \\ -2^9 3^{27} - 2^7 3^{28} - 2^6 3^{29} - 2^5 3^{30} - 2^3 3^{31} - 2^2 3^{32} - 2^0 3^{33} \end{aligned} \tag{13.7}$$

(13.8)

$$
\begin{aligned}
3^{174}\cdot \mathbf{8388607} = 2^{299} &- 2^{295}3^{0} - 2^{290}3^{1} - 2^{289}3^{2} - 2^{288}3^{3} - 2^{285}3^{4} \\
&-2^{281}3^{5} - 2^{279}3^{6} - 2^{277}3^{7} - 2^{273}3^{8} - 2^{272}3^{9} - 2^{271}3^{10} - 2^{270}3^{11} - 2^{267}3^{12} \\
&-2^{266}3^{13} - 2^{265}3^{14} - 2^{264}3^{15} - 2^{263}3^{16} - 2^{262}3^{17} - 2^{260}3^{18} - 2^{259}3^{19} - 2^{257}3^{20} \\
&-2^{256}3^{21} - 2^{255}3^{22} - 2^{252}3^{23} - 2^{250}3^{24} - 2^{249}3^{25} - 2^{248}3^{26} - 2^{247}3^{27} - 2^{245}3^{28} \\
&-2^{244}3^{29} - 2^{243}3^{30} - 2^{241}3^{31} - 2^{240}3^{32} - 2^{236}3^{33} - 2^{235}3^{34} - 2^{234}3^{35} - 2^{233}3^{36} \\
&-2^{232}3^{37} - 2^{229}3^{38} - 2^{227}3^{39} - 2^{225}3^{40} - 2^{224}3^{41} - 2^{223}3^{42} - 2^{221}3^{43} - 2^{219}3^{44} \\
&-2^{218}3^{45} - 2^{214}3^{46} - 2^{213}3^{47} - 2^{207}3^{48} - 2^{206}3^{49} - 2^{204}3^{50} - 2^{201}3^{51} - 2^{200}3^{52} \\
&-2^{198}3^{53} - 2^{197}3^{54} - 2^{196}3^{55} - 2^{195}3^{56} - 2^{193}3^{57} - 2^{190}3^{58} - 2^{187}3^{59} - 2^{185}3^{60} \\
&-2^{184}3^{61} - 2^{183}3^{62} - 2^{180}3^{63} - 2^{179}3^{64} - 2^{178}3^{65} - 2^{173}3^{66} - 2^{172}3^{67} - 2^{171}3^{68} \\
&-2^{170}3^{69} - 2^{169}3^{70} - 2^{168}3^{71} - 2^{166}3^{72} - 2^{165}3^{73} - 2^{163}3^{74} - 2^{162}3^{75} - 2^{160}3^{76} \\
&-2^{158}3^{77} - 2^{157}3^{78} - 2^{151}3^{79} - 2^{150}3^{80} - 2^{148}3^{81} - 2^{147}3^{82} - 2^{146}3^{83} - 2^{145}3^{84} \\
&-2^{143}3^{85} - 2^{139}3^{86} - 2^{138}3^{87} - 2^{131}3^{88} - 2^{130}3^{89} - 2^{128}3^{90} - 2^{126}3^{91} - 2^{123}3^{92} \\
&-2^{122}3^{93} - 2^{121}3^{94} - 2^{120}3^{95} - 2^{119}3^{96} - 2^{118}3^{97} - 2^{117}3^{98} - 2^{116}3^{99} - 2^{114}3^{100} \\
&-2^{113}3^{101} - 2^{112}3^{102} - 2^{108}3^{103} - 2^{107}3^{104} - 2^{105}3^{105} - 2^{102}3^{106} - 2^{101}3^{107} \\
&-2^{100}3^{108} - 2^{99}3^{109} - 2^{98}3^{110} - 2^{94}3^{111} - 2^{93}3^{112} - 2^{91}3^{113} - 2^{90}3^{114} - 2^{89}3^{115} \\
&-2^{87}3^{116} - 2^{86}3^{117} - 2^{84}3^{118} - 2^{83}3^{119} - 2^{81}3^{120} - 2^{80}3^{121} - 2^{78}3^{122} - 2^{74}3^{123} \\
&-2^{72}3^{124} - 2^{71}3^{125} - 2^{69}3^{126} - 2^{67}3^{127} - 2^{66}3^{128} - 2^{65}3^{129} - 2^{61}3^{130} - 2^{60}3^{131} \\
&-2^{59}3^{132} - 2^{58}3^{133} - 2^{57}3^{134} - 2^{56}3^{135} - 2^{54}3^{136} - 2^{53}3^{137} - 2^{52}3^{138} - 2^{49}3^{139} \\
&-2^{46}3^{140} - 2^{42}3^{141} - 2^{40}3^{142} - 2^{39}3^{143} - 2^{36}3^{144} - 2^{34}3^{145} - 2^{32}3^{146} - 2^{30}3^{147} \\
&-2^{29}3^{148} - 2^{28}3^{149} - 2^{24}3^{150} - 2^{22}3^{151} - 2^{21}3^{152} - 2^{20}3^{153} - 2^{19}3^{154} - 2^{18}3^{155} \\
&-2^{17}3^{156} - 2^{16}3^{157} - 2^{15}3^{158} - 2^{14}3^{159} - 2^{13}3^{160} - 2^{12}3^{161} - 2^{11}3^{162} - 2^{10}3^{163} \\
&-2^{9}3^{164} - 2^{8}3^{165} - 2^{7}3^{166} - 2^{6}3^{167} - 2^{5}3^{168} - 2^{4}3^{169} - 2^{3}3^{170} - 2^{2}3^{171} - 2^{1}3^{172} \\
&-2^{0}3^{173}
\end{aligned}
$$

FIGURE 6. Decreasing $\frac{A_n}{B_n} \to 1$, as $n \to \infty$ for $A_0 = 27$.

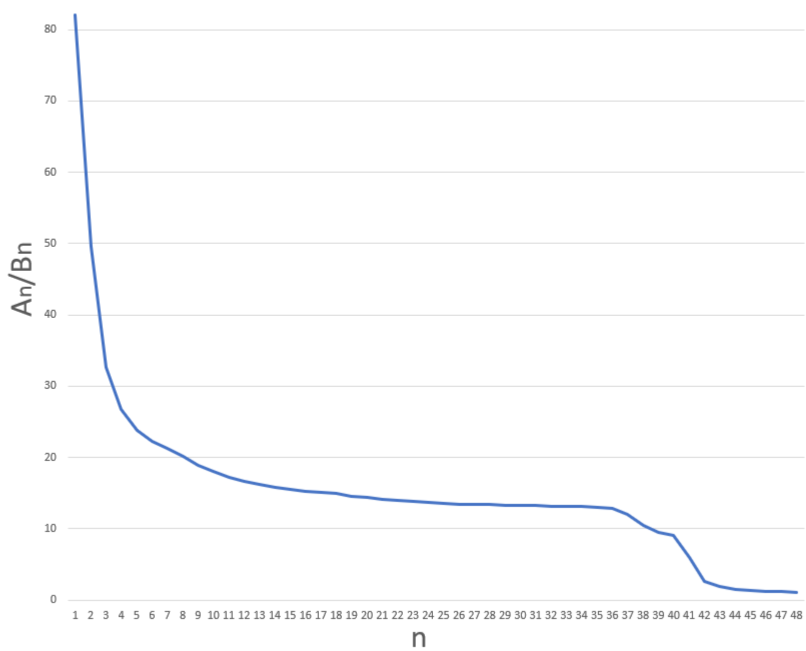


FIGURE 7. Decreasing $\frac{A_n}{B_n} \to 1$, as $n \to \infty$ for $A_0 = 121$.

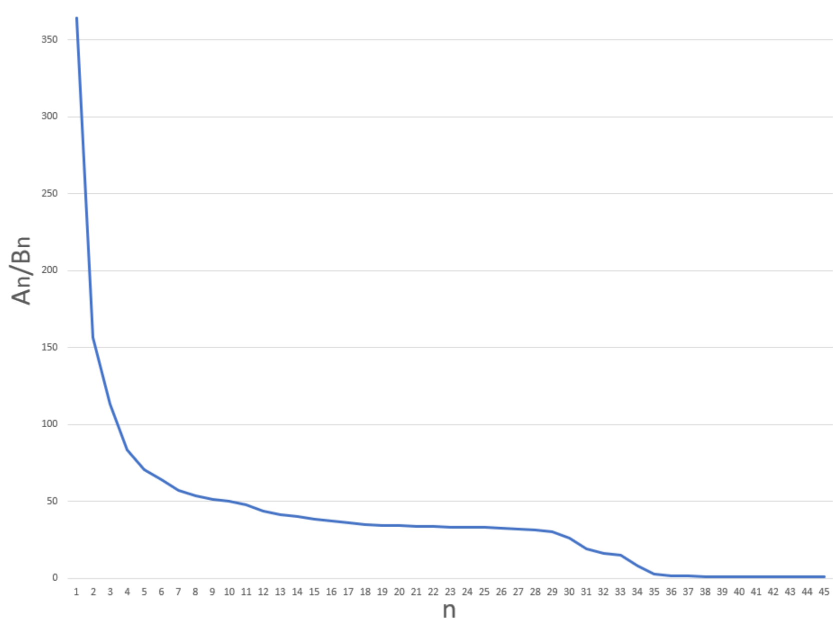
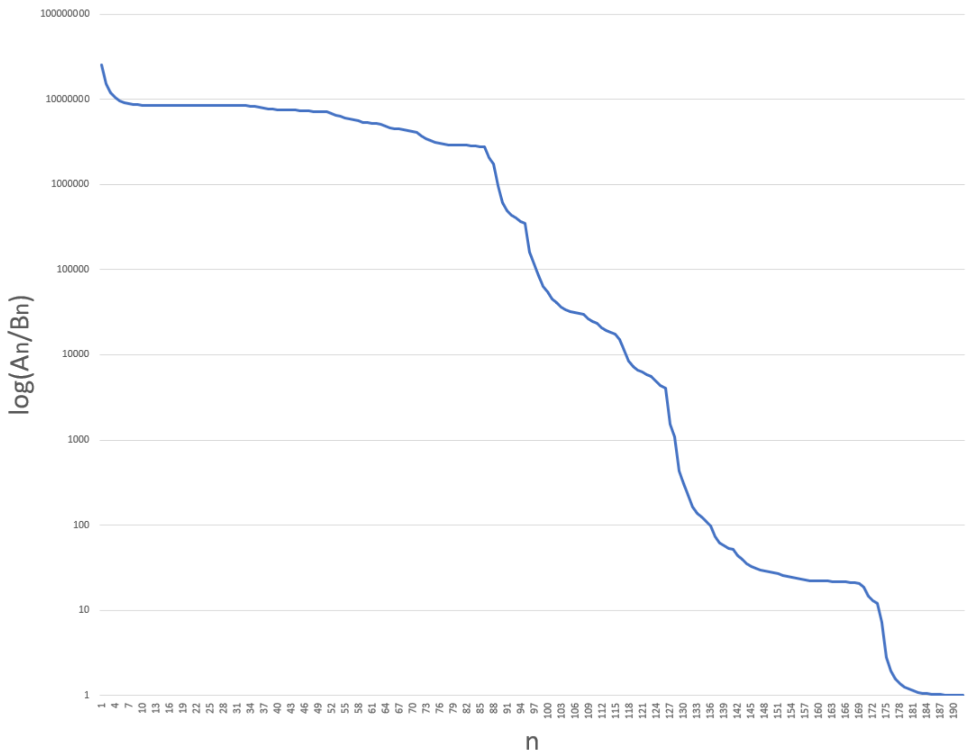


FIGURE 8. Decreasing $\frac{A_n}{B_n} \to 1$, as $n \to \infty$ for $A_0 = 8388607$.



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