Irrationality Proofs: From e to $\zeta(n \geq 2)$

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Abstract

We develop definitions and a theory for convergent series that have terms of the form $1/a_j$ where a_j is an integer greater than one and the series convergence point is less than one. These series have terms with denominators that can be used as number bases. The series for $e-2$ and $z_n = \zeta(n) - 1$ are of this type. Further, both series yield number bases that can represent all possible rational convergence points as single digits. As partials for these series are rational numbers, all partials can be given as single decimals using some a_j as a base. In the case of $e-2$, the last term of a partial yields such a base and partials form systems of nesting inequalities yielding a proof of the irrationality of $e-2$. In the case of z_n, using the z_2 case we determine that such systems of nesting inequalities are not formed, but we discover partials require bases greater than the denominator of their last term. We prove this for the general z_n and using it we give a proof of the irrationality of all z_n.

1 Introduction

Apery's $\zeta(3)$ is irrational proof [1] and its simplifications [3, 11] are the only proofs that a specific odd argument for $\zeta(n)$ is irrational. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs; see Poorten [12] for the history of Apery's proof and Havil [8] for an approachable introduction to Apery's original proof. Beukers's proofs replace Apery's mysterious recursive relationships with multiple integrals; see Huylebrouck [9] for an historical context for Beukers's proofs. Papers by Poorten and Beukers are in *Pi: A Source Book* [4] and *The Number* π [6] gives Beukers's proofs (condensed) and related material. Both the proofs of Apery and Beukers require the prime number theorem and subtle $\epsilon - \delta$ reasoning.

Thus we have the irrationality of all evens immediately proven irrational using a classic formula and exactly one odd; whereas, you would think that both evens and odds could be proven in the same way. Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. Apery's and other ideas can be seen in the long and difficult results of Rivoal and Zudilin [13, 15]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational do suggest a radically different approach is necessary.

In this paper we explore a different direction. We claim all $\zeta(n \geq 2)$ can be proven to be irrational by using what we call decimal sets and well known and relatively simple properties of decimal bases: [7, Chapter 9]. We still need the lesser cousin of the prime number theorem, Bertrand's postulate [5, 7], and some new, but relatively straight forward epsilon reasoning.

2 Decimals and series

We give definitions that make a connection between certain convergent infinite series and number bases. Here is the idea. Every partial sum for series of fractions is a rational number. We use the symbol $.(p)_q^k$ to designate that the partial with upper index k has for its first digit in base q the number (symbol) p. We use denominators of the fractions of the series as a source for number bases. For some k, this digit becomes fixed; we designate this with $.(p)_q^{k+}$. If a partial is not equal to a single digit in the number base q, it is between two such numbers. As all rational numbers in $(0, 1)$, $\mathbb{Q}(0, 1)$, can be represented as single decimal digits, if we can show a series convergent point is not equal to any such single digit, then we will have shown it is irrational.

Definition 1. A *plus one* series is a convergent infinite series with a convergent point less than one and terms of the form $1/a_j$ with a_j a strictly increasing sequence of integers all greater than one. Partials for such sums are given by s_k where k is the upper index; the infinite series convergent point is given by s.

Definition 2. A plus one series with denominators a_j is said to be complete if

$$B\{a_j\}_{j=1}^{\infty} = \mathbb{Q}(0,1),$$

where $B\{a_j\}_{j=1}^{\infty}$ is the union of all single decimal numbers formed with a_j as number bases.

As the partial sums of a plus-one series are all rational, a complete plus-one series must have partials that can be given as a single digit decimal using some term's denominator. The question is which denominator is used.

Definition 3. A plus-one series having partials s_k is said to be weak (k-less), flat (k-equal), or strong (k-greater) if s_k can be represented as a single decimal in a smallest base a_r where $r < k$, $r = k$, or $r > k$, respectively. If no such a_r exists the series is termed non-existent (k-null).

Partials of a complete plus-one series will always be k-less, k-equal, or k-greater. Incomplete series might be k-null.

Finally, plus-one series are convergent series. They can converge to a rational or irrational number in $(0, 1)$.

Definition 4. A plus-one series with convergent point s is said to be k-plus if there exists a smallest base a_r that can represent s as a single decimal. Such series are said to be k-null if no such a_r exists.

Theorem 1. *A complete k-null plus-one series converges to an irrational number.*

Proof. Suppose such a series converges to a rational number. Then that rational number can be represented in some base a_r as a single decimal digit. But a k-null series has no such a_r, a contradiction. □

Examples

Table 1 gives examples of partial sums for various plus one series. The telescoping series referenced is

$$\sum_{j=2}^{k} \frac{1}{j} - \frac{1}{j+1} = \sum_{j=2}^{k} \frac{1}{j(j+1)}.$$

This series converges to 1/2. It is easy to show that it is a complete, k-less series. The base 10 series referenced are geometric series given by repeating decimals. The first $.\overline{1}$ base 10 is such that its partials can be represented by powers of 10 given by their last term; this series converges to 1/9, a number that can't be represented as a single decimal in base 10; it is, then, k-null, as indicated in Table 2.. The second $.2\overline{9}$ has partials that can be represented by powers of ten given by their last term; this series converges to .3, a number represented by its first and all other terms. Both of these plus-one series are incomplete: using denominators of their terms as bases one can't represent $\mathbb{Q}(0,1)$ as single decimals. One can only represent finite decimals in base 10, 2 and 5.

Partial Sums	k-less	k-equal	k-greater
Incomplete		$.\overline{1}$, $.2\overline{9}$ base 10	
Complete	Telescoping	$e-2$	z_n

Table 1: Example series with decimal properties.

The number $e-2$, its infinite series, is a plus-one, k-equal, complete series. We will show these properties in the next section and use them to give a proof of the irrationality of this series.

Infinite series	k-plus	k-null
Incomplete	$.2\overline{9}$	$.\overline{1}$
Complete	Telescoping	$e-2$, z_n?

Table 2: Correlation between series properties and rational and irrational convergence points.

Properties and irrationality of $e-2$

Consider the series

$$e-2=\sum_{j=2}^{\infty}\frac{1}{j!}=\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots. \tag{2}$$

As $2 < e < 3$, this series is a one-plus series. We will show that it is a complete, k-equal series.

Lemma 1. *The series* (2) *is complete.*

Proof. We simply note

$$\frac{p(q-1)!}{q!} = \frac{p}{q} = .(p(q-1)!)_{q!}.$$

The decimal is a single decimal in base $q!$ as $p < q$ implies $p(q-1)! < q!$. □

Lemma 2. *The series* (2) *is k-equal.*

Proof. We need to show that if

$$s_k = \sum_{j=2}^{k} \frac{1}{j!},$$

then $s_k = .(x)_{k!}$. That is partials can be expressed as single decimals using the denominator of the last term in the partial as a number basis.

As $k!$ is a common denominator of all terms in this partial sum, $s_k = .(x)_{k!}$, for some x, $1 \leq x < k!$. The following induction argument shows that $k!$ is the least such factorial possible.

Clearly $2!$ is the least such factorial for the first partial. Suppose $k!$ is the least factorial for the kth partial. Let

$$s_{k+1} = \frac{x}{k!} + \frac{1}{(k+1)!} = \frac{y}{a!} \tag{3}$$

for some positive integers a and y. If $a \leq k$, then multiplying (3) by $k!$ gives an integer plus $1/(k+1)$ is an integer, a contradiction. So $a > k$, but $a = k+1$ works, so it is the least possible factorial. □

Lemma 3. *For each integer $k > 1$, there exists decimal digits x and $x+1$ base $k!$ such that*

$$.(x)_{k!} < e - 2 < .(x+1)_{k!}. \tag{4}$$

Proof. By Lemma 2, $s_k = .(x)_{k!}$. We have, using a geometric series,

$$0 < (e-2) - s_k = \sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} + \dots\right)$$

$$< \frac{1}{k!}\left(\frac{1}{(k+1)}+\frac{1}{(k+1)^2}+\dots\right)=\frac{1}{k}\frac{1}{k!}<\frac{1}{k!}.$$

That is $0 < e-2-.(x)_{k!} < 1/k!$. Adding $.(x)_{k!}$ and noting $.(x)_{k!}+1/k! = .(x+1)_{k!}$, we have (4). □

Lemma 3 implies the boundary decimals don't change with increasing partial upper index. In the case of $e-2$ nested intervals are formed. Recalling our superscript conventions, here are some examples:

$$.(1)_2^{1+} < e-2 < (1)_2^{1+}, \tag{5}$$

$$.(1)_2^{1+} < .(4)_6^{2+} < e-2 < .(5)_6^{2+} < (1)_2^{1+}, \tag{6}$$

and

$$.(1)_2^{1+} < .(4)_6^{2+} < .(17)_{24}^{3+} < e-2 < .(18)_{24}^{3+} < .(5)_6^{2+} < (1)_2^{1+}. \tag{7}$$

Theorem 2. *$e-2$ is irrational.*

Proof. Suppose $e-2$ is rational, then by Lemma 1 there exists a k such that $e-2=.(x)_{k!}$, but by Lemma 3 for some y

$$.(1)_2^{1+} < \dots < .(y)_{k!}^{(k-1)+} < e-2 = .(x)_{k!} < .(y+1)_{k!}^{(k-1)+} < \dots < (1)_2^{1+}, \tag{8}$$

but no single digit in base $k!$ can be between two other single digits in the same base, a contradiction. □

The series z_2 appears k-greater

We use the following symbols:

$$z_n = \zeta(n)-1 = \sum_{j=2}^{\infty}\frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k}\frac{1}{j^n}.$$

In this section we will use z_2 in hopes a finding a general pattern.

As with the series for $e-2$, we can form systems of inequalities for z_2. With upper index 3 we derive inequalities for bases 4 and 9:

$$(.1)_4^3 < (.3)_9^3 < s_3^2 = .(13)_{36}^3 < .(4)_9^3 < .(2)_4^3. \tag{9}$$

For upper index 4, we derive another set of inequalities:

$$.(1)_4^4 < .(3)_9^4 < .(6)_{16}^4 < s_4^2 = .(61)_{144} < .(7)_{16}^4 < .(4)_9^4 < .(2)_4^4. \tag{10}$$

Unlike the $e-2$ case, single fixed digits are not immediately created with each increment of the upper index. The inequalities don't immediately nest. Continuing with just the bases 4, 9, and 16, we observe

$$.(1)_4^5 < .(7)_{16}^5 < .(4)_9^5 < s_5^2 = .(1669)_{3600} < .(8)_{16}^5 = .(2)_4^5 < .(5)_9^5. \tag{11}$$

Base 16 and base 9 have been transposed and, on the right, base 16 and base 4 endpoints collide (i.e. are equal). The next two iterations are

$$.(1)_4^6 < .(7)_{16}^6 < .(4)_9^6 < s_6^2 = .(1769)_{3600} < .(8)_{16}^6 = .(2)_4^6 < .(5)_9^6 \tag{12}$$

and

$$.(4)_9^7 < .(8)_{16}^7 = .(2)_4^{7+} < s_7^2 = .(90281)_{176400} < .(5)_9^7 < .(9)_{16}^7 < .(3)_4^{7+}. \tag{13}$$

The left and right digits for base 4 have migrated to $.(2)_4$ and $.(3)_4$. As $.(2)_4 < z_2 < .(3)_4$, these left and right values for base 4 are fixed for $k \geq 7$. The decimal digit for this base is fixed, as indicated by the plus sign in the superscript for this base. The inequalities don't nest immediately and the nesting that does form changes.

But we do see a pattern of interest in these inequalities: this z_2 series seems to be, as indicated in Table 1, a k-greater series. We will show z_n (and z_2) has this property in Corollary 1, nota bene general n. We will also show z_n is complete in Lemma 4. These properties will enable us to give a proof that z_n (both odd and even n) have irrational convergence points.

The irrationality of z_n

First two definitions.

Definition 5. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n-1)/j^n\} = \{.1, \ldots, .(j^n-1)\} \text{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 in base j^n. The decimal set for j^n is

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 6.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

We next show this union of decimal sets give all rational numbers in $(0, 1)$.

Lemma 4. *The series z_n is complete.*

Proof. Every rational $a/b \in (0, 1)$ is included in a D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0, 1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

k	s_k^2	Prime factorization
3	$.(13)_{36}$	$36 = 2^2 3^2$
4	$.(61)_{144}$	$144 = 2^4 3^2$
5	$.(1669)_{3600}$	$3600 = 2^4 3^2 5^2$
6	$.(1769)_{3600}$	$3600 = 2^4 3^2 5^2$
7	$.(90281)_{176400}$	$176400 = 2^4 3^2 5^2 7^2$

Table 3: The reduced fractions (given as decimals) have denominators (basis) divisible by powers of 2 and a prime greater than $k/2$.

Next we will show z_n is k-greater. We use, once again, the z_2 case (with partials s_k^2) to look for helpful patterns. Table 3 gives some evidence that the reduced fractions giving partial sum totals have much larger denominators than the denominators of their last term: $36 > 3^2$; $144 > 4^2$; $3600 > 5^2$; $3600 > 6^2$; $176400 > 7^2$. We saw this earlier: (9), (10), (11), (12), and (13). Table 3 also suggests a strategy for proving this. If we can show partial sums of z_n are divisible by powers of 2 and some relatively large prime, as twice something greater than half is bigger than the whole, that would do it. Apostol's *Introduction to Analytic Number Theory* (Chapter 2, problem 21), solution in [10], gives the general technique used in this section.

Lemma 5. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{14}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (14) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 6. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 5. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{15}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 7. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate. □

Theorem 3. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$, that is z_n is k-greater.*

Proof. Using Lemma 7, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 7, we have assurance of the existence of a p that satisfies Lemma 6. Using Lemmas 5, 6, and 7 we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1.

$$s_k^n \notin \Xi_k^n \text{ or } s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$$

where $\mathbb{R}(0,1)$ is the set of real numbers in $(0,1)$.

Proof. This is a restatement of Theorem 3. □

Progress has been made. Consider the following heuristic. Using Lemma 4,

$$\lim_{k\to\infty} \Xi_k^n = \mathbb{Q}(0,1),$$

with Corollary 1 we have

$$\lim_{k\to\infty} \mathbb{R}(0,1) \setminus \Xi_k^n = \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1), \tag{16}$$

where $\mathbb{H}(0,1)$ is the set of irrational numbers in $(0,1)$.

We have then

$$\lim_{k\to\infty} s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n \implies z_n \in \mathbb{H}(0,1),$$

using $s_k^n \to z_n$ with (16). That is z_n is irrational.

To prove the irrationality of z_n, we characterize series that converge to rational (and irrational) numbers. This is the apparently new epsilon reasoning mentioned in the introduction. First a definition.

Definition 7. *Let $D_{j^n}^{\epsilon_j}$ be the set of all D_{j^n} decimal sets having an element within ϵ_j of s_j^n.*

Lemma 8. *If for every monotonically decreasing sequence ϵ_j such that*

$$\lim_{j\to\infty} \epsilon_j = 0,$$

we have

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset, \tag{17}$$

then z_n is irrational

Proof. We use proof by contraposition: $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$. Suppose z_n is rational then, using Lemma 4, $z_n \in D_{j^n}^*$. Define

$$\epsilon_j^* = z_n - s_j^n \text{ for } j \geq 2$$

and set

$$\epsilon_j = 2\epsilon_j^*.$$

Then

$$D_{j^n}^* \subset \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j},$$

so the intersection is not empty. □

The next lemma says that if a point is not a single decimal in base b then it is inside an interval between single decimals; hence, it is trapped within $1/b$ of these single decimal endpoints. This follows as decimal sets partition $(0,1)$ with intervals with widths equal to $1/b$.

Lemma 9. *If $.(a)_b \in (0,1)$ and $.(a)_b \notin D_{j^n}$ then there exists $x \in D_{j^n}$ such that*

$$.(a)_b \in (.(x-1), .(x))_{j^n},$$

where $(.(x-1), .(x))_{j^n}$ is an open set with end points $.(x-1)_{j^n}$ and $.(x)_{j^n}$. Further for any given $\epsilon > 0$,

$$|.(a)_b - .(x-1)_{j^n}| < \frac{1}{j^n} < \epsilon, \tag{18}$$

for large enough j.

Proof. D_{j^n} partitions the interval $(0,1)$ forcing $.(a)_b$ into such an interval. The distance between endpoints in such an open interval is $1/j^n$, so anything inside the interval is less than $1/j^n$ to an endpoint.

The right hand inequality in (18) follows from the Archimedean property of the reals [14]. □

Lemma 10. *For z_n there exists a sequence ϵ_j such that*

$$\bigcap_{j=2}^{\infty} D_j^{\epsilon_j} = \emptyset.$$

Proof. We construct a sequence ϵ_j that cumulatively excludes all possible rational convergence points. Let

$$\epsilon_j^* = \min\{|x - s_j^n| : x \in \Xi_j^n \}.$$

We know by Corollary 1 that $\epsilon_j^* > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon_3^*$. This excludes the decimal sets of Ξ_3^n at this our first iteration. Assume we can generally do this for the jth iteration. For the $j+1$st iteration, using Lemma 9, there exists a base in Ξ_{j+r}^n, for some r such that $\epsilon_{j+r}^* < \epsilon_j/2$. Set $\epsilon_{j+1} = \epsilon_{j+r}^*$. The procedure gives ϵ values that cumulatively exclude ever more decimal sets from $D_{j^n}^{\epsilon_j}$. Regroup the series. By Lemma 4, the exclusions are exhaustive, so

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset,$$

as needed. □

Theorem 4. *z_n is irrational.*

Proof. Let the sequence given in Lemma 10 be given by ϵ_{j_1} and let a general sequence needed for Lemma 8 be given by ϵ_j. Suppose

$$\frac{p}{q} \in \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j}. \tag{19}$$

That is suppose the intersection in (19) is not empty. As $\epsilon_{j_1} \to 0$ and $\epsilon_j \to 0$, for any fixed ϵ_{j_1} that excludes p/q there will be an ϵ_j such that $\epsilon_j < \epsilon_{j_1}$. This implies that p/q will be excluded using ϵ_j, contradicting (19). □

3 Conclusion

How does this proof compare to the work of Beukers? Why do we get a general result here and not with his techniques?

Beukers uses double integrals that evaluate to numbers involving partials for $\zeta(2)$. He uses

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dxdy = \text{various expressions related to } \zeta(2)$$

and uses this to calculate

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\, dxdy,$$

where $P_n(x)$ is the nth derivative of an integral polynomial.

These calculations yield integers A_n and B_n in

$$0 < |A_n + B_n \zeta(2)| d_n^2 < \left\{ \frac{\sqrt{5}-1}{2} \right\}^{5n} \zeta(2) < \left\{ \frac{5}{6} \right\}^n, \tag{20}$$

where d_n designates the least common multiple of the set of integers $\{1, \ldots, n\}$. This last, assuming $\zeta(2)$ is rational, forces an integer between 0 and 1, giving a contradiction. An upper limit for d_n requires the prime number theorem.

These themes repeat for $\zeta(3)$ with the complexity of the expressions at least doubling.

We don't use integrals to generate in effect an interval, a trap, like (20), but the relationships between terms and partials to generate partitions of $(0, 1)$ narrowing and leaving only irrational numbers. We use inherent and simple properties of z_n's partials and terms, Corollary 1, to avoid intractable complexity.

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