Oligopoly games for use in the classroom and laboratory

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Abstract

In support of the claim that the Nash equilibrium results from a flawed attempt to solve a game, this article studies two modifications of the classic oligopoly model of Cournot. The common cost function is quadratic, so that marginal cost may be increasing, constant or decreasing, and the (still linear) inverse demand functions allow for differentiated goods. A special feature of the model, not previously noted, is that at the optimal points the profit-output ratio of the industry is the same, regardless of the number of firms and of several other parameters. The choice of parameters is discussed and six model variants are analysed through numerical examples. Some comments on the use of this class of models in experimental economics are made.

Keywords: Bertrand, Cournot, Collusion, Oligopoly, Game theory, Vector maximisation, Experimental economics
JEL: C91, C92, D21, D43, L11, L13

> No one has the right, and few the ability, to lure economists into reading another article on oligopoly theory without some advance indication of its alleged contribution. The present paper accepts the hypothesis that oligopolists wish to collude to maximize joint profits. It seeks to reconcile this wish with facts, such as that collusion is impossible for many firms and collusion is much more effective in some circumstances than in others.
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> **G. J. Stigler (1964, p. 44)**

1. Introduction

You are about to read another article on oligopoly theory. Like Stigler's, it accepts the hypothesis that firms wish to collude to maximise profits, but it takes a different turn. It stresses that collusion *is* in fact the profit maximising strategy of the firms in the artificial worlds described by many oligopoly models, as these do not contain any impediments for collusion. The article's main aim is to support the claim, argued at length in Nieuwenhuis (2017a) and in a nutshell in Nieuwenhuis (2018), that the Nash equilibrium results from a mathematically flawed attempt to solve a game and that non-cooperative game theory must be eliminated from the textbooks on (micro)economic theory. It does so by analysing extensions of probably the most popular tool in the classroom and laboratory, the classic oligopoly model of Cournot (1927) with one homogeneous good, a linear inverse demand function and constant marginal cost common to all firms (henceforward to be called "the classic case"). It considers oligopolies with arbitrary numbers of firms that produce imperfectly substitutable varieties of some good (one homogeneous good included as a special case), sharing a quadratic cost function. The inverse demand functions are still linear; for simplicity's sake, the specification is symmetric in the way it treats the goods. It shows that at the optimal points the profit-output ratio of the industry is the same, regardless of the number of firms and of several other parameters. It discusses how parameter values may be chosen so as to obtain model variants with attractive features for use in the laboratory, and presents numerical examples to highlight the differences between and similarities of the outcomes of six model variants. Finally, it comments on the use of this class of models in the laboratory.

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2. The model

Consider an industry with profit maximising firms i ($i = 1,\ldots,I$), each of which produces a variety of some good. The firms have identical quadratic cost functions,

$$c(q_i) = c_1 q_i + \frac{1}{2} c_2 q_i^2, \tag{1}$$

where q_i is the quantity of the variety produced by Firm i. Adding fixed costs c_0 affects only the labels of the iso-profit (hyper)surfaces. The derivative of $c(q_i)$ is the linear marginal-cost function,

$$mc(q_i) = c_1 + c_2 q_i, \qquad c_1 > 0. \tag{2}$$

The parameter c_2 may be positive, zero or negative, corresponding to increasing, constant or decreasing marginal cost. Because of the last possibility, I assume $c_1 \gg 0$.

The inverse demand functions, too, are linear:[1]

$$p_i = f_i(\mathbf{q}) = d_0 - dQ - (t-d)q_i, \qquad i = 1,\ldots,I, \tag{3}$$

where $\mathbf{q}$ is the (column) vector of quantities and $Q := \sum_j q_j$. I shall use the entity Q as an indicator of industry output. The parameters satisfy $d_0 > c_1$, $0 < d \leq t$.[2] When $t > d$, the goods are imperfect substitutes. The condition $d_0 > c_1$ ensures that the prices of the goods exceed marginal cost when the firms produce nothing at all.

The symmetric specification of the inverse demand functions, next to the common cost function, implies that most outcomes of interest are points on the ray $q_1 = q_2 = \cdots = q_I$ (the axis of symmetry), which keeps the analysis simple.

With differentiated goods, that is when $t > d$, the demand functions are

$$q_i = g_i(\mathbf{p}) = \delta_{0,I} + \delta_I P - (\tau_I + \delta_I) p_i, \qquad i = 2,\ldots,I, \tag{4}$$

with $P := \sum_j p_j$, and where

$$\tau_I := \frac{(I-2)d + t}{\big((I-1)d+t\big)(t-d)}, \qquad I \geq 1, \tag{5a}$$

$$\delta_I := \frac{d}{\big((I-1)d+t\big)(t-d)}, \qquad I \geq 2, \tag{5b}$$

$$\delta_{0,I} := \frac{d_0}{(I-1)d+t}, \qquad I \geq 1. \tag{5c}$$

The parameters satisfy $\delta_{0,I} > 0$, $0 < \delta_I < \tau_I$.

The profit functions *in quantity space* are

$$\upsilon_i = u_i(\mathbf{q}) = (d_0 - c_1 - dQ)q_i - (t - d + \frac{1}{2}c_2)q_i^2, \qquad i = 1,\ldots,I. \tag{6}$$

They differ from the profit functions of the classic case only if $t - d + c_2/2 \neq 0$. However, even if $t - d + c_2/2 = 0$, the demand functions and, hence, the profit functions *in price space* do differ when $t > d$. More generally, every triplet (t,d,c_2) that satisfies $t - d + c_2/2 = z$ yields the same profit functions in quantity space but not in price space. This property of the model results from the linearity of both the marginal-cost function and the inverse demand functions. It implies that to every increase of the slope of the marginal-cost function there is an equivalent decrease of the slopes of the perceived marginal-revenue functions (for which see below).[3] I shall return to the issue when discussing the outcomes of model variants in Section 4.

[1] One may define $s := t - d$ and work with d and s instead. Both possibilities have advantages and disadvantages.

[2] The case $t < d$ is mathematically feasible, but economically implausible: it corresponds to demand functions with positive own-price and negative cross-price elasticities.

[3] For the Bertrand equilibrium the equivalent decrease differs from the one for the other points considered below.

The profit of a firm is zero if $q_i = 0$ or else if price equals average cost. The latter condition defines a (hyper)plane the non-negative segment of which I simply call the zero-profit plane of the firm. Let Z be the intersection of the ray $q_1 = q_2 = \cdots = q_I$ with the zero-profit plane of any one of the firms. Then Z is either the centroid of the common zero-profit plane (if $t - d + c_2/2 = 0$) or else the intersection of the zero-profit planes, the unique zero-profit point. All its coordinates equal

$$q_Z = \frac{2d}{2(I-1)d + 2t + c_2} k, \qquad k := \frac{d_0 - c_1}{d}. \tag{7}$$

Other quantities of interest, too, will be expressed as fractions of k.

The I problems of profit maximisation are interdependent, because all quantities are the arguments of all maximands: they constitute a vector maximisation problem (elsewhere I have used the term *simultaneous maximum problem*). The solution of such a problem is the Pareto optimal set. Here I consider mainly the centroid of the set, which I call *Col* (for Collusion point). As we shall see, in some cases *Col* is the Joint Profit Maximum, in other cases it is the point of the Pareto optimum where the joint profit is minimal. To find the coordinates of *Col*, note that along the ray $q_1 = q_2 = \cdots = q_I$ the profit of each firm is a quadratic function of the common quantity q, and that the zeros of the parabola are at the origin and at Z. Therefore *Col* is halfway between the origin and Z.

Two failed attempts to solve the vector maximisation problem constituted by the simultaneous maximisation of the profit functions are those by Cournot (1927) and by Bertrand (1883). In a Cournot oligopoly, each firm chooses its quantity while conditioning on the quantities of its rivals; in a Bertrand oligopoly, each firm chooses its price while conditioning on the prices of its rivals. The firms in a Bertrand oligopoly perceive the marginal-revenue functions

$$\begin{aligned} mr_i^B &= p_i - \frac{1}{\tau_I} g_i(\mathbf{p}) \\ &= f_i(\mathbf{q}) - \frac{1}{\tau_I} q_i. \end{aligned} \tag{8}$$

When $t = d$ (one homogeneous product), $1/\tau_I = 0$ so that $mr^B = p$. The firms in a Cournot oligopoly perceive the marginal-revenue functions

$$\begin{aligned} mr_i^C &= f_i(\mathbf{q}) - t q_i \\ &= p_i - t g_i(\mathbf{p}). \end{aligned} \tag{9}$$

When the firms collude, they perceive the marginal-revenue functions

$$mr_i^{Col} = f_i(\mathbf{q}) - \big((I-1)d + t\big) q_i \tag{10a}$$

$$= p_i + \frac{1}{(I-1)\delta_I - \tau_I} g_i(\mathbf{p}). \tag{10b}$$

These formulas result from assuming respectively that the quantities or the prices move in unison. The formulas are equivalent, meaning that the Collusion point is invariant to the choice of instrument. In fact, the Pareto optimal set is invariant to non-singular transformations of variables (as is the solution of every optimum problem).

Equating marginal revenue to marginal cost yields the "reaction functions." The Bertrand equilibrium B and the Cournot equilibrium C (and also *Col*) are at the intersection of the ray $q_1 = q_2 = \cdots = q_I$ with any one of the appropriate "reaction functions." Table 1 gives a number of outcomes. The prices are always a weighted average of c_1 and d_0.[4,5] The outcomes of the classic case are the limits of the outcomes of the general case when the number of firms grows without bound: the effects of non-constant marginal cost and of product differentiation become negligible.

A noteworthy feature of the Collusion point is the constant profit-output ratio,[6]

$$\frac{\upsilon_{Col}}{q_{Col}} = \frac{d_0 - c_1}{2}, \tag{11}$$

[4] With decreasing marginal cost, the weight on d_0 may be negative.

[5] The general formulas in the table are transparant because d_0 has the same dimension as c_1, and d and t have the same dimension as c_2. The equivalent formulas that use the parameters of the demand functions instead of those of the inverse demand functions are less transparant.

[6] For the quadratic form $ax^2 + bx + c$ the ratio of the optimum to the argument is $-b/2$.

regardless of the values of d, t and c_2 and of the number of firms I. In fact, this outcome is a special case of a more general result:

The Pareto optimal set of any oligopoly of the class considered in this paper is the set of points where the profit-output ratio of the industry equals $(d_0 - c_1)/2$.

The proof is in the Appendix. The feature sets the true solution apart from the other "solution concepts." It implies that, within the subset of models with the same value of $d_0 - c_1$, the joint profit is largest where the joint output is largest. In Section 4 I shall use the result in the comparison of the outcomes of six model variants.

Table 1: Main outcomes

	General formula	The classic case
Zero-profit point		
– quantity÷k	$\dfrac{2d}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{1}{I}$
– price	$\dfrac{2\big((I-1)d+t)\big)c_1+c_2d_0}{2\big((I-1)d+t)\big)+c_2}$	c_1
Bertrand equilibrium		
– quantity÷k	$\dfrac{d}{(I-1)d+t+\tau_I^{-1}+c_2}$	$\dfrac{1}{I}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+(\tau_I^{-1}+c_2)d_0}{(I-1)d+t+\tau_I^{-1}+c_2}$	c_1
– profit	$\Big(\dfrac{1}{\tau_I}+\dfrac{c_2}{2}\Big)q_B^2$	0
Cournot equilibrium		
– quantity÷k	$\dfrac{d}{(I-1)d+2t+c_2}$	$\dfrac{1}{I+1}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+(t+c_2)d_0}{(I-1)d+2t+c_2}$	$\dfrac{Ic_1+d_0}{I+1}$
– profit	$\Big(t+\dfrac{c_2}{2}\Big)q_C^2$	dq_C^2
Collusion point		
– quantity÷k	$\dfrac{d}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{1}{2I}$
– price	$\dfrac{\big((I-1)d+t\big)c_1+\big((I-1)d+t+c_2\big)d_0}{2\big((I-1)d+t\big)+c_2}$	$\dfrac{c_1+d_0}{2}$
– profit	$\Big((I-1)d+t+\dfrac{c_2}{2}\Big)q_{Col}^2$	Idq_{Col}^2

3. Choice of parameters: the classic case

Numerical examples help to reveal the information that the formulas in Table 1 contain. In this section I discuss the choice of values for the parameters d_0, d and c_1 of the classic case, because it takes a central place. The next section will show the impact of non-constant returns to scale and product differentiation on the outcomes.

Conveniently, choosing values for the parameters can be done in three steps. First, scale the quantities through the choice of k $(:= (d_0 - c_1)/d)$. Second, fix the price at the Collusion point through the choice of $c_1 + d_0$. Third, make an assumption about the elasticity of the inverse demand function at the Collusion point to identify c_1, d_0 and d.

Table 2 gives the unchanging pattern in the outcomes for 1–5 firms. I want to transform the table into a numerical example with certain desirable properties, which make the model suitable for application in the laboratory. One wish is that the Bertrand equilibrium B, Cournot equilibrium C and Collusion point Col be clearly separated; given the fixed ratios between the outcomes, this property can only be obtained by choosing the scale "sufficiently" large. A sufficiently large scale also helps to constrain the relative deviation from the true outcomes introduced by rounding them to the nearest integer, a practice that seems advisable in the laboratory. On the other hand, the figures must not be "unduly" large. It would also be nice for the outcomes to have a somewhat "realistic" flavor.

Table 2: The classic case: pattern in the outcomes

Number of firms (I)	1	2	3	4	5
Quantities$\div k$					
– q_Z	1	1/2	1/3	1/4	1/5
– q_B	1/2	1/2	1/3	1/4	1/5
– q_C	1/2	1/3	1/4	1/5	1/6
– q_{Col}	1/2	1/4	1/6	1/8	1/10
Prices					
– p_Z	c_1	c_1	c_1	c_1	c_1
– p_B	$\frac{c_1+d_0}{2}$	c_1	c_1	c_1	c_1
– p_C	$\frac{c_1+d_0}{2}$	$\frac{2c_1+d_0}{3}$	$\frac{3c_1+d_0}{4}$	$\frac{4c_1+d_0}{5}$	$\frac{5c_1+d_0}{6}$
– p_{Col}	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$	$\frac{c_1+d_0}{2}$

These admittedly vague desiderata leave ample room for other considerations. I have imposed the constraint that the parameters yield only integer outcomes for the cases of 1–5 firms. As to the quantities, it requires k to be a multiple of 120 $(= 2^3 \cdot 3 \cdot 5)$. It does not seem necessary to choose k larger than 120. As to the prices, it requires $c_1 + d_o$ to be a multiple of 60 $(= 2^2 \cdot 3 \cdot 5)$. The value of 60 implies $p_{Col} = 30$. Given the need to choose c_1 well in excess of 0 (to allow for decreasing marginal cost), this value leaves a rather small interval for the prices. So let's double it.

Lastly, the wish for a somewhat "realistic" flavor. Observe that the elasticity of the inverse demand function at the Collusion point, e_{Col}, is given by $-(d_0 - c_1)/(d_0 + c_1)$. A value of c_1 close to d_0 yields a value of e_{Col} close to zero and hence a low markup of price over (marginal) cost, whereas a value of c_1 close to zero yields a value of e_{Col} close to -1 and hence a high markup. Steering away from both extremes, I choose $e_{Col} = -0.5$, which yields a markup of 2 and implies $d_0 = 3c_1$.[7] In this way I arrive at $d_0 = 90$, $c_1 = 30$ and $d = 0.5$. Table 3 gives the outcomes for the quantities and prices with this choice of parameters. Note that the parameter d may be used to scale the quantities. For example, halving the value of d doubles all quantities: the change represents a pure demand shift, with the same relative rise of demand at every price.

[7] A lower value of the markup, for example by choosing $d_0 = 80$ and $c_1 = 40$ (with $d = 1/3$), would be more "realistic," but would shorten the interval for prices more than I like for the purpose of laboratory experiments.

Table 3: The classic case: a numerical example

I	1	2	3	4	5
Quantities					
– q_Z	120	60	40	30	24
– q_B	60	60	40	30	24
– q_C	60	40	30	24	20
– q_{Col}	60	30	20	15	12
Prices					
– p_Z	30	30	30	30	30
– p_B	60	30	30	30	30
– p_C	60	50	45	42	40
– p_{Col}	60	60	60	60	60

4. Six variants

I am now ready to consider model variants with non-constant returns to scale ($c_2 \neq 0$) and/or differentiated goods ($t > d$). It is convenient to choose c_2 and $t - d$ in proportion to d: this practice yields all quantities as fractions of k that are independent of d, so that a change of d still represents a pure demand shift. For c_2 I consider the values of $d/2$, 0 and $-d/2$, for $t - d$ the values of 0 and $d/4$. Table 4 is a survey of the six model variants, or *industries*, and their self-explanatory labels.

One element in the discussion of the outcomes is the comparison within an idustry across the numbers of firms, which have been treated as exogenous so far. Such a comparison naturally leads to the question what the assumption of joint profit maximisation implies. To answer it, the Tables 5–10 contain, next to the prices, the outcomes for total output and total profit of the industries. I shall often use the property of the model that the profit-output ratio of an industry is the same in every point of the Pareto optimal set. To avoid the infinite and infinitesimal I assume that there is some non-zero minimal firm size, and that firms can enter an industry only at the minimal size.

Table 4: Six model variants

Marginal cost	Nature of the good(s)	
	Homogeneous	Differentiated
Increasing	$c_2 = d/2$ $t - d = 0$ Label: HI	$c_2 = d/2$ $t - d = d/4$ Label: DI
Constant	$c_2 = 0$ $t - d = 0$ Label: HC	$c_2 = 0$ $t - d = d/4$ Label: DC
Decreasing	$c_2 = -d/2$ $t - d = 0$ Label: HD	$c_2 = -d/2$ $t - d = d/4$ Label: DD

Let me begin with the outcomes of the classic case, Industry HC, in Table 7. The pattern in the outcomes will be familiar to students of the "oligopoly problem." Two Bertrand oligopolists produce twice the monopoly quantity (a generic outcome of the classic case) at half the price (a consequence of the specific choice of parameters); the price equals marginal cost, and profits are down to zero. A further increase of the number of firms changes neither the total output nor the price. As the number of firms rises, the outcomes of the Cournot equilibrium move gradually from the monopoly outcome towards those of the Bertrand equilibrium and the zero-profit competitive outcome. The increase from one firm to ten firms closes most of the gap between the monopolistic and competitive outcomes;[8] actually, the Cournot equilibrium owes its popularity to this gradual transition, as it agrees with the intuition of many economists. Another way of looking at the same pattern is that the firms in the Cournot oligopoly perceive an incentive to merge or to collude: industry profit rises as the number of firms declines. The true profit maximising strategy for the firms, however, is to charge the monopoly price and to jointly produce the monopoly output. Because in the Pareto optimal set the profit-output ratio of the industry is the same over the whole range of firm sizes and numbers of firms, the joint profit is constant, too: the assumption of joint profit maximisation does not select a specific size distribution of firms. Non-constant returns to scale and/or product differentiation change this outcome, as we are about to see.

Let us next turn to Industry HI in Table 5. When marginal cost is increasing, the firms in the Bertrand oligopoly earn positive profits. The joint profit of two Bertrand oligopolists is substantially below the monopolist's profit, but has not fallen all the way down to zero; further increases of the number of firms drive the joint profit down to zero at a slower pace. Remarkably, the joint profit of two Cournot oligopolists exceeds the monopolist's profit; the cost savings obtained by spreading production over two firms outweigh the depressing effect of the additional firm on the price of the good. From two firms onward industry profit declines towards zero, again at a slower pace than in the Bertrand oligopoly.[9] The behaviour of industry profit in the "collusive" oligopoly is quite different: it moves upwards, back to its level in the classic case, as the number of firms increases. We know from the Appendix that the Pareto optimal set of this industry is concave to the origin. The centroid *Col* is its point where industry output and profit are largest: dividing total demand evenly among the firms is the most profitable arrangement of the industry. Firms that enter the industry wish to stay small, because in that way they avoid the adverse effects of increasing marginal cost. An increase of demand, for example through a drop of the value of d, is most profitably met by the entry of firms, not by the growth of existing firms.

The value of $t-d$ in Industry DC is half the value of c_2 in Industry HI, so that the industries have identical profit functions in quantity space. Therefore a number of rows of Table 8 are identical to the corresponding rows of Table 5, those for U_{Col}, U_C, Q_{Col}, Q_C and Q_Z, to be precise. However, the matching prices do differ, because the industries have different inverse demand functions. The Bertrand equilibrium is at a different point in both quantity space and price space. As to the "collusive" oligopoly, just like firms that enter Industry HI, firms that enter Industry DC wish to stay small, but for a different reason: by producing new varieties in small quantities they avoid the adverse effects of (relatively) fast decreasing marginal revenue.

Industry DI combines increasing marginal cost with "fast" decreasing marginal revenue, which may be typical of many traditional industries. As we have seen, both changes from Industry HC affect the quantities in the same direction. The figures in Table 6 confirm that the deviations of the quantities from their HC-counterparts are similar to, and larger than with one of the changes separately. As a consequence, this observation applies to the profits of the "collusive" oligopoly,

The picture changes drastically when we move on to Industry HD, which produces one homogeneous good using a technology with decreasing marginal cost (see Table 9). Bertrand oligopolists suffer losses as long as their number exceeds one. Merger increases the losses of the industry, unless all firms merge at once into one firm, which then starts acting as a monopolist and makes a large profit. A given number of Cournot oligopolists (more than one) perceive an incentive to merge that is stronger than in any other industry here considered. The firms in the "collusive" oligopoly, too, perceive an incentive to merge, albeit less strongly than the Cournot oligopolists; the reason is that their profit at the (exogenously fixed) initial number of firms is already optimal and exceeds by far the joint profit of the same number of Cournot oligopolists. We know from the Appendix that the Pareto optimal set of this industry is convex to

[8] For all three models in all six industries, the outcomes for the decapoly are close to the limiting values.

[9] Stated differently, industry profit rises faster in the Bertrand oligopoly than in the Cournot oligopoly when the number of firms declines. The difference may be related to the finding in the laboratory that 'Bertrand colludes more than Cournot.' For more on this matter, see Suetens and Potters (2007).

Table 5: Industry HI

I	1	2	3	4	5	10	100
Q_Z	96	107	111	113	114	117	120
Q_B	48	96	103	107	109	114	119
Q_C	48	69	80	87	92	104	118
Q_{Col}	48	53	55	56	57	59	60
p_Z	42	37	35	34	33	31	30
p_B	66	42	39	37	35	33	30
p_C	66	56	50	46	44	38	31
p_{Col}	66	63	62	62	61	61	60
U_B	1440	576	441	356	298	163	18
U_C	1440	1469	1333	1190	1065	681	87
U_{Col}	1440	1600	1662	1694	1714	1756	1796

Table 6: Industry DI

I	1	2	3	4	5	10	100
Q_Z	80	96	103	107	109	114	119
Q_B	40	75	88	95	99	109	119
Q_C	40	60	72	80	86	100	118
Q_{Col}	40	48	51	53	55	57	60
p_Z	40	36	34	33	33	31	30
p_B	65	48	43	40	38	34	30
p_C	65	56	51	48	45	39	31
p_{Col}	65	63	62	62	61	61	60
U_B	1200	984	781	645	548	312	35
U_C	1200	1350	1296	1200	1102	750	104
U_{Col}	1200	1440	1543	1600	1636	1714	1791

Table 7: Industry HC (The classic case)

I	1	2	3	4	5	10	100
Q_Z	120	120	120	120	120	120	120
Q_B	60	120	120	120	120	120	120
Q_C	60	80	90	96	100	109	119
Q_{Col}	60	60	60	60	60	60	60
p_Z	30	30	30	30	30	30	30
p_B	60	30	30	30	30	30	30
p_C	60	50	45	42	40	35	31
p_{Col}	60	60	60	60	60	60	60
U_B	1800	0	0	0	0	0	0
U_C	1800	1600	1350	1152	1000	595	71
U_{Col}	1800	1800	1800	1800	1800	1800	1800

Table 8: Industry DC

I	1	2	3	4	5	10	100
Q_Z	96	107	111	113	114	117	120
Q_B	48	89	100	105	108	114	119
Q_C	48	69	80	87	92	104	118
Q_{Col}	48	53	55	56	57	59	60
p_Z	30	30	30	30	30	30	30
p_B	60	40	36	34	33	32	30
p_C	60	51	47	44	42	37	31
p_{Col}	60	60	60	60	60	60	60
U_B	1440	889	598	449	360	180	18
U_C	1440	1469	1333	1190	1065	681	87
U_{Col}	1440	1600	1662	1694	1714	1756	1796

Table 9: Industry HD

I	1	2	3	4	5	10	100
Q_Z	160	137	131	128	126	123	120
Q_B	80	160	144	137	133	126	121
Q_C	80	96	103	107	109	114	119
Q_{Col}	80	69	65	64	63	62	60
p_Z	10	21	25	26	27	28	30
p_B	50	10	18	21	23	27	30
p_C	50	42	39	37	35	33	30
p_{Col}	50	56	57	58	58	59	60
U_B	2400	-1600	-864	-588	-444	-199	-18
U_C	2400	1728	1322	1067	893	490	53
U_{Col}	2400	2057	1964	1920	1895	1846	1805

Table 10: Industry DD

I	1	2	3	4	5	10	100
Q_Z	120	120	120	120	120	120	120
Q_B	60	109	116	118	119	120	120
Q_C	60	80	90	96	100	109	119
Q_{Col}	60	60	60	60	60	60	60
p_Z	15	23	25	26	27	29	30
p_B	53	29	27	27	28	29	30
p_C	53	45	41	39	38	34	30
p_{Col}	53	56	58	58	59	59	60
U_B	1800	595	248	133	83	19	0
U_C	1800	1600	1350	1152	1000	595	71
U_{Col}	1800	1800	1800	1800	1800	1800	1800

the origin. The centroid *Col* is its point where industry output and profit are smallest. They reach their maximum at any of the monopoly points, because decreasing marginal cost is exploited maximally by concentrating all production in one firm. An increase of demand, for example through a drop of the value of d, is most profitably met by increasing the output of this one firm.

Industry DD, with decreasing marginal cost and product differentiation, may be characteristic of many modern industries. The values of $t-d$ and $c_2/2$ that I have chosen are such that their sum is zero: the profit functions in quantity space are identical to those in Industry HC. I do not repeat here the part of the discussion of Industry DC on this matter. In the "collusive" oligopoly, the amount of profit is constant across the number of firms; once more, joint profit maximisation does not select a specific size distribution of firms.

In five of the six industries, the Bertrand oligopoly and the Cournot oligopoly yield other outcomes for the optimal number of firms than the "collusive" oligopoly does. I leave it to the reader to judge which of the three models agrees best with a cursory observation of the world.

5. Some comments on laboratory experiments

Oligopoly models like the ones studied here are popular tools in the economics laboratory for testing theories of behaviour in situations of few, interacting participants. In contrast to what the use of the term "laboratory" suggests, however, the proceedings in the economics laboratory differ fundamentally from those in the physics laboratory. Whereas in the physics laboratory the participants (for example, elementary particles) "know" the laws of nature and the experimenters are struggling to find out what the laws are, in the economics laboratory the experimenters have set the "laws of nature" and the participants (often undergraduate students) are struggling to find them out. How reasonable is it to expect from amateurs that they are able to grasp, within an hour or so, the mechanics of an artificial world that has taken professional economists almost two centuries to fully understand?

It will surely help to give the amateurs a head start by instructing them extensively, on the model and the means at their disposal to reach good decisions. A concern of the designer of the experiment is to supply the participants with adequate information without unveiling the solution. However, I think that suggesting certain procedures for attacking the problem is quite justified. The procedures need not be sophisticated; after all, Huck et al. (2004) have shown that a simple trial-and-error method often leads to the centroid of the Pareto optimal set. Meanwhile, the designer must beware of leading the endeavors of the participants in a particular direction. This aspect gains weight in connection with another one. Participants in experiments have often been, and in future experiments will be recruited from undergraduate students. Many of them, I suspect, will be economics students, with prior exposure to economic theory and, even worse, maybe also to the fallacy of Nash equilibrium.

Even when prospective participants have received extensive instructions, it may be a good idea to familiarise them as monopolists with the model and the experimental setting. A bad performance of some participant as a monopolist puts into perspective the outcomes of her later plays of games.

My last, but not least important comment on current practice in the economics laboratory is this. In the physics lab, the experimenters create the conditions in which the phenomena predicted by their models are likely to occur. Experimenters in the economics lab, however, have frequently failed to do so. The first instruction that the participants in an oligopoly game receive goes often like this:

> During the experiment you are not allowed to talk to other participants. If something is not clear, please raise your hand and one of us will help you.

Nash (1951) is to blame for this ban on communication. He suggested a new "solution concept" for a game, which would apply when the players of the game were unable to communicate and cooperate. Essentially, Nash replaced a simultaneous maximum problem by a set of conditional maximum problems by postulating that each player conditions on the (*endogeneous*) actions of the other players; when applied to an oligopoly game, the Nash equilibrium is the Bertrand equilibrium or the Cournot equilibrium, depending on whether the firms use the prices or the quantities as instruments. Mathematically, the postulate amounts to ignoring the *non-zero* partial cross derivatives of the profit functions when the first-order conditions for a solution are derived.[10] On top of this obvious mathematical flaw, the

[10] See Nieuwenhuis (2017a) for an extensive treatment of issues concerning the Nash equilibrium.

ban on communication is not in keeping with the specification of the model. Variables corresponding to the actions of communication and cooperation are not present in the model, let alone constraints on such actions. Certain constraints may be lurking in the background, but in the model under empirical scrutiny their Lagrange multipliers are zero. Stated differently, communication and cooperation are free actions in the artificial world of the model. Therefore, the first instruction better be replaced by something like this:

> The experiment consists of a number of plays of a game. During the experiment, you and the other players have access to a chatroom, where you may discuss any issues concerning the plays of the game. However, if you have questions concerning the experimental setting, please raise your hand and one of us will help you.

The objection above is not to deny the interest of experiments in which the participants may not communicate. It merely stresses that the model does not apply to this situation, and that the Nash equilibrium is not a mathematically consistent yardstick to judge the outcomes.

6. Concluding remark

In a review of the experimental literature, Haan et al. (2006) find that *The ability to communicate among sellers has a strong and positive effect on the ability to collude.* The finding is good news for the proponents of the rationality postulate as the starting point of economic theory. In real life, there appears to be more coordination of actions than is compatible with non-cooperative game theory. Because an experimental setting that allows for easy communication is a better approximation of many real-world situations than the alternative, the finding is consistent with this observation. Individually rational decision makers seem to understand well that in many situations they serve their private interests best by acting in unison with others. The theory now known as *cooperative game theory* is the basis of the theory of rational decision making, unqualified by adjectives like "individual" versus "collective," or "non-cooperative" versus "cooperative."

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Appendix A. The Pareto optimal set

The oligopoly problem without the assumption of joint profit maximisation is a vector maximisation problem. Here I present the solution and prove the proposition in Section 2,

The Pareto optimal set of any oligopoly of the class considered in this paper is the set of points where the profit-output ratio of the industry equals $(d_0 - c_1)/2$.

To prepare the way, write the profit functions in quantity space as

$$\upsilon_i = u_i(\mathbf{q}) = (d_0 - c_1 - dQ)q_i - zq_i^2, \qquad i = 1, \ldots, I, \tag{A.1}$$

$$z := t - d + \frac{1}{2}c_2. \tag{A.2}$$

This way of writing them covers the cases of both (non-)constant marginal cost and product differentiation, and any of their combinations. The formulas for q_{Col} and u_{Col} in the new notation are

$$q_{Col} = \frac{d_0 - c_1}{2(Id + z)}, \tag{A.3a}$$

$$u_{Col} = (Id + z)q_{Col}^2. \tag{A.3b}$$

I shall give a proof only for a duopoly, which enables me to use the results of Nieuwenhuis (2017a, Section 4.1) (from which I have also adopted Figure A.1). The method of proof applies to any number of firms. The short proof is followed by a lenghtier exposition of the Pareto optimal set.

One part of the first-order conditions of the problem of maximising I (continuously differentiable) functions of $J \geq I$ continuous variables is that the matrix of first-order derivatives of the functions have deficient row rank. I do not consider the other part of the first-order conditions here, nor the second-order conditions. Let $\mathbf{U}$ be the matrix of first-order derivatives of the profit functions. In the present case the condition amounts to $|\mathbf{U}| = 0$; geometrically, the set of points satisfying the condition is the variety where the iso-profit curves of the firms have a tangent line in common. The non-negative quadrant contains two segments of the variety; the Pareto optimal set is the segment closer to the origin. The monopoly points of the firms and the Collusion point belong to the set; at these points the profit-output ratios of the firms and the industry do indeed equal $(d_0 - c_1)/2$.

I must show that if P is *any* point of the Pareto optimal set, the profit-output ratio of the industry equals $(d_0 - c_1)/2$ at P. To this end, consider the industry's profit along the ray $q_2 = \alpha q_1$. The point on the ray where industry profit is largest is a point of the Pareto optimal set, as the summation vector $[1 \;\; 1]$ is a left eigenvector of $\mathbf{U}$ with eigenvalue 0 and hence $|\mathbf{U}| = 0$ at the point. Substitute $q_2 = \alpha q_1$ in the profit functions of the firms:

$$\upsilon_1 = v_1(q_1, \alpha) := (d_0 - c_1)q_1 - \big((1+\alpha)d + z\big)q_1^2, \tag{A.4a}$$

$$\upsilon_2 = v_2(q_1, \alpha) := \alpha(d_0 - c_1)q_1 - \big((\alpha + \alpha^2)d + \alpha^2 z\big)q_1^2, \tag{A.4b}$$

so that

$$\upsilon_1 + \upsilon_2 =: \Upsilon = V_\alpha(q_1, \alpha) := (1+\alpha)(d_0 - c_1)q_1 - \big((1+\alpha)^2 d + (1+\alpha^2)z\big)q_1^2. \tag{A.4c}$$

The ratio of the maximum to its argument, Υ^*/q_1^*, is $(1+\alpha)(d_0 - c_1)/2$, so that the profit-output ratio of the industry is in fact $(d_0 - c_1)/2$. From the very nature of the problem it is clear that there are no other points along the ray $q_2 = \alpha q_1$ where the profit-output ratio of the industry is $(d_0 - c_1)/2$. End of proof.

To gain a better understanding of the result, rewrite (A.4b) and (A.4c) to

$$w_2(q_1, \alpha) := (d_0 - c_1)q_1 - \big((1+\alpha)d + \alpha z\big)q_1^2, \qquad (\text{for } \alpha > 0) \tag{A.4b'}$$

$$W_\alpha(q_1, \alpha) := (d_0 - c_1)q_1 - \Big((1+\alpha)d + \frac{1+\alpha^2}{1+\alpha}z\Big)q_1^2. \tag{A.4c'}$$

(A.4a), (A.4b') and (A.4c') are quadratic functions of the form $ax^2 + bx$ with a common value of b but different values of a. The value of a in the third function is a weighted average of its values in the first two functions, with weights of $1/(1+\alpha)$ and $\alpha/(1+\alpha)$, respectively. Therefore the point on the ray where industry profit is maximal is generally in between the points where the profits of the firms reach their maxima. As the ray $q_2 = \alpha q_1$ revolves around the origin from the q_1-axis ($\alpha = 0$) towards the q_2-axis ($\alpha = \infty$), the weight of Firm 1's profit function declines from 1 to 0. Simultaneously, the intersection of the ray with the zero-profit line of Firm 1, $Z_{1,\alpha}$, moves from the point $Z_1 = Z_{1,0} := [(d_0 - c_1)/(d+z), 0)]$ towards the point $B_1 = Z_{1,\infty} := [0, k]$, where $k := (d_0 - c_1)/d$. Parallel to this line (segment), at half the distance from the origin, is the maximal-profit line of Firm 1; along the line, profit declines linearly from the monopoly profit at $C_1 = P_{1,0} := [(d_0 - c_1)/2(d+z), 0]$ towards zero at $P_{1,\infty} := [0, k/2]$. The maximal-profit line of Firm 2 is the mirror image of the one of Firm 1 with respect to the ray $q_1 = q_2$.

In the classic case, that is when $z = 0$, the maximal-profit lines coincide, and the line segment constitutes the Pareto optimal set; along the line segment, industry output and profit are constant. Only in this case is the ray $q_2 = \alpha q_1$, tangent to an iso-profit curve of the industry, also the common tangent line of the iso-profit curves of the firms at every point of the Pareto optimal set.

In the general case of $z \neq 0$, the maximal-profit lines intersect at *Col*. The industry has a maximal-profit curve, the Pareto optimal set, that runs from C_1 through *Col* to C_2, the mirror image of C_1 with respect to $q_1 = q_2$; the intermediate segments of the curve are in between the maximal-profit lines of the firms. Because the profits of the firms are generally not maximal at the same point of $q_2 = \alpha q_1$, the ray is generally *not* the common tangent line.

It remains to see how industry output and profit evolve along the curve. Compare the output of a firm at *Col*,

$$q_{Col} = \frac{d_0 - c_1}{2(2d+z)}, \tag{A.3a'}$$

to the output of a firm halfway between C_1 and C_2 , which equals half the monopolist's output:

$$q_{mono}/2 = \frac{d_0 - c_1}{4(d+z)}. \tag{A.5}$$

If $z > 0$, then $q_{Col} > q_{mono}/2$, so that the curve is concave to the origin; it proves to be a segment of a hyperbola. *Col* is the point of the curve where industry output and profit are largest. If $z < 0$, then $q_{Col} < q_{mono}/2$, so that the curve is convex to the origin; it proves to be a segment of an ellipse. *Col* is the point of the curve where industry output and profit are smallest, they are largest at any of the monopoly points.

If the profit-output ratios of all firms equal $(d_0 - c_1)/2$, the matrix $\mathbf{U}$ has a special form. For $q_i \neq 0$, the profit-output ratio of Firm i is

$$\pi_i := \frac{u_i(\mathbf{q})}{q_i} = d_0 - c_1 - dQ - zq_i, \qquad i = 1, \ldots, I. \tag{A.6}$$

Then

$$\pi_i = \frac{d_0 - c_1}{2} \iff d_0 - c_1 = 2(dQ + zq_i), \qquad i = 1, \ldots, I. \tag{A.7}$$

The first-order derivatives of the profit functions are $\partial u_i/\partial q_j =: u_{ij} = -dq_i, j = 1, \ldots, I,\ i \neq j$, and

$$\frac{\partial u_i}{\partial q_i} =: u_{ii} = d_0 - c_1 - dQ - (2z+d)q_i, \qquad i = 1, \ldots, I. \tag{A.8}$$

From (A.7) and (A.8),

$$\pi_i = \frac{d_0 - c_1}{2} \iff u_{ii} = d(Q - q_i), \qquad i = 1, \ldots, I. \tag{A.9}$$

Substitute this expression in the matrix $\mathbf{U}$. For the triopoly (just as an example) the outcome is

$$\mathbf{U} = d \begin{bmatrix} Q - q_1 & -q_1 & -q_1 \\ -q_2 & Q - q_2 & -q_2 \\ -q_3 & -q_3 & Q - q_3 \end{bmatrix}. \tag{A.10}$$

Adding the second and third row to the first one yields a row of zeros, so that $|\mathbf{U}| = 0$.

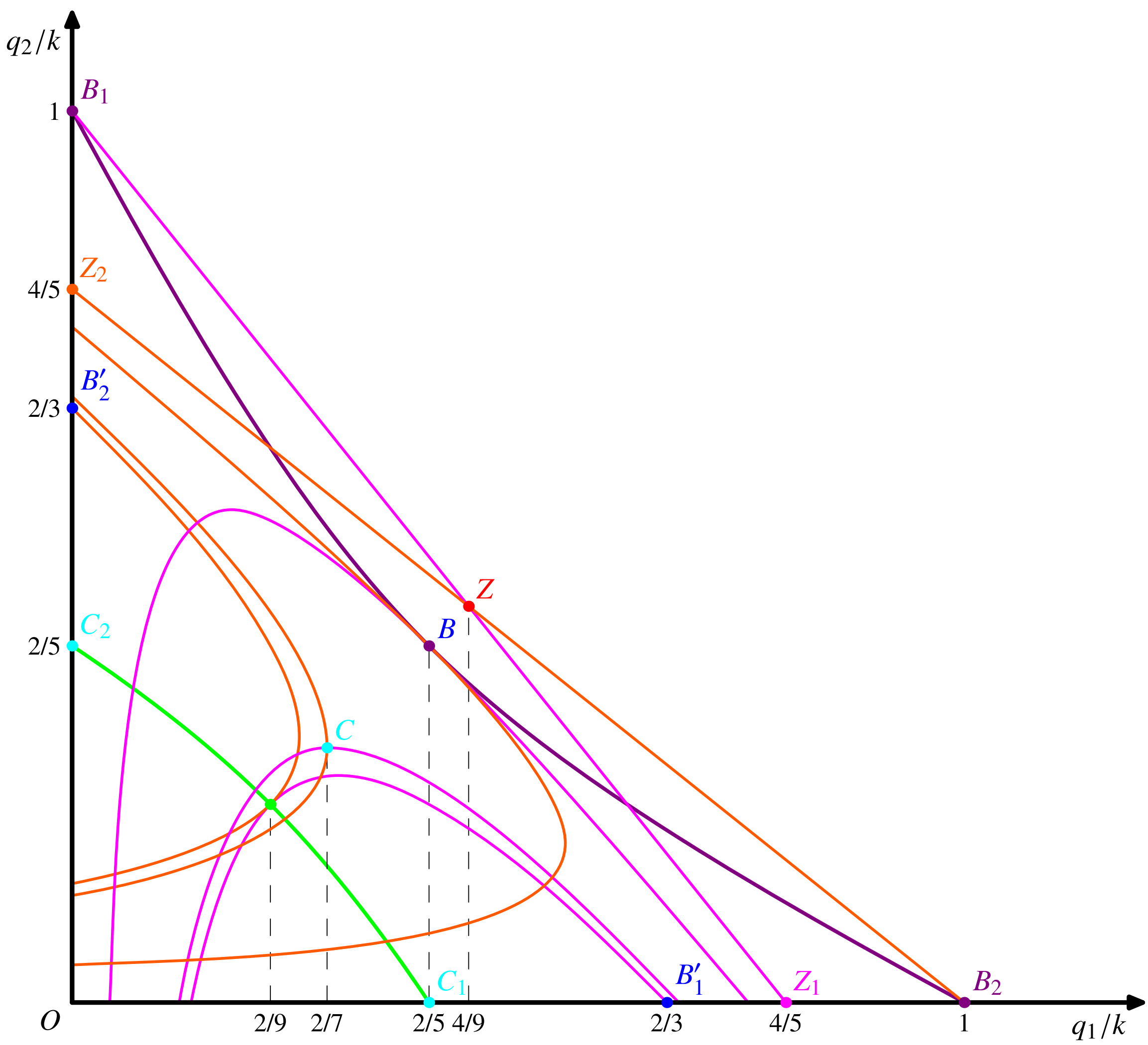

Figure A.1: Solution and "equilibria" when $z = c_2/2 = d/4$ (homogeneous product and increasing marginal cost)

Notes: The curve through B_1, B and B_2 solves $|U| = 0$, but is not part of the Pareto optimum. The curve through C_1 and C_2 solves $|U| = 0$, and is the Pareto optimum; its midpoint is the Collusive equilibrium. $[B_iZ_i]$ is the Zero-profit line of Firm *i*. Z is the Zero-profit point, $B_i{}'$ the end point of Firm i's Bertrand reaction function, C_i the end point of Firm i's Cournot reaction function (also Firm i's monopoly point), B the Bertrand equilibrium and C the Cournot equilibrium.
Source: Nieuwenhuis (2017a, Figure 4).

What does the Pareto optimal set look like when the number of firms exceeds two? In a triopoly, there are three monopoly points and three curves like C_1C_2 connecting them. The Pareto optimal set is the surface area, topologically a triangle, enclosed by the curves. The surface area is convex to the origin, flat, or concave to the origin for $z < 0$, $z = 0$, or $z > 0$, respectively. For a tetrapoly, the Pareto optimal set is a volume, topologically a tetrahedron, the four faces of which are the "triangles" of the included triopolies. And so on, beyond graphical representation, for still larger numbers of firms. We have a perfect example of a result, for the first time stated and proved by De Finetti (2017, Section 12),

The locus of "optimum" points with respect to n *functions is, topologically, a simplex of* $\mathrm{n}-1$ *dimensions, the* n *faces of which are the loci of "optimum" with respect to* $\mathrm{n}-1$ *<of the> functions, the* $\binom{\mathrm{n}}{2}$ *edges of which those for* $\mathrm{n}-2$ *<of the> functions, and so on, up to the* n *vertices, "optimum" points with respect to the* n *functions separately.*

Here, the "locus of "optimum" points" is what we call the Pareto optimal set nowadays.