Irrationality Proofs: From e to $\zeta(n \geq 2)$

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Abstract

We first give a proof that e is irrational. The proof uses the denominators of the terms of the series $e - 2$ as decimal bases. All rational numbers in $(0, 1)$ can be represented as single decimals digits using these bases. We prove that partials sums for this series require the largest denominator in their terms – the last term. This allows systems of inequalities to be formed that eliminate ever more possible rational convergence points. In the limit all possible rational convergence points are eliminated and e is proven to be irrational. We next observe that the denominators of $\zeta(n) - 1 = z_n$ can be used as number bases and that these number bases cover with single digit decimals all possible rational convergence points, just like the case of $e - 2$. We next prove that partial sums of these series can't be expressed using single digit decimals using as bases the denominators of their terms: the partials escape their terms completely, unlike $e - 2$ with its partials expressible with single digits from the denominator of its last term. This is suggestive that z_n can be proven to be irrational. Finally, we show that this covering and escaping quality of z_n yields a proof that all z_n are irrational.

1 Introduction

Apery's $\zeta(3)$ proof and its simplifications are the only proofs that a specific odd argument for $\zeta(n)$ is irrational [1, 4, 6, 9, 11]. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2]:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs [3] a lot; see Poorten [12] for the history of Apery's proof and Havil [8] for an approachable introduction to Apery's original proof. Beukers's proofs replace Apery's mysterious recursive relationships with multiple integrals and are easier to understand; see Huylebrouck [9] for an historical context for Beukers's proofs. Papers by Poorten and Beukers are in *Pi: A Source Book* [4] and Eymard and LaFon *The Number* π [6] gives Beukers's proofs and related material. Both the proofs of Apery and Beukers require the prime number theorem and subtle $\epsilon - \delta$ reasoning.

Thus we have the irrationality of all evens immediate from a classic formula and exactly one odd proven to be irrational; whereas you would think that both evens and odds could be proven in the same way. Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. It is not for a lack of trying. Apery's and other ideas can be seen in the long and difficult results of Rivoal and Zudilin [13, 16]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational, seem less than encouraging.

In this paper we explore a different direction. We claim all $\zeta(n \geq 2)$ can be proven to be irrational by using what we call decimal sets and well known and relatively simple properties of decimal bases: [7, Chapter 9]. We still need the lesser cousin of the prime number theorem, Bertrand's postulate, and some new, but relatively straight forward epsilon reasoning.

2 Motivation

As the use of decimals in irrationality proofs is new, we first motivate the ideas. We show how using decimals to prove e is irrational suggests that $\zeta(n)$ should be irrational too.

The case of e

Every fraction a/b can be given as a decimal $.(a)$ base b where a is a symbol in base b. We will use $.(a)_b$ to designate this. So, for example, $1/2 + 1/6 = 4/6 = .(4)_6$. This reduces to $.(2)_3$, but for our purposes we want to limit bases to the form $k!$. As $3! = 6$, this sum is given within this constraint.

Our concern is to prove

$$e - 2 = \sum_{j=2}^{\infty} \frac{1}{j!} = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots$$

is irrational. This is just e minus the first two terms, so if $e - 2$ is proven to be irrational, e will be too.

We first show that all rational numbers in $(0, 1)$ can be expressed as single digits in base $k!$.

Lemma 1. *Every rational $p/q \in (0, 1)$ can be expressed as a single digit in some base $k!$.*

Proof. Let $k = q$ and note

$$\frac{p(q-1)!}{q!} = \frac{p}{q} = .(p(q-1)!)_{q!}.$$

The decimal is a single decimal in base $q!$ as $p < q$ implies $p(q-1)! < q!$. $\square$

As $e - 2 < 1$, terms of $e - 2$ *cover* rational possible convergence points. As all partial sums of $e - 2$ are themselves rational numbers, it is of interest to know the relationship between the terms of $e - 2$ and their partials. In particular, can partials be expressed with single decimal digits in the number bases given by the denominators of the partials terms? In the case of $e - 2$ the last term can express partials. We show this next.

Lemma 2. *Let*

$$s_k = \sum_{j=2}^{k} \frac{1}{j!},$$

then $s_k = .(x)_{k!}$, for some $1 \leq x < k!$ and $k!$ is the least factorial.

Proof. As $k!$ is a common denominator of all terms in s_k, s_k can be expressed as a fraction having this denominator; that is there exists some integer x, $1 \leq x < k!$. The following induction argument shows that $k!$ is the least factorial possible.

Clearly $2!$ works for the first partial. Suppose $k!$ works for the kth partial. So the $k + 1$ partial can be expressed with

$$\frac{x}{k!} + \frac{1}{(k+1)!} = \frac{y}{a!} \tag{2}$$

for some positive integers a and y. If $a \leq k$, then multiplying (2) by $k!$ gives and integer plus $1/(k+1)$ is an integer, a contradiction. So $a > k$, but $a = k+1$ works (it's a common denominator), so it is the least possible factorial. ☐

Each partial is represented by a single decimal digit. This implies that each partial has a tail and that tail must be given by additional digits. Lemma 3 shows that these tails are trapped in between two single decimal digits in base $k!$.

Lemma 3.

$$s_k < s_k + \sum_{j=k+1}^{\infty} \frac{1}{j!} = e - 2 < s_k + \frac{1}{k!}. \tag{3}$$

Proof. Using the geometric series, we have

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)(k+2)} + \dots\right)$$

$$< \frac{1}{k!}\left(\frac{1}{(k+1)} + \frac{1}{(k+1)^2} + \dots\right) = \frac{1}{k}\frac{1}{k!}.$$

So

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} < \frac{1}{k}\frac{1}{k!} < \frac{1}{k!}$$

and (3) follows. ☐

Lemma 3 implies the x decimal in $.(x)_y^z$ doesn't change with increasing upper index, z, of the partial; $z = k$, where k is the upper index of the partial. The symbol $.(x)_y^{z+}$ means the digit x in base y doesn't change for all partials with upper index z and greater.

Theorem 1. *$e - 2$ is irrational.*

Proof. Using Lemma 3, all partials, given by dots, are trapped between 1/2 and $1/2 + 1/2 = 1$:

$$.(1)_2^{1+} < \cdots < (1)_2^{1+}. \tag{4}$$

Incrementing the upper index we get tighter and tighter traps for $e - 2$:

$$.(1)_2^{1+} < .(4)_6^{2+} < \cdots < .(5)_6^{2+} < (1)_2^{1+}; \tag{5}$$

and

$$.(1)_2^{1+} < .(4)_6^{2+} < .(17)_{24}^{3+} < \cdots < .(18)_{24}^{3+} < .(5)_6^{2+} < (1)_2^{1+}. \qquad (6)$$

Suppose $e-2$ is rational, then by Lemma 1 there exists a k such that $e-2 = .(x)_{k!}$, but for some y we must have

$$.(1)_2^{1+} < \cdots < .(y)_{k!}^{(k-1)+} < e-2 = .(x)_{k!} < .(y+1)_{k!}^{(k-1)+} < \cdots < (1)_2^{1+} \quad (7)$$

and no single digit in base $k!$ can be between two other single digits in the same base, a contradiction. □

The case of $\zeta(n)$

We use the following symbols:

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

Using just developed for $e-2$, we can form systems of inequalities for each upper index of z_2. With upper index 3 we derive inequalities for bases 4 and 9:

$$(.1)_4^3 < (.3)_9^3 < s_3^2 = .(13)_{36}^3 < .(4)_9^3 < .(2)_4^3. \qquad (8)$$

For upper index 4, we derive another set of inequalities:

$$.(1)_4^4 < .(3)_9^4 < .(6)_{16}^4 < s_4^2 = .(61)_{144} < .(7)_{16}^4 < .(4)_9^4 < .(2)_4^4. \qquad (9)$$

Unlike the $e-2$ case, single fixed digits are not created.

The inequalities in (8) and (9) nest. If it were the case that this nesting continued, then we could exclude ever more rational values as possible convergence points; the terms cover possible rational convergence points (Lemma 4); we could prove z_2 is irrational just like we proved $e-2$ is irrational. But z_2 intervals do not continue to nest. Continuing with just the bases 4, 9, and 16, we observe

$$.(1)_4^5 < .(7)_{16}^5 < .(4)_9^5 < s_5^2 = .(1669)_{3600} < .(8)_{16}^5 = .(2)_4^5 < .(5)_9^5.$$

Base 16 and base 9 have been transposed and, on the right, base 16 and base 4 endpoints collide (i.e. are equal). The next two iterations are

$$.(1)_4^6 < .(7)_{16}^6 < .(4)_9^6 < s_6^2 = .(1769)_{3600} < .(8)_{16}^6 = .(2)_4^6 < .(5)_9^6$$

and

$$.(4)_9^7 < .(8)_{16}^7 = .(2)_4^{7+} < s_7^2 = .(90281)_{176400} < .(5)_9^7 < .(9)_{16}^7 < .(3)_4^{7+}.$$

The left and right digits for base 4 have migrated to $.(2)_4$ and $.(3)_4$. As $.(2)_4 < z_2 < .(3)_4$, these left and right values for base 4 are fixed for $k \geq 7$.

Looking at the inequalities for z_2, the bases for partial sums exceed those of the terms used. We will show that s_k^n is not an element of sets of single decimals in the bases of its terms, their denominators (Corollary 1); nota bene general n. We claim that these properties of partials *escaping* terms and terms *covering* rationals are enough to show the irrationality of all z_n. We use these properties to show partials get arbitrarily close to numbers of ever greater precision, Theorem **??**; this implies irrationality.

3 Terms cover rationals

First two definitions.

Definition 1. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \text{ base } j^n.$$

That is d_{j^n} *consists of all single decimals greater than 0 and less than* 1 *in base* j^n*. The decimal set for* j^n *is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 2.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

We next show this union of decimal sets give all rational numbers in $(0, 1)$.

Lemma 4.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0, 1),$$

where $\mathbb{Q}(0, 1)$ *designates all rational numbers in the interval* $(0, 1)$*.*

Proof. Every rational $a/b \in (0,1)$ is included in a D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

As $0 < z_n < 1$ for $n \geq 2$, Lemma 4 shows, for large enough k, Ξ_k^n will contain any possible rational convergence point for any given z_n.

The first two example series, z_2 and $e - 2$, have terms that cover possible rational convergence points. They both converge to irrational numbers. Covering rational convergence points does not insure irrationality though, as the next examples show.

Example 1. The telescoping series

$$\sum_{k=2}^{\infty} \frac{1}{k} - \frac{1}{k+1} = \sum_{k=2}^{\infty} \frac{1}{k(k+1)}$$

covers rational points. If $a/b \in (0,1)$, $a < b$ and

$$\frac{a}{b} = \frac{a(b+1)}{b(b+1)} \in d_{b(b+1)}.$$

But this telescoping series converges to a rational number: 1/2.

Example 2. The geometric series or the series for such numbers as $.\overline{1}$ base 4, don't cover possible rational convergence points. For example $1/3 \notin d_{4^k}$, for any $k \geq 1$.

Example 3. Such numbers as $.2\overline{9}$ base 10 converge to a number covered by the terms, although not all rational numbers are covered.

4 Partials escape terms

We show partial sums of z_n can't be expressed as a single decimal using for a base the denominators of any of the partial sum's terms. We use the simple fact that a reduced fraction can't be expressed as a single digit decimal in a base less than its denominator. We just need to show, then, that the reduced denominator of s_k^n exceeds k^n, the denominator of the last term in a partial sum with upper index k.

Table 1 gives some evidence that the reduced fractions giving partial sum totals have much larger denominators than the denominators of their last

k	s_k^2	Prime factorization
3	$.(13)_{36}$	$36 = 2^2 3^2$
4	$.(61)_{144}$	$144 = 2^4 3^2$
5	$.(1669)_{3600}$	$3600 = 2^4 3^2 5^2$
6	$.(1769)_{3600}$	$3600 = 2^4 3^2 5^2$
7	$.(90281)_{176400}$	$176400 = 2^4 3^2 5^2 7^2$

Table 1: The reduced fractions (given as decimals) are divisible by powers of 2 and a prime greater than $k/2$.

term: $36 > 3^2$; $144 > 4^2$; $3600 > 5^2$; $3600 > 6^2$; $176400 > 7^2$. It also suggests a strategy for proving this. If we can show partial sums of z_n are divisible by powers of 2 and some relatively large prime, as twice something greater than half is bigger than the whole, that would do it. Apostol's *Introduction to Analytic Number Theory* (Chapter 2, problem 21), solutions in [10], gives the general technique used in this section.

Lemma 5. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{10}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (10) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 6. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 5. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{11}$$

As $(k, p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 7. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate. □

Theorem 2. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 7, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k - 1$ is even and we are assured of the existence of prime p such that $k - 1 > p > (k - 1)/2$. As $k - 1$ is even, $p \neq k - 1$ and $p > (k - 1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 7, we have assurance of the existence of a p that satisfies Lemma 6. Using Lemmas 5, 6, and 7 we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1.

$$s_k^n \notin \Xi_k^n \text{ or } s_k^n \in \mathbb{R}(0, 1) \setminus \Xi_k^n$$

where $\mathbb{R}(0, 1)$ is the set of real numbers in $(0, 1)$.

Proof. This is a restatement of Theorem 2. □

5 Examples

Table 2 gives examples of properties of series and suggests the pattern: irrationals have terms that cover possible rational convergence points and partials that escape their terms.

Series	Covers	Escapes	Convergence point
z_2	Yes	Yes (k)	not covered (irrational)
$e-2$	Yes	Yes (k-1)	not covered (irrational)
Telescoping	Yes	No	covered (rational)
$.\overline{1}$ base 4	No	Yes (k-1)	not covered (rational)
$.2\overline{9}$ base 10	No	Yes (k-1)	covered (rational)

Table 2: The two series that converge to a rational number, the geometric and telescoping, have one pattern; those that converge to an irrational number have another.

6 Towards Greater Precision

Progress has been made. Consider the following heuristic.

Using Lemma 4,

$$\lim_{k\to\infty} \Xi_k^n = \mathbb{Q}(0,1),$$

with Corollary 1 we have

$$\lim_{k\to\infty} \mathbb{R}(0,1) \setminus \Xi_k^n = \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1), \tag{12}$$

where $\mathbb{H}(0,1)$ is the set of irrational numbers in $(0,1)$.

We have then

$$\lim_{k\to\infty} s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n \implies z_n \in \mathbb{H}(0,1),$$

using $s_k^n \to z_n$, (12) and Corollary 1. That is z_n is irrational.

It seems reasonable that if s_k^n's require and are close to numbers requiring larger bases than those contained in $\{2^n, 3^n, \ldots, k^n\}$ then the numbers close to these partials are not single decimals in these bases, so too for z_n. That is the partials s_k^n and hence z_n are getting arbitrarily close to numbers requiring ever greater bases. In set topological terms, the limits points for s_k^n must reside in the complement of Ξ_k^n. We now give a formal proof.

Definition 3. *Let $D_{j^n}^{\epsilon_j}$ be the set of all D_{j^n} decimal sets having an element within ϵ_j of s_j^n.*

Example 4. .5, a single decimal, is a limit point of $.4\overline{9}_n$, where the subscript indicates the repetition of 9's. Ordering the convergence point base and partial bases for this example, one has 10^* (for .5),10 (for .4), 10^2 (for .49), 10^3 (for .499) ,..., where the superscript asterisk indicates the convergence point base. A sequence of epsilons of $D_{10^j}^{\epsilon_j}$ can be calculated: .4 is .1 from .5; .49 is .01 from .5; .499 is .001 from .5; etc.. Thus a decreasing and eliminative sequence, ϵ_j, is defined: $D_{10^3}^{.001}$ eliminates D_{10} and D_{10^2}, for example. We can make a different sequence that never eliminates base 10 – we know this must be possible. In fact for any $\epsilon > 0$, there exists an N_ϵ such that $.5 - .4\overline{9}_n < \epsilon$, for all $n > N_\epsilon$.

Using Example 4, we characterize series that converge to rational (and irrational) numbers.

Lemma 8. *If for every monotonically decreasing sequence ϵ_j such that*

$$\lim_{j\to\infty} \epsilon_j = 0,$$

we have

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset, \tag{13}$$

then z_n is irrational

Proof. We use proof by contraposition: $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$. Suppose z_n is rational then $z_n \in D_{j^n}^*$, using Lemma 4. Define

$$\epsilon_j^* = z_n - s_j^n \text{ for } j \geq 2$$

and set

$$\epsilon_j = 2\epsilon_j^*.$$

Then

$$D_{j^n}^* \subset \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j},$$

so the intersection is not empty. ☐

Example 5. Some irrational numbers are approximated very well (up to a point) using a low precision rational number. Using + for an irrational tail, .(25)0000000000+, a single decimal in base 100, is approximated by $.(25)_{100}$ to 10^{-10}. To get a closer approximation, a piece of the tail can be converted to a new decimal head: $.(250000000000x)+$ gives a better rational approximation, where x is a non-zero, fixed digit in base 10^{11}. A string of x's consisting of all 9s is not a rational tail. In contrast to series converging to rational numbers, Example 4, series converging to an irrational number get closer to higher precision numbers, numbers requiring larger bases. For the x decimal digit of this example, its accuracy must be at least as good as its precision, but there is always a better approximation using a larger base. This is proven in Lemma 9.

Lemma 9 simply says that if a point is not a single decimal in a base then it is inside an interval between single decimals; hence, it is trapped within $1/b$, b of these endpoints. This follows as decimal sets partition $(0,1)$ with intervals with widths equal to their precision, $1/b$, b the base.

Lemma 9. *If $.(a)_b \in (0,1)$ and $.(a)_b \notin D_{j^n}$ then there exists $x \in D_{j^n}$ such that*

$$.(a)_b \in (.(x-1), .(x))_{j^n},$$

where $(.(x-1), .(x))_{j^n}$ is an open set with end points $.(x-1)_{j^n}$ and $.(x)_{j^n}$. Further for any given $\epsilon > 0$,

$$|.(a)_b - .(x-1)_{j^n}| < \frac{1}{j^n} < \epsilon, \tag{14}$$

for large enough j.

Proof. D_{j^n} partitions the interval $(0,1)$ forcing $.(a)_b$ into such an interval. The distance between endpoints in such an open interval is $1/j^n$, so anything inside the interval is less than $1/j^n$ to an endpoint. The right hand inequality in (14) follows from the Archimedean property of the reals [14]. □

We suspect series that cover and escape their cover, in the sense developed above, have ever better approximations with greater bases. Any finite base approximation can't equal the convergence point.

Lemma 10. *For z_n there exists a sequence ϵ_j such that*

$$\bigcap_{j=2}^{\infty} D_j^{\epsilon_j} = \emptyset.$$

Proof. We need to define a sequence ϵ_j. Let

$$\epsilon_j^* = \min\{|x - s_j^n| : x \in \Xi_j^n \}.$$

We know by Corollary 1 that $\epsilon_j^* > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon_3^*$. This excludes the decimal sets of Ξ_3^n at this our first iteration. Assume we can generally do this for the jth iteration. For the $j+1$st iteration, using Lemma 9, there exists a base in Ξ_{j+r}^n, for some r such that $\epsilon_{j+r}^* < \epsilon_j/2$. Set $\epsilon_{j+1} = \epsilon_{j+r}^*$. The procedure gives ϵ values that cumulatively exclude ever more decimal sets from $D_{j^n}^{\epsilon_j}$. Regroup the series. By Lemma 4, the exclusions are exhaustive, so

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset,$$

as needed. □

Theorem 3. *z_n is irrational.*

Proof. Let the sequence given in Lemma 10 be given by ϵ_{j_1} and let a general sequence needed for Lemma 8 be given by ϵ_j. Suppose

$$\frac{p}{q} \in \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j}. \tag{15}$$

That is suppose the intersection in (15) is not empty. As $\epsilon_{j_1} \to 0$ and $\epsilon_j \to 0$, for any fixed ϵ_{j_1} that excludes p/q there will be an ϵ_j such that $\epsilon_j < \epsilon_{j_1}$. This implies that p/q will be excluded using ϵ_j, contradicting (15). □

7 Conclusion

How does this proof compare to the work of Beukers? Why do we get a general result here and not with his techniques?

Beukers uses double integrals that evaluate to numbers involving partials for $\zeta(2)$. He uses

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1 - xy}\, dxdy = \text{various expressions related to } \zeta(2)$$

and uses this to calculate

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\, dxdy,$$

where $P_n(x)$ is the nth derivative of an integral polynomial.

These calculations yield integers A_n and B_n in

$$0 < |A_n + B_n\zeta(2)|d_n^2 < \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2) < \left\{\frac{5}{6}\right\}^n, \tag{16}$$

where d_n designates the least common multiple of the set of integers $\{1,\ldots,n\}$. This last, assuming $\zeta(2)$ is rational, forces an integer between 0 and 1, giving a contradiction. An upper limit for d_n requires the prime number theorem.

These themes repeat for $\zeta(3)$ with the complexity of the expressions at least doubling.

We don't use integrals to generate in effect an interval, a trap, like (16), but the relationships between terms and partials to generate partitions of $(0,1)$ narrowing and leaving only irrational numbers. We use inherent and simple properties of z_n's partials and terms, Corollary 1, to avoid intractable complexity.

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Berggren, L., Borwein, J., Borwein, P. (2004). *Pi: A Source Book*, 3rd ed. New York: Springer.

[5] Erdös, P. (1932). Beweiss eines Satzes von Tschebyschef. *Acta Litt. Sci. Reg. Univ. Hungar, Fr.-Jos., Sect. Sci. Math.* 5: 194-198.

[6] Eymard, P., Lafon, J.-P. (2004). *The Number* π. Providence, RI: American Mathematical Society.

[7] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[8] J. Havil (2012). *The Irrationals.* Princeton, NJ: Princeton Univ. Press.

[9] Huylebrouck, D. (2001). Similarities in irrationality proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$, *Amer. Math. Monthly* 108(10): 222–231.

[10] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[11] Nesterenko, Y. V. (1996). A few remarks on $\zeta(3)$, *Math. Zametki* 59(6): 865–880.

[12] van der Poorten, A. (1978/9). A proof that Euler missed... Apery's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intelligencer* 1(4): 195–203.

[13] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[14] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.

[15] Sondow, J. (2006). A geometric proof that e is irrational and a new measure of its irrationality. *Amer. Math. Monthly* 113(7): 637–641.

[16] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.