Oligopoly Games for Use in the Classroom and Laboratory

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Abstract

To illustrate that a Nash equilibrium results from a flawed attempt to solve a game, this article studies two extensions of the classic oligopoly model of Cournot. The common cost function is quadratic, and the (still linear) inverse demand functions allow of differentiated goods. The industry has a maximal-profit set that is characterised by a constant profit-output ratio, independent of the number of firms and the slopes of the marginal-cost function and inverse demand functions. The choice of parameters is discussed and six model variants are analysed numerically; in five of them, the incentives for merger according to noncooperative game theory are at odds with the rationale of economics. Some comments are made on the use of the model in experimental economics.

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> No one has the right, and few the ability, to lure economists into reading another article on oligopoly theory without some advance indication of its alleged contribution. The present paper accepts the hypothesis that oligopolists wish to collude to maximize joint profits. It seeks to reconcile this wish with facts, such as that collusion is impossible for many firms and collusion is much more effective in some circumstances than in others.
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> **G. J. Stigler (1964, p. 44)**

1. Introduction

There is something weird about the notion of Nash equilibrium. Although game theory has been designed to deal with decision problems in which the effects of each player's actions on the outcomes for the other players cannot be ignored, noncooperative game theory does just that: the "first-order conditions" for a Nash equilibrium ignore the partial cross-derivatives of the payoff functions. Ignoring the cross-derivatives is in fact a grave mathematical error: it amounts to treating the actions inconsistently and leads to contradictions. The two-page note Nieuwenhuis (2018) merely underlines that the conditions for a Nash equilibrium are incompatible with the first-order approximations of the payoff functions. Nieuwenhuis (2017a) exposes the error in noncooperative game theory at greater length, arguing that a game is a vector maximisation problem from which a Nash equilibrium derives by treating variables erroneously as constants in certain places, and illustrates the fallacy of Nash equilibrium with three examples, in oligopoly theory, in the Prisoner's Dilemma, and in dynamic general equilibrium models with imperfect competition and rational expectations. As an example within the first example, the paper analyses the classic duopoly model of Cournot (1927)—homogeneous product, linear inverse demand function and cost function—extended to the case of a common quadratic cost function.

The present article analyses a further extension of the "classic case" to illustrate that a Nash equilibrium results from a mathematically flawed attempt to solve a game, thus insisting on the removal of noncooperative game theory

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from the textbooks on microeconomic theory. It considers oligopolies with any number of firms that produce varieties of some good (one homogeneous good included as a special case), sharing a quadratic cost function. The inverse demand functions are still linear; for simplicity's sake, the specification is symmetric in the way it treats the goods. Like Stigler's, the article accepts the hypothesis that firms wish to collude to maximise profits, but it takes a different turn. It stresses that collusion *is* in fact the profit maximising strategy of the firms in the artificial world of the model, as this world does not contain any impediments for collusion. It defines the *maximal-profit set* of an industry as the set of points where industry profit is maximal along a ray through the origin, and shows that it is characterised by a constant profit-output ratio, independent of the number of firms and of the slopes of the marginal-cost function and inverse demand functions. The maximal-profit set and the Pareto optimal set have a number of points in common, but in general do not coincide. The article discusses how parameter values may be chosen so as to obtain model variants with attractive features for use in the laboratory, and presents numerical examples to highlight the differences between and similarities of the outcomes of six model variants; in five variants, the incentives for merger according to noncooperative game theory appear at odds with the tenet of rational, optimising behaviour. Finally, it comments on the use of this class of models in the economics laboratory.

2. The model

Consider an industry with profit maximising firms i, $i = 1,\dots,I$, each of which produces a variety of some good. The firms have identical quadratic cost functions,

$$c(q_i) = c_1 q_i + \frac{c_2}{2} q_i^2, \tag{1}$$

with q_i the quantity of the variety produced by Firm i. The linear marginal-cost function is

$$mc(q_i) = c_1 + c_2 q_i, \tag{2}$$

where $c_2 \lesseqgtr 0$, so that marginal cost may be increasing, constant or decreasing. The last case requires $c_1 \gg 0$ for marginal cost to be positive over a certain range.

The inverse demand functions, too, are linear:

$$p_i = f_i(q_1,\dots,q_I) = d_0 - dQ - (t-d)q_i, \qquad i = 1,\dots,I, \tag{3}$$

where $Q := \sum_j q_j$. I shall use the entity Q as an indicator of industry output. The parameters satisfy $d_0 > c_1$, $0 < d \leq t$. When $t > d$, the goods are imperfect substitutes. The condition $d_0 > c_1$ ensures that the prices of the goods exceed marginal cost when the firms produce nothing at all. The symmetric specification of the inverse demand functions, next to the common cost function, implies that most outcomes of interest are points on the ray $q_1 = q_2 = \dots = q_I$ (the axis of symmetry of the model), which simplifies the analysis.

With differentiated goods, that is when $t > d$, the ordinary demand functions are

$$\begin{aligned}
q_i = g_i(p_1,\dots,p_I) &= \delta_{0,I} + \delta_I P - (\tau_I + \delta_I)p_i, && i = 1,\dots,I, \\
\delta_{0,I} &= \frac{d_0}{(I-1)d+t}, && I \geq 1, \\
\tau_I &= \frac{(I-2)d+t}{\big((I-1)d+t\big)(t-d)}, && I \geq 1, \\
\delta_I &= \frac{d}{\big((I-1)d+t\big)(t-d)}, && I \geq 2,
\end{aligned} \tag{4}$$

where $P := \sum_j p_j$. The parameters satisfy $\delta_{0,I} > 0$, $0 < \delta_I < \tau_I$.

The systems of inverse or ordinary demand functions may also be *partially* inverted to a system of *mixed* demand functions, with the prices of some goods and the quantities of the other goods as left-hand side variables. In general, there are 2^I equivalent systems of demand functions. The present, symmetric specification reduces the number of analytically distinct cases to $I+1$, one case for each number of firms that use the prices of their products as instruments.

The profit of Firm i is revenue minus cost, $p_i \cdot q_i - c(q_i)$. The profits of all firms must be maximised subject to the constraint of the cost function (1) and to the constraints of the inverse demand functions (3) or, equivalently, the ordinary demand functions (4) or some system of mixed demand functions. These are *extensive* forms of the problem, all equivalent to one another. The I problems of profit maximisation are interdependent, because the arguments of the maximands are interrelated through the demand functions: they constitute a vector maximisation problem[2] or game. The game, in any extensive form, may be treated with the methods of De Finetti (2017a) or Kuhn and Tucker (1950, Section 6). The alternative and more common approach is to use a system of demand functions for eliminating one half of the number of arguments from the maximands to arrive at a *normal* form of the game; in each normal form, the arguments of the maximands may be varied independently of one another. Every normal form, combined with the matching system of demand functions, is equivalent to an extensive form; hence, all normal forms are equivalent to one another. The choice of which variables to eliminate is arbitrary in the sense that, with a *correct* mathematical treatment, it does not affect the solutions for the prices and quantities.[3]

The choice of which variables to eliminate is not arbitrary in every aspect. In fact, there is a most convenient choice, which is to eliminate the prices. It yields easy to understand expressions, because the dimensions of the parameters of the cost function and the inverse demand functions agree: c_1 has the same dimension as d_0 does, c_2 has the same dimension as d and t do. The profit functions *in quantity space* are

$$\upsilon_i = u_i(q_1,\dots,q_I) = (d_0 - c_1 - dQ)q_i - \left(t - d + \frac{c_2}{2}\right)q_i^2, \qquad i = 1,\dots,I. \tag{5}$$

They differ from the profit functions of the classic case only if $t - d + c_2/2 \neq 0$. However, even if $t - d + c_2/2 = 0$, the ordinary demand functions and, hence, the profit functions *in price space* do differ when $t > d$. More generally, every triplet (t, d, c_2) that satisfies $t - d + c_2/2 = z$ yields the same profit functions in quantity space but not in price space. This property of the model results from the linearity of the marginal-cost function and the inverse demand functions. It means that to every increase of the slope of the marginal-cost function there corresponds a decrease of the slopes of the perceived marginal-revenue functions (for which see below) so as to yield the same profit functions in quantity space. I shall return to the issue when discussing the outcomes of model variants in Section 4.

The profit of Firm i is zero if $q_i = 0$ or else if $p_i = c(q_i)/q_i$. The latter condition defines a (hyper)plane the nonnegative segment of which I simply call the zero-profit plane of the firm. Let Z be the intersection of the ray $q_1 = q_2 = \cdots = q_I$ with the zero-profit plane of any one of the firms. Then Z is either the centroid of the common zero-profit plane (if $t - d + c_2/2 = 0$) or else the intersection of the zero-profit planes, the unique zero-profit point. All its coordinates equal

$$q_Z = \frac{2d}{2(I-1)d + 2t + c_2}k, \qquad k := \frac{d_0 - c_1}{d}.$$

Other quantities of interest, too, will be expressed as fractions of k.

The solution of a vector maximisation problem is the Pareto optimal set. Here I consider mainly the centroid of the set, which I call *Col* (for Collusion point). As we shall see, in some cases *Col* is the Joint Profit Maximum, in other cases it is the point of the Pareto optimum where the joint profit is minimal. To find the coordinates of *Col*, note that along the ray $q_1 = q_2 = \cdots = q_I$ the profit of each firm is a quadratic function of the common quantity q, and that the zeros of the parabola are at the origin and at Z. Therefore *Col* is halfway between the origin and Z.

Two failed attempts to solve the vector maximisation problem constituted by the simultaneous maximisation of the profit functions are those by Cournot (1927) and by Bertrand (1883). In a Cournot oligopoly, each firm chooses its quantity while conditioning on the quantities of its rivals; in a Bertrand oligopoly, each firm chooses its price while conditioning on the prices of its rivals. The firms in a Bertrand oligopoly perceive the marginal-revenue functions

$$mr_i^B = p_i - \frac{1}{\tau_I} g_i(p_1,\dots,p_I) \tag{6a}$$

$$= f_i(q_1,\dots,q_I) - \frac{1}{\tau_I} q_i. \tag{6b}$$

[2]Elsewhere I have used the term *simultaneous maximum problem*.

[3]Else the eliminations would not even have been permissible.

When $t = d$ (one homogeneous product), $1/\tau_I = 0$ so that $mr^B = p$. The firms in a Cournot oligopoly perceive the marginal-revenue functions

$$mr_i^C = f_i(q_1,\dots,q_I) - tq_i \tag{7a}$$
$$= p_i - tg_i(p_1,\dots,p_I). \tag{7b}$$

The firms in the "collusive" oligopoly perceive the marginal-revenue functions

$$mr_i^{\mathrm{Col}} = f_i(q_1,\dots,q_I) - \big((I-1)d + t\big)q_i \tag{8a}$$
$$= p_i + \frac{1}{(I-1)\delta_I - \tau_I} g_i(p_1,\dots,p_I). \tag{8b}$$

These expressions result from assuming respectively that the quantities or the prices move in unison. (8a) and (8b) are equivalent, meaning that the Collusion point is unaffected by the choice of instruments. In fact, the Pareto optimal set is unaffected by nonsingular transformations of variables (as is the solution of every optimisation problem). The Cournot equilibrium and Bertrand equilibrium, by contrast, two examples of a Nash equilibrium, will differ from one another. In general, the Nash equilibria of all normal forms will be different. The dependence of a Nash equilibrium on the *arbitrary* choice of instruments points to the flawed treatment of the vector maximisation problem, and is a major defect of noncooperative game theory as a theory of rational behaviour.

Subsequently equating marginal revenue according to the functions (6b), (7a) or (8a) to marginal cost (2) yields the "reaction functions" in implicit form. The Bertrand equilibrium B and the Cournot equilibrium C (and also Col) are at the intersection of the ray $q_1 = q_2 = \cdots = q_I$ with any one of the appropriate "reaction functions." Table 1 gives a number of outcomes. The prices are always a weighted average of c_1 and d_0. The outcomes of the classic case are the limits of the outcomes of the general case when the number of firms grows without bound: the effects of nonconstant marginal cost and of product differentiation become negligible.

A noteworthy feature of the Collusion point is the constant profit-output ratio,[4]

$$\frac{\upsilon_{\mathrm{Col}}}{q_{\mathrm{Col}}} = \frac{d_0 - c_1}{2},$$

independent of I, the number of firms, and of c_2, d and t, the slopes of the marginal-cost function and inverse demand functions. In fact, this outcome is a special case of a more general result. Define the *maximal-profit set* of an industry as the set of points where industry profit is maximal along the rays $\alpha_1 q_1 = \alpha_2 q_2 = \ldots = \alpha_I q_I$, $\alpha_i \geq 0$ for all i. There holds

FACT: *The maximal-profit set of every oligopoly in this paper is the set of points where the profit-output ratio of the industry equals* $(d_0 - c_1)/2$.

The maximal-profit set and the Pareto optimal set have the Collusion points of the included oligopolies in common, but they do not coincide; the Appendix delves deeper into this matter. Still it is true that, within the Pareto optimal sets of the oligopolies with the same value of $d_0 - c_1$, the joint profit is largest where the joint output is largest. In Section 4 I shall use the result in the comparison of the outcomes of six model variants.

3. Choice of parameters: the classic case

Numerical examples help to reveal the information that the expressions in Table 1 contain. The present section discusses the choice of values for the parameters c_1, d_0 and d of the classic case, because it takes a central place. The next section will show the impact of nonconstant marginal cost and product differentiation on the outcomes.

Conveniently, choosing values for the parameters can be done in three steps. First, scale the quantities through the choice of k $(:= (d_0 - c_1)/d)$. Second, fix the price at the Collusion point through the choice of $c_1 + d_0$. Third, make an assumption about the elasticity of the inverse demand function at the Collusion point to identify c_1, d_0 and d.

[4]For the quadratic form $ax^2 + bx$, the ratio of the extremum to its argument is $b/2$.

Table 1: Main outcomes

	General formula	The classic case
Zero-profit point		
– quantity$\div k$	$\dfrac{2d}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{1}{I}$
– price	$\dfrac{2\big((I-1)d+t\big)c_1+c_2d_0}{2\big((I-1)d+t\big)+c_2}$	c_1
Bertrand equilibrium		
– quantity$\div k$	$\dfrac{d}{(I-1)d+t+\tau_I^{-1}+c_2}$	$\dfrac{1}{I}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+(\tau_I^{-1}+c_2)d_0}{(I-1)d+t+\tau_I^{-1}+c_2}$	c_1
– profit	$\Big(\dfrac{1}{\tau_I}+\dfrac{c_2}{2}\Big)q_B^2$	0
Cournot equilibrium		
– quantity$\div k$	$\dfrac{d}{(I-1)d+2t+c_2}$	$\dfrac{1}{I+1}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+(t+c_2)d_0}{(I-1)d+2t+c_2}$	$\dfrac{Ic_1+d_0}{I+1}$
– profit	$\Big(t+\dfrac{c_2}{2}\Big)q_C^2$	dq_C^2
Collusion point		
– quantity$\div k$	$\dfrac{d}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{1}{2I}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+\big((I-1)d+t+c_2\big)d_0}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{c_1+d_0}{2}$
– profit	$\Big((I-1)d+t+\dfrac{c_2}{2}\Big)q_{\text{Col}}^2$	Idq_{Col}^2

Table 2 gives the unchanging pattern in the outcomes for 1–5 firms. I want to transform the table into a numerical example with certain desirable properties, which make the model suitable for application in the laboratory. One wish is that the Bertrand equilibrium B, Cournot equilibrium C and Collusion point Col be clearly separated; given the fixed ratios between the outcomes, this property can only be obtained by choosing the scale "sufficiently" large. A sufficiently large scale also helps to constrain the relative deviation from the true outcomes introduced by rounding them to the nearest integer, a practice that seems advisable in the laboratory. On the other hand, the figures must not be "unduly" large. It would also be nice for the outcomes to have a somewhat "realistic" flavour.

The admittedly vague desiderata leave ample room for other considerations. I have imposed the constraint that the parameters yield only integer outcomes for the cases of 1–5 firms. As to the quantities, it requires k to be a multiple of 120 ($= 2^3 \cdot 3 \cdot 5$). It does not seem necessary to choose k larger than 120. As to the prices, it requires $c_1 + d_o$ to be a multiple of 60 ($= 2^2 \cdot 3 \cdot 5$). The value of 60 implies $p_{Col} = 30$. Given the need to choose c_1 well in excess of 0 (to allow of decreasing marginal cost), this value leaves a rather small interval for the prices. So let's double it.

Table 2: The classic case: pattern in the outcomes

Number of firms (I)	1	2	3	4	5
Quantities$\div k$					
– q_Z	1	1/2	1/3	1/4	1/5
– q_B	1/2	1/2	1/3	1/4	1/5
– q_C	1/2	1/3	1/4	1/5	1/6
– q_{Col}	1/2	1/4	1/6	1/8	1/10
Prices					
– p_Z	c_1	c_1	c_1	c_1	c_1
– p_B	$\frac{c_1+d_0}{2}$	c_1	c_1	c_1	c_1
– p_C	$\frac{c_1+d_0}{2}$	$\frac{2c_1+d_0}{3}$	$\frac{3c_1+d_0}{4}$	$\frac{4c_1+d_0}{5}$	$\frac{5c_1+d_0}{6}$
– p_{Col}	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$

Lastly, the wish for a somewhat "realistic" flavour. Observe that the elasticity of the inverse demand function at the Collusion point, e_{Col}, is given by $(c_1-d_0)/(c_1+d_0)$. A value of c_1 close to d_0 yields a value of e_{Col} close to zero and hence a low markup of price over (marginal) cost, whereas a value of c_1 close to zero yields a value of e_{Col} close to -1 and hence a high markup. Steering away from both extremes, I choose $e_{\text{Col}} = -0.5$, which yields a markup of 2 and implies $d_0 = 3c_1$. In this way I arrive at $c_1 = 30$, $d_0 = 90$ and $d = 0.5$. Table 3 gives the outcomes for the quantities and prices with this choice of parameters. Note that the parameter d may be used to scale the quantities. For example, halving the value of d doubles all quantities: the change represents a pure demand shift, with the same relative rise of demand at every price.

Table 3: The classic case: a numerical example

I	1	2	3	4	5
Quantities					
– q_Z	120	60	40	30	24
– q_B	60	60	40	30	24
– q_C	60	40	30	24	20
– q_{Col}	60	30	20	15	12
Prices					
– p_Z	30	30	30	30	30
– p_B	60	30	30	30	30
– p_C	60	50	45	42	40
– p_{Col}	60	60	60	60	60

4. Six variants

I am now ready to consider model variants with nonconstant marginal cost ($c_2 \neq 0$) and/or differentiated goods ($t > d$). It is convenient to choose c_2 and $t-d$ in proportion to d: this practice yields all quantities as fractions of k that are independent of d, so that a change of d still represents a pure demand shift. For c_2 I consider the values of $d/2$, 0 and $-d/2$, for $t-d$ the values of 0 and $d/4$. Table 4 gives the labels of the six model variants, or *industries*.

Table 4: Six model variants

Nature of the good(s)	Marginal cost		
	Increasing	Constant	Decreasing
	($c_2 = d/2$)	$c_2 = 0$	($c_2 = -d/2$)
Homogeneous: $t-d=0$	HoIn	HoCon	HoDe
Heterogeneous ($t-d=d/4$)	HeIn	HeCon	HeDe

One element in the discussion of the outcomes is the comparison within an industry across the numbers of firms, which have been treated as exogenous so far. Such a comparison naturally leads to the question what the assumption of joint profit maximisation implies. To answer it, the Tables 5–10 contain, next to the prices, the outcomes for total output Q and total profit U of the industries. I shall use the property of the model that within the Pareto optimal set, industry profit is largest where total output is largest. To avoid the infinite and infinitesimal, I assume that there is some non-zero minimal firm size and that firms can enter an industry only at the minimal size.

Let me begin with the outcomes of the classic case, Industry HoCon, in Table 7. A Bertrand duopoly produces twice the monopoly quantity, which is a generic outcome of the classic case. The price equals marginal cost at half the monopoly price (a consequence of the specific choice of parameters), and profits are down to zero. A further increase of the number of firms changes neither the total output nor the price. As the number of firms rises, the Cournot equilibrium moves gradually from the monopoly outcome towards the Bertrand equilibrium and the zero-profit competitive outcome; the increase from one firm to ten firms closes most of the gap between the monopolistic and competitive outcomes.[5] Actually, the Cournot oligopoly owes its popularity to this gradual transition, as it agrees with the intuition of many economists. Another way of looking at the same pattern, however, is that the firms in the Cournot oligopoly perceive an incentive to merge or to collude: industry profit rises as the number of firms declines. But the true profit maximising strategy for the firms is in fact to charge the monopoly price and to jointly produce the monopoly output. Because in the Pareto optimal set the profit-output ratio of this industry is the same over the whole range of firm sizes and numbers of firms, the joint profit is constant, too: the assumption of joint profit maximisation does not select a specific size distribution of firms. Nonconstant marginal cost and/or product differentiation usually change this outcome, as will appear below.

Let me next turn to Industry HoIn in Table 5. When marginal cost is increasing, the firms in the Bertrand oligopoly earn positive profits. The joint profit of Bertrand duopolists is substantially below the monopolist's profit, but has not fallen all the way down to zero; further increases of the number of firms drive the joint profit down to zero at a slower pace. Note that the joint profit of Cournot duopolists exceeds the monopolist's profit; the cost savings obtained by spreading production over two firms outweigh the depressing effect of the additional firm on the price of the good. From two firms onward industry profit declines towards zero, again at a slower pace than in the Bertrand oligopoly.[6] The behaviour of industry profit in the "collusive" oligopoly is quite different: it moves upwards, back to its level in the classic case, as the number of firms increases. As shown in the Appendix, the Pareto optimal set of this industry is concave to the origin. The centroid *Col* is its point where industry output and profit are largest: dividing total

[5]For all three models in all six industries, most outcomes for the decapoly are close to the limiting values.

[6]Stated differently, industry profit rises faster in the Bertrand oligopoly than in the Cournot oligopoly when the number of firms declines. The difference may be related to the finding in the laboratory that 'Bertrand colludes more than Cournot.' For more on this matter, see Suetens and Potters (2007).

Table 5: Industry HoIn

I	1	2	3	4	5	10
Q_Z	96	107	111	113	114	117
Q_B	48	96	103	107	109	114
Q_C	48	69	80	87	92	104
Q_{Col}	48	53	55	56	57	59
p_Z	42	37	35	34	33	31
p_B	66	42	39	37	35	33
p_C	66	56	50	46	44	38
p_{Col}	66	63	62	62	61	61
U_B	1440	576	441	356	298	163
U_C	1440	1469	1333	1190	1065	681
U_{Col}	1440	1600	1662	1694	1714	1756

Table 6: Industry HeIn

I	1	2	3	4	5	10
Q_Z	80	96	103	107	109	114
Q_B	40	75	88	95	99	109
Q_C	40	60	72	80	86	100
Q_{Col}	40	48	51	53	55	57
p_Z	40	36	34	33	33	31
p_B	65	48	43	40	38	34
p_C	65	56	51	48	45	39
p_{Col}	65	63	62	62	61	61
U_B	1200	984	781	645	548	312
U_C	1200	1350	1296	1200	1102	750
U_{Col}	1200	1440	1543	1600	1636	1714

Table 7: Industry HoCon

I	1	2	3	4	5	10
Q_Z	120	120	120	120	120	120
Q_B	60	120	120	120	120	120
Q_C	60	80	90	96	100	109
Q_{Col}	60	60	60	60	60	60
p_Z	30	30	30	30	30	30
p_B	60	30	30	30	30	30
p_C	60	50	45	42	40	35
p_{Col}	60	60	60	60	60	60
U_B	1800	0	0	0	0	0
U_C	1800	1600	1350	1152	1000	595
U_{Col}	1800	1800	1800	1800	1800	1800

Table 8: Industry HeCon

I	1	2	3	4	5	10
Q_Z	96	107	111	113	114	117
Q_B	48	89	100	105	108	114
Q_C	48	69	80	87	92	104
Q_{Col}	48	53	55	56	57	59
p_Z	30	30	30	30	30	30
p_B	60	40	36	34	33	32
p_C	60	51	47	44	42	37
p_{Col}	60	60	60	60	60	60
U_B	1440	889	598	449	360	180
U_C	1440	1469	1333	1190	1065	681
U_{Col}	1440	1600	1662	1694	1714	1756

Table 9: Industry HoDe

I	1	2	3	4	5	10
Q_Z	160	137	131	128	126	123
Q_B	80	160	144	137	133	126
Q_C	80	96	103	107	109	114
Q_{Col}	80	69	65	64	63	62
p_Z	10	21	25	26	27	28
p_B	50	10	18	21	23	27
p_C	50	42	39	37	35	33
p_{Col}	50	56	57	58	58	59
U_B	2400	-1600	-864	-588	-444	-199
U_C	2400	1728	1322	1067	893	490
U_{Col}	2400	2057	1964	1920	1895	1846

Table 10: Industry HeDe

I	1	2	3	4	5	10
Q_Z	120	120	120	120	120	120
Q_B	60	109	116	118	119	120
Q_C	60	80	90	96	100	109
Q_{Col}	60	60	60	60	60	60
p_Z	15	23	25	26	27	29
p_B	53	29	27	27	28	29
p_C	53	45	41	39	38	34
p_{Col}	53	56	58	58	59	59
U_B	1800	595	248	133	83	19
U_C	1800	1600	1350	1152	1000	595
U_{Col}	1800	1800	1800	1800	1800	1800

demand evenly among the firms is the most profitable arrangement of the industry. Firms that enter the industry wish to stay small, because in that way they avoid the adverse effects of increasing marginal cost. An increase of demand, for example through a drop of the value of d, is most profitably met by the entry of firms, not by the growth of existing firms.

The value of $t-d$ in Industry HeCon is half the value of c_2 in Industry HoIn, so that the industries have identical profit functions in quantity space. Therefore a number of rows of Table 8 are identical to the corresponding rows of Table 5, those for U_{Col}, U_C, Q_{Col}, Q_C and Q_Z, to be precise. The matching prices do differ, because the industries have different inverse demand functions. The Bertrand equilibrium is at a different point in both quantity space and price space. As to the "collusive" oligopoly, just like firms that enter Industry HoIn, firms that enter Industry HeCon wish to stay small, but for a different reason: they want to avoid the adverse effects of (relatively) fast decreasing marginal revenue. An increase of demand is most profitably met by the entry of firms that bring new varieties to the market, not by producing more of existing varieties.

Industry HeIn combines increasing marginal cost with "fast" decreasing marginal revenue, which may be typical of many traditional industries. As we have seen, both changes from Industry HoCon affect the quantities in the same direction. The figures in Table 6 confirm that the deviations of the quantities from their HoCon-counterparts are similar to, and larger than with one of the changes separately. As a consequence, this observation applies to the profits of the "collusive" oligopoly, too.

The picture changes drastically when we move on to Industry HoDe, which produces one homogeneous good using a technology with decreasing marginal cost (see Table 9). Bertrand oligopolists suffer losses as long as their number exceeds one. Merger increases the losses of the industry, unless all firms merge at once into one firm, which then starts acting as a monopolist and makes a large profit. A given number of Cournot oligopolists (more than one) perceive a stronger incentive to merge than in any other industry here considered. The firms in the "collusive" oligopoly, too, perceive an incentive to merge, albeit less strongly than the Cournot oligopolists do; the reason is that industry profit at the (exogenously fixed) initial number of firms is already maximal and exceeds by far the joint profit of the same number of Cournot oligopolists. As shown in the Appendix, the Pareto optimal set of this industry is convex to the origin. The centroid Col is its point where industry output and profit are smallest. They reach their maxima at any of the monopoly points, because decreasing marginal cost is exploited maximally by concentrating all production in one firm. An increase of demand, for example through a drop of the value of d, is most profitably met by increasing the output of this one firm.

Industry HeDe, with decreasing marginal cost and product differentiation, may be characteristic of many modern industries. The values of $t-d$ and $c_2/2$ that I have chosen are such that their sum is zero: the profit functions in quantity space are identical to those in Industry HoCon. I will not repeat the part of the discussion of Industry HeCon on this matter. In the "collusive" oligopoly, the amount of profit is constant across the number of firms; once more, joint profit maximisation does not select a specific size distribution of firms.

The flawed treatment of the vector maximisation problems by noncooperative game theory results in distorted incentives for merger: in five of the six industries, the Bertrand oligopoly and the Cournot oligopoly yield other outcomes for the optimal number of firms than the "collusive" oligopoly does. In my opinion, the "collusive" oligopoly agrees best with a cursory view of the world. But of course, this observation carries no weight in a dispute on a mathematical issue.

5. Some comments on laboratory experiments

Oligopoly models like the ones studied here are popular tools in the economics laboratory for testing theories of behaviour in situations with few, interacting participants. In contrast to what the use of the term "laboratory" suggests, however, the proceedings in the economics laboratory differ fundamentally from those in the physics laboratory. Whereas in the physics laboratory the participants (for example, elementary particles) "know" the laws of nature and the experimenters are struggling to find out what the laws are, in the economics laboratory the experimenters have set the "laws of nature" and the participants (often undergraduate students) are struggling to find them out. Can one rationally expect beginners to grasp, within an hour or so, the mechanics of an artificial world that many an accomplished economist still does not understand properly?

It will surely help to give the beginners a head start by instructing them extensively, on the model and the means at their disposal to reach good decisions. A concern of the designer of the experiment is to supply the participants with adequate information without unveiling the solution. However, I think that suggesting certain procedures for attacking the problem is quite justified. The procedures need not be sophisticated; after all, Huck et al. (2004) have shown that, for the industries HoCon and HoIn, a simple noisy trial-and-error method always leads to the centroid of the Pareto optimal set. Meanwhile, the designer must beware of leading the endeavors of the participants in a particular direction. This aspect gains weight in connection with another one. Participants in experiments have often been, and in future experiments will be recruited from undergraduate students. Many of them, I suspect, will be economics students, with prior exposure to economic theory and the fallacy of Nash equilibrium.

Even when prospective participants have received extensive instructions, it may be a good idea to familiarise them as monopolists with the model and the experimental setting. A bad performance of some participant as a monopolist puts into perspective the outcomes of her later plays of games.

My last, but not least important comment on current practice in the economics laboratory is this. In the physics lab, the experimenters create the conditions in which the phenomena predicted by their models are likely to occur. Experimenters in the economics lab, however, have frequently failed to do so. The first instruction that the participants in an oligopoly game receive goes often like this:

> During the experiment you are not allowed to talk to other participants. If something is not clear, please raise your hand and one of us will help you.

Nash (1951) is to blame for this ban on communication. He suggested a new "solution concept" for a game, which would apply when the players of the game were unable to communicate and cooperate. Essentially, Nash replaced a simultaneous maximum problem with a set of conditional maximum problems by postulating that each player conditions on the *endogeneous* actions of the other players; when applied to an oligopoly game, the Nash equilibrium is the Bertrand equilibrium (Cournot equilibrium) when the firms use the prices (quantities) as instruments. Mathematically, the postulate amounts to ignoring the partial cross-derivatives of the profit functions when the first-order conditions for a solution are derived, *as if they are zero*.[7] The resulting conditions are incompatible with the first-order approximations of the profit functions, for—when evaluated at a Nash equilibrium—the cross-derivatives turn out to be *non-zero*. We have a prime example of a contradiction here; oddly enough, in economic theory and game theory this familiar result has not been recognised as the sign of a logical flaw. Regardless of this mathematical error, the ban on communication in experiments is not in keeping with the specification of the model. Variables corresponding to the actions of communication and cooperation are not present in the model, let alone constraints on such actions. Certain constraints may be lurking in the background, but in the model under empirical scrutiny their Lagrange multipliers are zero. Stated differently, communication and cooperation are free activities in the artificial world of the model. Therefore, the first instruction better be replaced by something like this:

> The experiment consists of a number of plays of a game. During the experiment, you and the other players have access to a chatroom, where you may discuss any issues concerning the plays of the game. However, if you have questions concerning the experimental setting, please raise your hand and one of us will help you.

The criticism above is not to deny the interest of experiments in which the participants may not communicate. It merely stresses that the model does not apply to this situation, and that a Nash equilibrium is not a mathematically consistent yardstick to judge the outcomes. Elements of reality absent from the model cannot affect the solution. Impediments for collusion must be modeled explicitly, for example along the lines suggested by Stigler (1964).

6. Concluding remarks

Ever since Cournot (1838), economists have built multi-agent models in the following way. First, they derived the first-order conditions of each agent's conditional maximum problem, treating only the agent's actions as endogenous and conditioning on the actions of the other agents. Next, they assembled all first-order conditions into one system of equations, adding some new (!) constraints (the market equilibrium conditions) if needed. Von Neumann and

[7] See Nieuwenhuis (2018) for a brief exposition, or Nieuwenhuis (2017a) for an extensive treatment of issues concerning the Nash equilibrium.

Morgenstern (1947), and De Finetti (1937a,b) before them, noted that the conditional maximum problems were interdependent and actually constituted a vector maximisation problem, or game. Shortly after, Nash (1951) postulated that the agents, when they were unable to cooperate, would in fact condition on the *endogenous* actions of their rivals; thus he disassembled the game into the old set of conditional maximum problems. However, the justification of the postulate relies on a particular interpretation of the problem. To see the fallacy of the notion of Nash equilibrium, just concentrate on the mathematics and solve the game devoid of any interpretation. The classic oligopoly model of Cournot and its simple extensions serve well as examples in an introduction to the method of vector maximisation, meanwhile exposing the flaw in noncooperative game theory.

Nash (1951) uses a fixed-point theorem to prove the existence of a Nash equilibrium; its status of fixed point of some mapping gives the Nash equilibrium an aura of stability. In our oligopoly games, the existence of a Nash equilibrium is shown constructively, by solving a system of linear equations. The simplicity of the games makes it easy to study the properties of a Nash equilibrium. Its dependence, as noted, on the arbitrary choice of instruments nullifies any claim of stability: The prices at the Cournot equilibrium do not constitute a Nash equilibrium in price space, so that (according to noncooperative game theory itself) the firms will be tempted to change their prices until they arrive at the Bertrand equilibrium, where they observe that the implied quantities do not constitute a Nash equilibrium in quantity space, so that . . . , and so on.

In a review of the experimental literature, Haan et al. (2006) find that *The ability to communicate among sellers has a strong and positive effect on the ability to collude.* The finding is good news for the proponents of the rationality postulate as the starting point of economic theory. In real life, there appears to be more coordination of actions than is compatible with noncooperative game theory. Because an experimental setting that allows of easy communication is a better approximation of many real-world situations than the alternative, the finding is consistent with this observation. Individually rational decision makers seem to understand well that in many situations they serve their private interests best by acting in unison with others.

'There is such a thing as being just plain wrong' (Richard Dawkins in *The Selfish Gene*), and that's what noncooperative game theory is. Economics cannot hold on to a failed solution of a vector maximisation problem as its prediction of the outcome of rational behaviour in situations where actions are interdependent. The theory now known as *cooperative* game theory is the basis of the theory of rational decision making, unqualified by adjectives like "cooperative" versus "noncooperative," or "individual" versus "collective."

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Appendix A. The solution of the game

The oligopoly problem without the assumption of joint profit maximisation is a vector maximisation problem. Here I present the solution. To simplify the formulas, let us change the units of measurement. Define the scaled quantities $x_i := q_i/k$, $i = 1, \dots, I$, $k := (d_0 - c_1)/d$, and divide the profit functions (5) by k^2/d to redefine them to

$$\upsilon_i = u_i(x_1, \dots, x_I) = (1 - X)x_i - rx_i^2, \qquad i = 1, \dots, I, \tag{A.1}$$
$$r = \frac{t - d + c_2/2}{d},$$

where $X := \sum_j x_j$. The expressions for x_{Col} and υ_{Col} are

$$x_{\text{Col}} = \frac{1}{2(I + r)}, \tag{A.2a}$$
$$\upsilon_{\text{Col}} = (I + r)x_{Col}^2. \tag{A.2b}$$

According to (A.2), $-1 < r$ must hold for optimal monopoly output and profit to be positive and finite.

The maximal-profit set

Let me first show the equivalent of the fact stated in Section 2, p. 4,

FACT: *The maximal-profit set of oligopoly* (A.1) *is the set of points where the industry's profit-output ratio equals* $1/2$.

PROOF: The proof is for a duopoly, but the method of proof applies to any number of firms. Consider the industry's profit along the ray $x_2 = \alpha x_1$. Substitute $x_2 = \alpha x_1$ in the profit functions of the firms and add the results:

$$\upsilon_{1,\alpha} = v_1(x_1|\alpha) := x_1 - (1 + \alpha + r)x_1^2, \tag{A.3a}$$
$$\upsilon_{2,\alpha} = v_2(x_1|\alpha) := \alpha x_1 - \big(\alpha + \alpha^2(1 + r)\big)x_1^2, \tag{A.3b}$$
$$\Upsilon_\alpha := \upsilon_{1,\alpha} + \upsilon_{2,\alpha} = (1 + \alpha)x_1 - \big((1 + \alpha)^2 + (1 + \alpha^2)r\big)x_1^2. \tag{A.3c}$$

The ratio of the maximum to its argument, Υ_α^*/x_1^*, is $(1 + \alpha)/2$, so that the profit-output ratio of the industry is in fact $1/2$. Because the profit-output ratio is a strictly decreasing function of industry output, there is no other point along the ray $x_2 = \alpha x_1$ where the profit-output ratio of the industry is $1/2$. *Q.E.D.*

To gain a better understanding of the result, rewrite (A.3b) and (A.3c) to

$$w_2(x_1|\alpha) := x_1 - \big(1 + \alpha(1 + r)\big)x_1^2, \qquad \alpha > 0, \tag{A.3b'}$$
$$W(x_1|\alpha) := x_1 - \Big(1 + \alpha + \frac{1 + \alpha^2}{1 + \alpha}r\Big)x_1^2. \tag{A.3c'}$$

(A.3a), (A.3b') and (A.3c') are quadratic functions of the form $ax^2 + bx$ with a common value of b but different values of a. The value of a in the third function is a weighted average of its values in the first two functions, with the weights of $1/(1 + \alpha)$ and $\alpha/(1 + \alpha)$, respectively. Therefore the point on the ray where industry profit is maximal is generally in between the points where the profits of the firms reach their maxima. As the ray $x_2 = \alpha x_1$ revolves around the origin from the x_1-axis ($\alpha = 0$) towards the x_2-axis ($\alpha = \infty$), the weight of Firm 1's profit function declines from 1 to 0. Simultaneously, the intersection of the ray with the zero-profit line of Firm 1, $Z_{1,\alpha}$, moves from the point $Z_1 = Z_{1,0} := [1/(1 + r), 0]$ towards the point $B_1 = Z_{1,\infty} := [0, 1]$. Parallel to this line (segment), at half the distance from the origin, is the maximal-profit line of Firm 1; along the line, profit declines linearly from the monopoly profit at $C_1 = P_{1,0} := [1/\big(2(1 + r)\big), 0]$ towards zero at $P_{1,\infty} := [0, 1/2]$. The maximal-profit line of Firm 2 is the mirror image of the one of Firm 1 with respect to the ray $x_1 = x_2$.

In the classic case, or more generally when $r = 0$, the maximal-profit lines coincide, and the line $1 - 2X = 0$ constitutes the Pareto optimal set; along the line, industry output and profit are constant. Only in this case is the ray $x_2 = \alpha x_1$ the common tangent line of the isoprofit curves at every point of the Pareto optimal set.

In the general case of $r \neq 0$, the maximal-profit lines intersect at *Col*. The industry has a maximal-profit curve, which runs from C_1 through *Col* to C_2, the mirror image of C_1 with respect to $x_1 = x_2$; the intermediate segments of the curve are in between the maximal-profit lines. The ray $x_2 = \alpha x_1$ is tangent to an isoprofit curve of the industry at $P_{1,\alpha}$; it is generally *not* the common tangent line of the isoprofit curves of the firms, as these profits usually reach their maxima at other points of the ray.

It remains to see how industry output and profit evolve along the maximal-profit curve. Compare the output of a firm at *Col*,

$$x_{\text{Col}} = \frac{1}{2(2+r)}, \tag{A.2a'}$$

to the output of a firm halfway between C_1 and C_2 , which equals half the monopolist's output,

$$\frac{x_{\text{mono}}}{2} = \frac{1}{4(1+r)}.$$

If $r > 0$, then $x_{\text{Col}} > x_{\text{mono}}/2$, so that the curve is concave to the origin; it proves to be a segment of an ellipse (see below). *Col* is the point of the curve where industry output and profit are largest. If $r < 0$, then $x_{\text{Col}} < x_{\text{mono}}/2$, so that the curve is convex to the origin; it proves to be a segment of a hyperbola (see below). *Col* is the point of the curve where industry output and profit are smallest, they are largest at any of the monopoly points.

For any number of firms, putting the sum of the profit functions equal to $X/2$ yields the equation of the maximal-profit (hyper)surface,

$$2X^2 + 2r\sum_j x_j^2 - X = 0. \tag{A.4}$$

If $r = 0$, the quadric surface reduces to $X(2X-1) = 0$. Because $X \neq 0$, the solution equals the Pareto optimal set $2X-1 = 0$. For a duopoly, the Discriminant Δ of the cone section is $-16r(2+r)$. If $r > 0$ (and if $r < -2$), then $\Delta < 0$: the equation represents an ellipse. If $-2 < r < 0$, then $\Delta > 0$: the equation represents a hyperbola. For larger numbers of firms, the maximal-profit (hyper)surfaces are ellipsoids or hyperboloids of revolution.

The Pareto optimal set

Let me next turn to the Pareto optimal set. One part of the first-order conditions of the problem of maximising I (continuously differentiable) functions of $J \geq I$ continuous variables is that the matrix of first-order derivatives of the functions have deficient row rank. I do not consider the other part of the first-order conditions here, nor the second-order conditions. Let $\mathbf{U}$ be the matrix of first-order derivatives of the profit functions. In the present case the condition amounts to $|\mathbf{U}| = 0$; geometrically, the set of points satisfying the condition is the variety where the isoprofit curves of the firms have a tangent line in common.[8] The nonnegative quadrant contains two segments of the variety; the Pareto optimal set is the segment closer to the origin.

When there are I firms, $|\mathbf{U}|$ is a polynomial of degree I in I variables. If $r = 0$, the condition $|\mathbf{U}| = 0$ reduces to $(1-X)^{I-1}(1-2X) = 0$, which represents the Pareto optimal set and Zero-profit set; the Pareto optimal set and maximal-profit set coincide. For a duopoly, the Discriminant Δ of the cone section is $16(1+r)^2 r(2+r)$. If $r > 0$ (and if $r < -2$), then $\Delta > 0$: the equation represents a hyperbola. If $-2 < r < 0$, then $\Delta < 0$: the equation represents an ellipse. The Pareto optimal set and maximal-profit set do not coincide. They do have the Collusion points of the included oligopolies in common, but at the intermediate segments the maximal-profit set is slightly farther away from the origin than the Pareto optimal set. The property of a constant profit-output ratio applies only to the maximal-profit set. Still, the evolutions of industry profit and output along the Pareto optimal curve and maximal-profit curve are qualitatively similar. Figure A.1) sketches the solution of $|\mathbf{U}| = 0$ for the duopoly HoIn.

What does the Pareto optimal set look like when the number of firms exceeds two? In a triopoly, there are three monopoly points and three curves like C_1C_2 connecting them. The Pareto optimal set is a surface area, topologically

[8] At a point where $|\mathbf{U}| = 0$, there exists a vector of variations $\mathrm{d}\mathbf{x}$ such that $\mathbf{U}\mathrm{d}\mathbf{x} = 0$; the differential quotients $\mathrm{d}x_i/\mathrm{d}x_j$ derived from the vector, in the literature known under the misnomer of *conjectural variations*, are the marginal rates of substitution along this common tangent line. Noncooperative game theory ignores the off-diagonal elements of $\mathbf{U}$ and defines a Nash equilibrium by the conditions $u_{ii}\mathrm{d}x_i = 0, i = 1,\ldots,I$: at a Nash equilibrium, the isoprofit curves are *perpendicular* instead of tangent. Contrary to tangency, perpendicularity is not generally invariant under nonsingular transformations of variables.

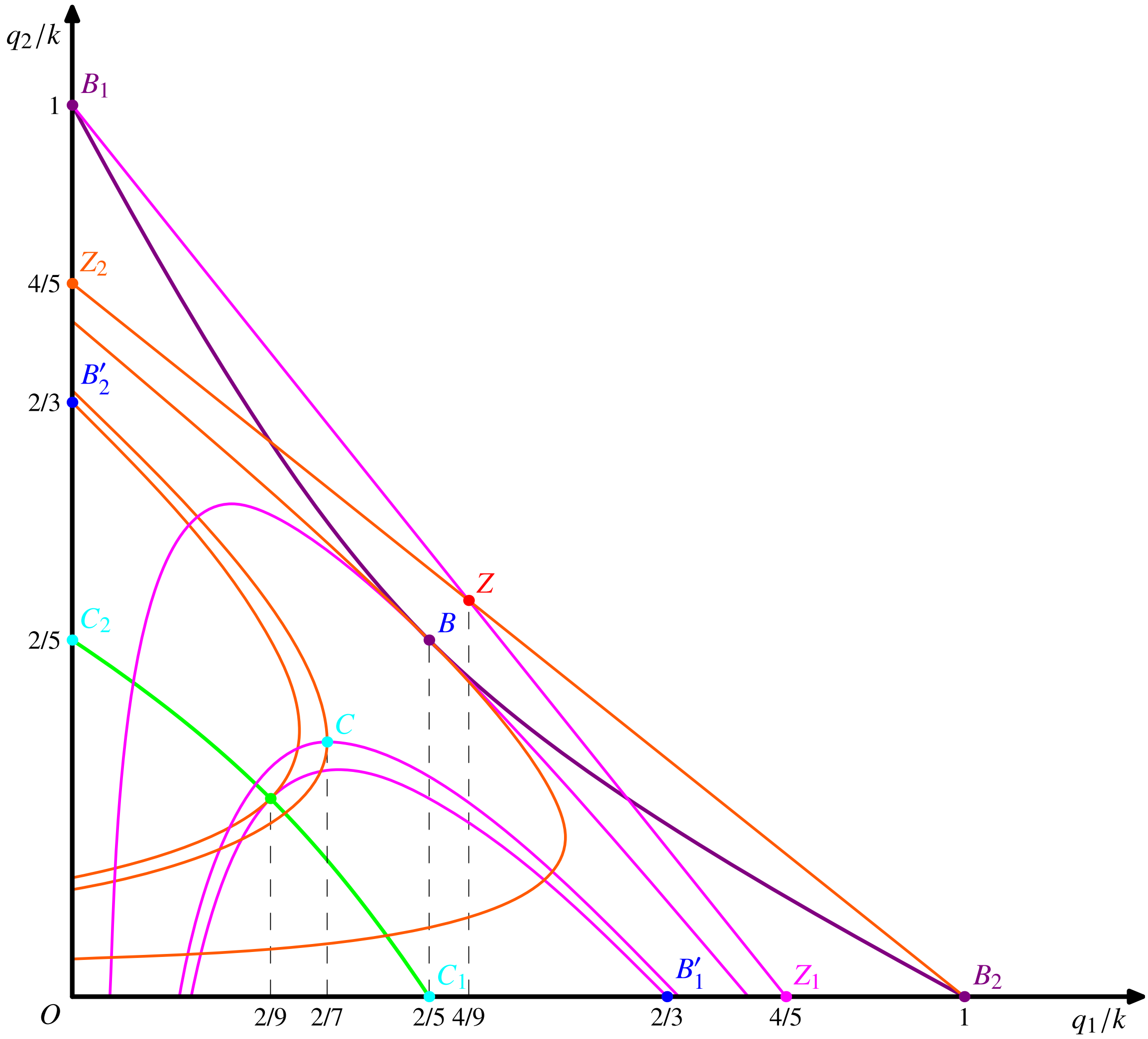

Figure A.1: Industry HoIn ($r = c_2/2d = 1/4$)

Notes: The curve through B_1, B and B_2 solves $|\mathbf{U}| = 0$, but is not part of the Pareto optimum. The curve through C_1 and C_2 solves $|\mathbf{U}| = 0$, and is the Pareto optimum; its midpoint is the Collusive equilibrium. $[B_iZ_i]$ is the Zero-profit line of Firm i. Z is the Zero-profit point, $B_i{}'$ the end point of Firm i's Bertrand reaction function, C_i the end point of Firm i's Cournot reaction function (also Firm i's monopoly point). At the Bertrand equilibrium B the isoprofit curves have a tangent line in common, but still they intersect. At the Cournot equilibrium C the isoprofit curves are perpendicular to one another.
Source: Nieuwenhuis (2017a, Figure 4).

a triangle, the three sides of which are the curves of the included duopolies. The surface area is convex to the origin, flat, or concave to the origin for $r < 0$, $r = 0$, or $r > 0$, respectively. For a tetrapoly, the Pareto optimal set is a volume, topologically a tetrahedron, the four faces of which are the "triangles" of the included triopolies. And so on, beyond graphical representation, for still larger numbers of firms. We have a perfect example of a result, for the first time stated and proved by De Finetti (2017b, Section 12),

The locus of "optimum" points with respect to n *functions is, topologically, a simplex of* $\mathrm{n}-1$ *dimensions, the* n *faces of which are the loci of "optimum" with respect to* $\mathrm{n}-1$ *<of the> functions, the* $\binom{\mathrm{n}}{2}$ *edges of which those for* $\mathrm{n}-2$ *<of the> functions, and so on, up to the* n *vertices, "optimum" points with respect to the* n *functions separately.*

Here, the "locus of "optimum" points" is what we call the Pareto optimal set nowadays.