Magnetic symmetry of geometrical optics

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We propose that there exist magnetic symmetry in the geometrical optics. What are the consequences of the magnetic symmetry existence to the formulation of refractive index and its related curvature?

I. INTRODUCTION

Dirac proposed, due to symmetrical reasoning in the Maxwell's theory of electromagnetism, there exist magnetic monopole which has magnetic charge[1,2]. Inspired with Dirac idea of monopole, the magnetic symmetry was formulated as the $SU(2)$ gauge theory, non-Abelian theory of the $(4+n)$-dimensions of unified space[3]. The $(4+n)$-dimensions of unified space is actually the $(3+1)$-dimensions of external space, i.e. the $(3+1)$-dimensions of curved spacetime, plus the n-dimensions of (curved) internal space[4].

To the best of our knowledge, the geometrical optics (e.g. eikonal equation) is formulated without including magnetic symmetry[5–11]. The eikonal equation can be derived from the Maxwell equations[9]. Because of Maxwell's theory is $U(1)$ gauge theory, Abelian theory, and the eikonal equation can be derived from the Maxwell's theory, *we argue that the geometrical optics is also $U(1)$ gauge theory, Abelian theory and there exist magnetic symmetry in the geometrical optics.*

The situation of the $SU(2)$ gauge theory, non-Abelian theory of the $(4+n)$-dimensions of unified space[3] looks like the $SU(N)$ Yang-Mills theory where the choice of the simple Lie group $G = U(1)$ reduces the Yang-Mills theory to Maxwell's theory[12].

Because of the $U(1)$ gauge theory can be generalized to $SU(2)$ gauge theory[13], it has the consequence that we can obtain the magnetic symmetry in the $U(1)$ gauge theory from the magnetic symmetry of the $SU(2)$ gauge theory.

We will apply the gauge potential of the $U(1)$ magnetic symmetry to the geometrical optics, especially in the formulation of eikonal equation, the refractive index and the curvature relation.

II. POTENTIAL AND FIELD STRENGTH TENSOR

In the geometrical optics approximation (short wavelength, $\lambda \to 0$[11]), the four-vector potential, A_α, and the field strength tensor, $F_{\alpha\beta}$, can be represented respectively as[6,7]

$$A_\alpha = a_\alpha \; e^{i\psi} \tag{1}$$

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \tag{2}$$

where ψ_α is *the phase*, *the eikonal*, a large quantity which is "almost linear" in the coordinates and time[10], a_α is a slowly varying amplitude, a slowly varying function of the coordinates and time[10] and ∇_α denotes a spacetime covariant derivative[5]. Here, we assume that the geometrical optics "lives" in the $(4+n)$-dimensions of unified space, the indices α, β run from 1 to $4+n$.

The equation in ray propagation in a medium with refractive index, n, is[9,10]

$$|\vec{\nabla}\psi_1| = n \tag{3}$$

Because ψ_1 is also called *the eikonal*[10] or *the optical length*, a real scalar function of position[9], then the equation (3) is called *the eikonal equation*[9,10], where[10]

$$\psi_1 = \frac{c}{\omega}\psi + ct \tag{4}$$

Here, ψ is same as ψ in eq.(1).

Because the eikonal or the optical length, ψ_1, "lives" in the $(4+n)$-dimensions of unified space, we need to transform ψ_1 to ψ_μ and the gradient operator, $\vec{\nabla}$, in eq.(3) to the four-gradient, ∂ (the covariant four-gradient, ∂_μ, or the contravariant four-gradient, ∂^μ)[14]. So, eqs.(3), (4) become

$$n_{\mu\nu} = |\partial_\nu \psi_\mu| \tag{5}$$

$$\psi_\mu = \frac{c}{\omega}\psi + ct \tag{6}$$

In the Maxwell's theory, the four-vector potential, A^ρ, and and the field strength tensor, $F^{\rho\tau}$, can be represented respectively as[15]

$$A^\rho = \left(\frac{V}{c}, A_x, A_y, A_z\right) \tag{7}$$

$$F^{\rho\tau} = \partial_\rho A^\tau - \partial_\tau A^\rho \tag{8}$$

where $\partial_\rho = \partial/\partial x_\rho$ is the differentiation with respect to the covariant vector, x_ρ. We see that the formulations of the field strength tensor in the geometrical optics (2) and in the Maxwell's theory (8) look similar.

III. $SU(2)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

Magnetic symmetry of the $SU(2)$ gauge theory in the $(4+n)$-dimensions of unified space is formulated in relations with the symmetry of the unified metric[3]. The

Killing vector fields must satisfy[3]

$$\pounds_m \, g_{AB} = 0 \tag{9}$$

where m is *vector field*, $\pounds_m$ is the Lie derivative along the direction of m, $g_{AB}(A, B = 1, 2, .., 4+n)$ is metric in the $(4+n)$-dimensional unified space[3].

The eq.(9) has consequence that

$$D_\mu \hat{m} = \partial_\mu \hat{m} + g\vec{B}_\mu \times \hat{m} = 0 \tag{10}$$

where $\hat{m}$ is *the multiplet* and $\vec{B}_\mu$ is *the gauge potential* of the n-dimensional isometry group[3]

$$\vec{B}_\mu = A_\mu \hat{m} - \frac{1}{g}\hat{m} \times \partial_\mu \hat{m} \tag{11}$$

A_μ is *the (Abelian) component of $SU(2)$ gauge potential*, $\vec{B}_\mu$.

The corresponding field strength, $\vec{G}_{\mu\nu}$, to the gauge potential (11) is[3]

$$\vec{G}_{\mu\nu} = \partial_\mu \vec{B}_\nu - \partial_\nu \vec{B}_\mu + g\vec{B}_\mu \times \vec{B}_\nu \tag{12}$$

We see that the formulation of the field strength tensor in the $SU(2)$ gauge theory (12) is different from eqs.(8) and (2), because of there exist the second term of (12).

IV. $U(1)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

In an analogue that the choice of $G = U(1)$ reduces $SU(2)$ Yang-Mills theory to the $U(1)$ Maxwell's theory, from eqs.(11), (12) we obtain that the $U(1)$ gauge potential and the related $U(1)$ field strength tensor can be represented respectively as

$$\vec{B}_\mu^{U(1)} = A_\mu^{U(1)} \; \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \tag{13}$$

$$\vec{G}_{\mu\nu}^{\,U(1)} = \partial_\mu \vec{B}_\nu^{\,U(1)} - \partial_\nu \vec{B}_\mu^{\,U(1)} \tag{14}$$

where $A_\mu^{U(1)}$ is the (Abelian) component of the $U(1)$ gauge potential, $\vec{B}_\mu^{U(1)}$ and $\hat{m}^{U(1)}$ is *the multiplet* of the n-dimensional $U(1)$ group.

Because, $\vec{G}_{\mu\nu}^{\,U(1)}$ in (14), $F^{\rho\tau}$ in (8) and $F_{\alpha\beta}$ in (2) are in principle same i.e. *the fields*, so we can replace $F^{\rho\tau}$ or $F_{\alpha\beta}$ with $\vec{G}_{\mu\nu}^{\,U(1)}$[16]. In other words, we can replace A^ρ or A_α with $\vec{B}_\mu^{\,U(1)}$. If we replace A_α with $\vec{B}_\mu^{U(1)}$ (by considering the harmonic form of notation) then we obtain

$$a_\mu \; e^{i\psi} = \vec{B}_\mu^{\,U(1)} \tag{15}$$

V. THE REFRACTIVE INDEX-CURVATURE OF UNIFIED SPACE

In general relativity, light rays follow *null geodesics*[17], i.e. the line-element of the "world" of space-time, ds, vanishes[18]. The null geodesics are *the tracks of rays of light*[18]. Mathematically, the tracks of rays of light are expressed in *the Fermat's principle*.

To simplify the problem, the Fermat's principle is formulated in case of a static gravitational field[11], isotropic and spherically symmetric metric[19]. The Fermat's principle is

$$\delta \int_{r_1}^{r_2} n \; dr = 0 \tag{16}$$

We can derive, from the Fermat's principle (16), the relation between refractive index and curvature as below[10]

$$\frac{1}{R} = \vec{N} \cdot \frac{\vec{\nabla} n}{n} \tag{17}$$

where $\vec{N}$ is the unit vector along the principal normal, R is the radius of curvature and n is the refractive index. We see from eq.(17), the rays are therefore bent in the direction of increasing refractive index[20].

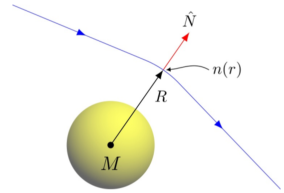

Fig. 1 The illustration of eq.(17).

In the geometry of (3+1)-dimensions of curved space-time, eq.(17) can be written as[21,22]

$$R_{\mu\nu\rho\sigma} = gN_\sigma \; \partial_\rho \ln n_{\mu\nu} \tag{18}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann-Christoffel curvature tensor.

The form of eq.(18) does not change if we treat it in the $(4+n)$-dimensions of unified space. So, we can say that eq.(18) is the relation between refractive index and Riemann-Christoffel curvature tensor in the $(4+n)$-dimensions of unified space.

VI. THE GAUGE POTENTIAL-CURVATURE IN UNIFIED SPACE

We see from eq.(15)

$$e^{i\psi} = \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{19}$$

Using Euler's formula, eq.(19) can be written as

$$\cos\psi + i \sin\psi = \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{20}$$

To simplify the problem, we only take the real part of (20), then eq.(20) becomes

$$\cos\psi = \vec{B}_\mu^{U(1)} a_\mu^{-1} \quad (21)$$

$$\psi = \arccos \vec{B}_\mu^{U(1)} a_\mu^{-1} \quad (22)$$

Substituting eqs.(13), (22) into eq.(4), we obtain

$$\psi_\mu = \frac{c}{\omega} \arccos\left(A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \quad (23)$$

Substituting eq.(23) into eq.(24), we obtain

$$\left|\partial_\nu \left\{ \frac{c}{\omega} \arccos\left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \right\}\right| = n_{\mu\nu} \quad (24)$$

Substituting eq.(24) into (18), we obtain

$$gN_\sigma\ \partial_\rho \ln \left|\partial_\nu \left\{ \frac{c}{\omega} \arccos \left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \right\}\right| = R_{\mu\nu\rho\sigma} \quad (25)$$

VII. DISCUSSION AND CONCLUSION

We see from eqs.(24), (25) that there exist the magnetic symmetry (magnetic monopole) represented by $\hat{m}^{U(1)}$ in the geometrical optics, especially in the eikonal equation, the refractive index and the Riemann-Christoffel curvature tensor relation formulated in the $(4+n)$-dimensions of unified space.

$A_\mu^{U(1)}$ and $\hat{m}^{U(1)}$ contribute to the refractive index and in turn to the refractive index and the Riemann-Christoffel curvature tensor relation. In other words, the refractive index and the Riemann-Christoffel curvature tensor consist of the (Abelian) component of the $U(1)$ gauge potential and the multiplet of the n-dimensional $U(1)$ group.

What does it mean physically that the refractive index and the curvature are decomposed into the (Abelian) component of the $U(1)$ gauge potential and the multiplet of the n-dimensional $U(1)$ group?

Related with the gravitational lensing problem (when the light passes near a massive mass object, the curvature of space-time due to such a massive mass object will deflect the light path), what is the consequence of the decomposition of the refractive index to the angle of deflection of the light path?

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