Computability in the Theory of Theories

By Philip E. Gibbs

Abstract: Information is the basic material of reality, but the processing of information raises paradoxes due to the self-referential nature of computability. This principle can be embraced in a theory of theories leading to the emergence of quantum mechanics, geometry and physics.

"If the laws of physics are to be seen as a universal behaviour of some class of systems then it is necessary to ask what class to choose. We can regard any possible mathematical system as a theory of physics. I suggest that the laws of physics are a universal behaviour to be found in the class of all possible mathematical systems. This is known as The Theory of Theories" – From *"The Cyclotron Notebooks"* by Philip Gibbs*,* 3 March 1996. [1]

Physicists have long dreamt of finding a single set of equations from which all laws of physics could in principle be derived, but what would it mean if they found it? Might there be an enlightening foundational principle by which the solution could be explained? No matter how unique it were, there would always remain the question of why that principle is so special. How could it give such power to those equations, conjuring existence out of nothing? Why wouldn't any solution just raise further "how" and "why" questions without end?

As the search for answers ran dry, some thinkers resorted to an alternative idea. Perhaps there are no *special* equations. In this new philosophical landscape, all possible mathematical constructions are equal. Reality is relative. There is no magic principle that makes the nature of our universe more unique than any other option. Instead of a universe out of nothing they give us one drawn from a "multiverse" of everything mathematically possible.

This Theory of Theories could be modelled as a formal sum over a collection of mathematical universes $\mathfrak{M}$. Elements of $\mathfrak{M}$ could be determined by symbolic expressions that define some system of equations, or perhaps by computer algorithms that compute the structure of a universe. An observable $\mathcal{O}$ would be some property of these numbers and their expectation values could be calculated.

$$\langle \mathcal{O} \rangle = \frac{\sum_{\mathcal{U}\in\mathfrak{M}} \mathcal{O}(\mathcal{U})\ m(\mathcal{U})}{\sum_{\mathcal{U}\in\mathfrak{M}} m(\mathcal{U})}$$

Here $m(\mathcal{U})$ is some weight or measure attached to each universe $\mathcal{U}$ but how do we decide what that should be and would the calculation depend on the choice of symbolic language? Would the sums even converge and be meaningful? If there are answers to these questions, they must not depend on arbitrary choices. That would wipe out the philosophical arguments in favour of this approach, and if the calculation cannot be related to known physics in some way then it is a mathematical dead-end.

The problem of specifying the measure $m(\mathcal{U})$ and its apparent non-uniqueness is often touted as an almost immediate show-stopper, but this is not the case. The measure problem can be resolved using information theory. Any Universe $\mathcal{U}$ in the ensemble $\mathfrak{M}$ can be specified with a finite quantity of information $I = \frac{S(\mathcal{U})}{\ln(2)}$. This quantity is the length in bits of the shortest algorithm required to compute $\mathcal{U}$ (Kolmogorov complexity). The factor of $\ln(2)$ is a conventional conversion from bits to natural units of information (nats). Although this depends on the symbolic language used to specify the algorithm, it does so only up to an additive constant for sufficiently complex structures. The universality of computability means we can write a language interpreter of finite length to switch from the given language to the most efficient in terms of program length. Speed of computation is not relevant.

Although we think of data as quantised in discrete bits, information is actually a continuous quantity with no minimum value. A quantity of n bits specifies an integer in the range 0 to $2^n - 1$, i.e. one out of $N = 2^n$ options. By generalisation of the definition, the amount of information gained when an entity is specified from a set of size N is $I = log_2(N)$ in bits. In a random variable context this is true if the prior probabilities for each element in the set being selected are equal to $p = \frac{1}{N}$. More generally still, if we are told that a fact with a prior probability p is true then the amount of information we have gained (in bits) is $I = -log_2(p)$ or in nats, $S = -\ln(p)$.

This link between probability and information is the key to solving the measure problem. The measure function $m(\mathcal{U})$ needs to be proportional to the probability of selecting universe $\mathcal{U}$ from $\mathfrak{M}$. This selection provides us with the minimum amount of information required to specify the algorithm for $\mathcal{U}$. Inverting the relationship between probability and information we find

$$m(\mathcal{U}) \propto p(\mathcal{U}) = e^{-S(\mathcal{U})}$$

$$\langle \mathcal{O} \rangle = \frac{\sum_{\mathcal{U}\in\mathfrak{M}} \mathcal{O}(\mathcal{U})\ e^{-S(\mathcal{U})}}{\sum_{\mathcal{U}\in\mathfrak{M}} e^{-S(\mathcal{U})}}$$

So in our ensemble of possible universes, information quantity plays a role analogous to action in a quantum path integral. This is the uniquely correct way to proceed. You may worry that the choice of symbolic language for the algorithm could spoil uniqueness but as already mentioned, this will change the information $S(\mathcal{U})$ content only by a language dependent additive constant S_L which multiplies the ratio for $\langle \mathcal{O} \rangle$ by a factor e^{-S_L} in both numerator and denominator. There is no overall effect provided universes with high complexity dominate the sum. This is indeed the case because the number of universes increases with information content to match the decreasing measure factor.

So the measure problem is solved, but the Theory of Theories has other issues and some could be much harder to resolve. Not least are questions of computability and decidability. Whenever we use the concept of computation as a fundamental feature of existence, we must face up to the logical paradoxes that arise because algorithms can be self-referential, as shown by Church and Turing. Is this a problem that will force us to reject the hypothesis, or is it a feature that actually corresponds to reality? Physics allows for computation and information processing. This is demonstrated by the existence of computers and the human mind, both of which work within the laws of physics. Self-awareness is the defining characteristic of consciousness. The process of evolution which led to the emergence of conscious agents is based on the natural processing of information encoded in DNA [4]. By natural processes, this led to the human brain which itself is able to design and build more computers of enormous power. It would be a folly to reject a physical hypothesis just because it has features of computability built in at the foundational level. Instead we should try to check to see if any potential contradictions can be avoided. Perhaps computability may even act as a guiding principle to help us understand the emergence of physics from information theory.

In the ensemble of possible universes an algorithm is specified as a standalone source code string for a computer program. Source code can of course have input data hardwired inline. Just as it is convenient for physicists to separate initial conditions from dynamical equation, it is similarly convenient for computer scientists to artificially separate input data from algorithm code.

Definition 1: An algorithm is a computer program written in some specific universal programming language which takes a finite input string and returns a finite output string. There is no bound on how large the program's memory heap can grow or how long it can take to run.

To make this definition complete we would have to define the precise rules of a programming language. Turing did this by defining a Turing machine while Church used the lambda calculus, but for current purposes it will be assumed that the reader is familiar with how programming languages work.

A possible universe is an output from running an algorithm so we can write

$$\mathcal{U} = \text{output} = \text{code(input)}$$

The output, input and code are all finite (but long) streams of information. Many different combinations of input and code can produce the same output, but the Kolmogorov complexity of the output is the smallest amount of information (combined input and code) that produces it. Can we ever be sure that we have found the shortest algorithm? If we have a very short algorithm it may be possible to show that it is indeed the shortest, but in general this problem is not merely hard. It is not computable. This is a consequence of the halting problem as described by Alan Turing who discovered that there is no general algorithm that can determine if a given computer program will stop. His result is central to the thinking in this essay so it is worthwhile to reproduce an outline of Turing's proof which he demonstrated in detail based on his definition of computability.

Definition 2: A halt algorithm halt(input) is one that takes a combined input stream for the code and input of any algorithm. It returns a single bit equal to 1 if the program will halt and 0 if it will not halt (i.e if it enters into an infinite loop).

A combined stream is a single stream of information bits formed from two separate streams in such a way that that either of the two can be recovered inside a program. There is a natural one-to-one correspondence between bit streams and non-negative integers in their binary representation. One suitable way to combine streams is to use the injective (and therefore reversible) integer function

$$\text{combine}(a, b) = (a + b)^2 + a$$

A halt algorithm would execute halt(combine(input,code)) to determine whether or not the algorithm code will halt when given specific input.

Try to think about ways a halt program might work. It is possible to write a program that will always return a 1 if it is given an algorithm that halts. This can be done by writing an interpreter algorithm that effectively runs the code to see if it ends. When that happens it can output a 1 and stop. The same program could detect if the code comes back to a previous state and return a 0 in this case because it knows the program must repeat indefinitely, but to detect more complex infinite loops with no repetition is harder. Remember that the quantity of data that can be generated internally by the algorithm is unbounded so it may never return to a previous state. Perhaps if the program has not stopped after some computable number of steps as a function of the length of the program then we can assume it never will. Can such an approach work? Turing found that the answer is no.

Turing's Halting Theorem: There does not exist any halt algorithm.

Proof: Proceed by contradiction assuming that a halt algorithm halt(input) exists. Let confound(input) by an algorithm that halts if the input is a single zero bit, but which enters an infinite loop otherwise. Combine algorithms to create the algorithm turing(input) as follows

turing(input) = confound(halt(combine(input,input)))

Now consider whether execution of turing(turing)) will halt. I.e. will confound(halt(combine(turing,turing))) halt? By construction of the confound algorithm, the answer is yes iff halt(combine(turing,turing)) will NOT halt, i.e. iff turing(turing) will not halt. The conclusion that turing(turing) will halt iff turing(turing) will not halt is a self-contradiction. Therefore the premise that a halt program exists must be false. *QED*.

This is a far-reaching result with deeply rooted implications. It means that no computable function increases more rapidly than the longest time it takes for an algorithm to halt as a function of the length of its code. We could even attempt to write a halt algorithm that performs a brute-force search for a symbolic proof that an algorithm either halts or does not halt. The fact that this cannot succeed implies that there are undecidable statements, confirming the proof of Godel's undecidability theorems. The paradox is an inescapable feature of our universe. It arises whenever we combine the self-referencing nature of computing necessary for consciousness with the concept that the universe emerges from a statistical ensemble of computable structures. Fortunately a paradox is not a logical contradiction. There is no need to give up, even if it does mean that **the algorithmic complexity that determines the weight of a universe in the ensembles of possible universes is not computable**.

Up to this point the Theory of Theories as described here looks very much like Max Tegmark's Mathematical Universe Hypothesis (MUH) [2]. Certainly there is much in common, but there are also crucial differences with the Theory of Theories approach. According to the MUH we live within one external physical reality [3], a mathematical structure selected by random or anthropomorphic principles from the level-4 multiverse of consistent mathematical structures. The laws of physics that we know are the defining characteristic of this structure. They are supposed to be simple because the measure in the space of possibilities favours simplicity. Indeed they might be even simpler were it not for the requirement that self-aware structures (i.e. us) need to be able to evolve in this universe. The "many worlds" of quantum mechanics is a different level-3 multiverse contained within a single level-4 universe. We can experience the physical effects of interference between those quantum worlds, but the alternatives in the level-4 multiverse are inaccessible because our external reality at this level is fixed in relation to us. (Level 2 and Level 1 multiverses are merely different cosmological solutions in an eternal inflation theory or other form of infinite universe. They can be ignored here.)

The term "multiverse" has often been used interchangeably to describe these different levels without the necessary distinctions being made. For this reason I prefer to avoid the term. Others have objections to the concept itself while I simply worry that the baggage it carries would confuse what I am trying to say.

In the Theory of Theories the picture is in fact quite different from the MUH. There is no hierarchy of multiverses and no fixed external physical reality. There is just one level of mathematical possibilities describing a single "multiverse". Indeed, this controversial terminology is superfluous. Mathematical structures can interfere within our reality just as quantum worlds can interfere, because in fact there is no distinction between the two. The laws of physics themselves are uncertain. There is just one universe influenced by the combined effect of all possibilities, all accessible to us in principle. The simplicity, symmetry and unity of nature emerges from a universality principle in the ensemble of complex systems [6]. In fact structures determined by large amounts of information dominate in the ensemble rather than the simpler ones, because their number outweighs their diminished weight.

We can start with the working assumption that everything beyond our state of mind is uncertain, including the familiar laws of physics. The world around us is described by a large number of facts whose truth is only knowable as a probability. Our brains can only register a relatively small number of facts, but logical consistency is a powerful constraint so a much larger number of facts about the universe are needed to build a complete model, some more certain than others. Some things that we normally regard as certain may not be. This can include details of past events that left no detectable trace, or distant structures in the universe that we will never reach.

A fact is normally true or false, but it is useful to consider more general facts that can take k mutually exclusive states where k is some positive integer, not necessarily just the two states of Boolean logic. Our knowledge of any fact is probabilistic and described by a probability vector $(p_i, i = 1, \dots, k)$ such that the sum of its components is equal to one. If two facts with k_1 and k_2 states are considered together than the result can be regarded as just one vector with $k = k_1 \times k_2$ states. N Boolean facts can likewise be combined into a single state vector with $k = 2^N$. To begin with, let's put aside the algorithmic side of the universe and think of it as one big probability vector like a statistical physics system $\mathcal{U} = \boldsymbol{p} = (p_i, i = 1, \dots, k)$. The ensemble of universes $\mathfrak{M}$ is the continuum of possible probability values. The measures $m(\mathcal{U})$ form a probability vector indexed by all possible probability vectors, so the system has the potential for self-reference. This double layered structure could be related to second quantisation, and by self-reference to iterated or multiple quantisation [7].

To complete this simplified model of reality, a formula for the measure $m(\mathcal{U}) = m(\boldsymbol{p})$ is needed. This should be determined by the amount of information $S(U) = S(\boldsymbol{p})$ in the probability vector, then $m(\mathcal{U}) = e^{-S(U)}$. If the k-valued state is known with certainty then the information in natural units is $S(\mathcal{U}) = \ln(k)$, but how much information is there when the state is uncertain? This question was answered by Claude Shannon in his seminal work on information theory where he gave the answer [5]

$$S(\boldsymbol{p}) = \sum_i p_i \ln(k p_i)$$

This quantity is a minimum of zero when all the probabilities are equal $p_i = \frac{1}{k}$, and is a maximum of $\ln(k)$ when one of the probabilities is equal to 1, all others 0.

This gives us

$$\langle \mathcal{O} \rangle = \frac{\int d^k\boldsymbol{p}\ \ \mathcal{O}(\boldsymbol{p})e^{-\sum_i p_i \ln(p_i)}}{\int d^k\boldsymbol{p}\ \ e^{-\sum_i p_i \ln(p_i)}}$$

This is equivalent to a classical statistical physics system with Gibbs entropy, but if we transform using $p_i = \overline{\psi_i}\psi_i$ and introduce a factor of i into the exponential by way of a Wick rotation, then it can also be interpreted as the emergence of Hilbert space and quantum mechanics.

$$\langle \mathcal{O} \rangle = \frac{\int d^k\boldsymbol{\psi}\, d^k\overline{\boldsymbol{\psi}}\ \mathcal{O}(\boldsymbol{\psi})e^{i\sum_i \overline{\psi_i}\psi_i \ln(\overline{\psi_i}\psi_i)}}{\int d^k\boldsymbol{\psi}\, d^k\overline{\boldsymbol{\psi}}\ e^{i\sum_i \overline{\psi_i}\psi_i \ln(\overline{\psi_i}\psi_i)}}$$

This model is too simple to fully implement the requirement of the Theory of Theories. Hilbert space alone is not sufficient to implement algorithms. However it does show that quantum theory is one of the most primordial features of nature, preceding geometry.

To go further, consider how universes can be combined algebraically. Recall that probability vectors for two sets of facts with k_1 and k_2 states can be combined into one probability vector with $k_1 \times k_2$ states. If the facts are independent then the probabilities are simply multiplied $p_{ij} = p_i p_j$. In Hilbert space form, this amounts to taking tensor products of wave-functions $\psi = \psi_1 \otimes \psi_2$. Wave functions can also be added in superpositions so long as they are renormalized before calculating probabilities. These operations can be repeated, so algebraically it makes sense to consider the freely generated associative algebra generated using these operations when starting from a single Hilbert space $\mathcal{H}_k$. This algebra is known as the **tensor algebra** of $\mathcal{H}_\mathrm{k}$ and is denoted $T(\mathcal{H}_k)$. Its elements are in fact a combination of tensors over the Hilbert space with one of each rank starting with scalars (The subscript k can be dropped for brevity.)

$$T(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \dots$$

This algebra is itself a Hilbert space with an extra multiplicative structure so it also fits into the ensemble of possible universes.

The tensor algebra $T(\mathcal{H})$ is well known to mathematicians and has been extensively studied. Using its commutators, the **free Lie algebra** $L(\mathcal{H})$ can be constructed, whose **universal enveloping algebra** takes us back to the tensor algebra $U(L(\mathcal{H})) \simeq T(\mathcal{H})$. $L(\mathcal{H})$ is in fact a fascinating object which deserves to be better known among physicists. Like $T(\mathcal{H})$ it is composed of a sequence of tensor algebras of increasing rank. Whereas the individual tensor components of $T(\mathcal{H})$ can be viewed as open ended chains of Hilbert spaces tensored together, the components of $L(\mathcal{H})$ taken over the **Lyndon basis** are closed necklaces of Hilbert spaces tensored round a cyclic loop. For this reason it is known as a **necklace Lie-algebra**. The free Lie algebra is related to mathematical structures relevant to quantum gravity such as **the Milnor invariants of the link group.**

Mathematicians know that $T(\mathcal{H})$ can be better understood by giving it a further algebraic operation known as a coproduct. This, along with a counit and antipode, promotes it to a **Hopf algebra**. Hopf algebras have duals which switch the role of the product and coproduct. The dual $T^*(\mathcal{H})$ has a commutative product known as a **shuffle algebra** because it operates by summing over all reorderings of tensor products that keep the order of the original factors unchanged. Shuffle products are connected to **iterated integrations** so this observations leads to the important discovery that the tensor algebra $T(\mathcal{H}_k)$ is isomorphic to an algebra of curves in a k-dimensional space. Shuffle products and iterated integrations are also related to **multi-polylogarithms** and the **Dyson series** that arise in the computation of **Feynman diagrams**.

Unfortunately it is not possible to expand on the mathematics here due to limited space, but the keywords should be sufficient to locate the necessary expositions online (see also the end-note). Even without a full understanding of the mathematics the implications should be clear. From a starting point of information, uncertainty and an ensemble of all possible universes we can proceed in natural mathematical steps to the emergence of quantum theory and geometry. Is it possible that a complete framework for fundamental physics could emerge? I believe the answer is yes, but more structure is needed and the origin of this additional structure is computability.

In a digital computer the machine state at any time is stored in the form of data bits in various forms of memory. In the execution of an algorithm, some of this data represents variables that change. Other data represents the code for the algorithm that is fixed. At each execution step an operation is performed on the changing data that could be written as a set of large Boolean expressions, i.e a combination of AND, OR and NOT operations on the bits. This can be converted to a truth table. If the computer has k states, then each step executes a function for the set of those k possibilities back to itself. This function is iterated until some end condition is reached.

In a probabilistic situation where both the state and the algorithm are uncertain, the computation process would follow the actions of a markov chain. In a quantum computation the equivalent would be the iterated action of a unitary matrix. The evolution of the quantum state in quantum mechanics is therefore equivalent to the processing of a quantum computer.

The memory capacity of any digital computer is in practice finite, but the general concept of computability requires that the data size can extend without limits. A program must also be able to run for an unlimited amount of time. How does this apply to the universe as a computer? If the universe is destined to expand without limit then it can be a complete universal computer.

An algebraic model of computable universes cannot be just an ensemble of probability vectors. It must also include all possible operators, iterated to allow computation. Reality is not just a probability vector of universes. It is an algebraic structure with operations that allow self-referencing through recursion. I conjecture that by developing these ideas of computability and self-reference algbraically in a theory of theories, the familiar frameworks of theoretical physics will emerge.

Much has been written about the lack of successfully predictions in fundamental physics beyond the standard model in recent decades. Since the 1970s all work on theoretical physics, including supersymmetry and string theory, has failed to make contact with experiment. "Alternatives" such as technicolour and loop quantum gravity have fared no better. Some commentators blame the theorists for being lost in their admiration of the mathematical beauty found in these theories. I think this criticism is unwarranted. It should be clear to anyone who steps back far enough to take in the bigger picture, that the real problem is a dearth of new experimental results at the energy frontier. The hoped-for exotic particles at LEP, the Tevatron and LHC were simply not there. Instead the standard model has been confirmed with unexpected precision. Only neutrino masses, dark matter, dark energy, inflation and a few other anomalies hint at something that we cannot grasp in the darkness beyond this dessert. Theorists have simply done their job by exploring the implications of consistency applied to the requirements of unifying the standard model with gravity. They have carefully classified the possible theories that could have been the right unification. The history of string theory followed a natural progression of discoveries that were inevitable because they were the only consistent way forward. The mathematical beauty of what they found has been more of a conclusion than a guiding principle, even if theorists themselves sometimes believe otherwise.

Theorists cannot be held to account for failing to predict the actual outcome of experiments when we now know that the correct answer up to energies tested so far was simply "nothing new." Predictions of low energy supersymmetry were wrong, but there was nothing else there that they could have been right about. Nature, after centuries of generosity, has now decided to throw humankind a harder challenge. The theorists' work on quantum gravity still stands. We have learnt that its phenomenology lies beyond the reach of any experiment tried so far, and possibly of any we are able to attempt in the near future. Trying to test quantum gravity with a particle accelerator is like observing Mars through a telescope to investigate the atomic structure of matter.

Yet there is still an element of failure of a different kind. Physicists have so far failed to find the mathematical foundations of quantum field theory, supersymmetry, string theory and loop quantum gravity. The mathematics must exist even if those ideas are wrong for physics. Why is this so hard? In my opinion they are failing because they still cannot relinquish certain cherished philosophical beliefs. Reductionism, determinism, objective physical reality, simplicity, naturalness and causality: There is a clear message from quantum nature that these concepts do not belong in a fundamental theory. Nearly everyone is trying to look back to a theoretical foundation for a single reality that is independent of the observer. They are facing the wrong way. Instead a theory based on information, uncertainty, symmetry, complexity, computability and a single-level universe of multiple possibilities will lead directly to the emergence of the mathematical structures that appear in these advanced physical theories.

The path is not easy to tread. I have offered here the beginnings of a **quantum information calculus** that emerges as a universality principle from the ensemble of all possible computable universes. It leads to the complete symmetry of necklace Lie algebras which map the emergence of geometry through iterated integration. The mathematical sophistication required probably rises well beyond what I have outlined in this essay, but if there are some with the courage to give up old notions and others that are willing to listen, then it can be understood. I don't know if nature will then be kind enough to provide the empirical observations that will confirm it, but at least we will have the necessary foundations to investigate the options ahead with confidence. Who knows what else it might be good for?

References

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[4] Philip E. Gibbs, “Consciousness Bootstrapped”, FQXi essay https://fqxi.org/community/forum/topic/2808 (2017)

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[7] Philip E Gibbs, “An Acataleptic Universe”, FQXi essay https://fqxi.org/community/forum/topic/1603 (2013)

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End Note

The **Free Lie Algebra** is likely to be an important algebraic object in the foundations of physics, but it is not well known to physicists. Here I will give a technical introduction in a form that they may find more understandable, and I will explain its significance to string theory in particular.

Let $T_k^n = \otimes^n V_k$ be the space of rank-n tensors from the k-dimensional vector space V_k over $\mathbb{C}$

An index rotation operator R with $R^n = 1$ can be defined to act on T_k^n such that

$$R(v_1 \otimes v_2 \otimes \dots \otimes v_n) = v_2 \otimes \dots \otimes v_n \otimes v_1, \qquad v_r \in V_k$$

A cyclic subspace is defined $C_k^n \subset T_k^n = \{t \in T_k^n | R(t) = e^{\frac{2\pi i}{n}} \times t\}$

The indexes for base elements of this space C_k^n are identified with aperiodic necklaces.

A projection operator P_n from T_k^n to C_k^n is to given by

$$P_n = \frac{1}{n}\sum_{r=1}^{n} R^r e^{-\frac{2\pi i r}{n}}$$

Define an infinite dimensional graded space $L_k = C_k^1 \oplus C_k^2 \oplus \dots$

A bilinear bracket operator can be defined on L_k by

$$[A, B] = P_{n+m}(A \otimes B - B \otimes A), \qquad A \in C_k^n, B \in C_k^m$$

This is extended to all elements of L_k using the bilinear property of the bracket.

L_k is an infinite dimensional Lie algebra with the bracket [A,B]. **It is the free Lie algebra generated from V_k.**

This can be verified by checking the Jacobi identity.

The dual of the universal algebra of the free Lie algebra L_k^* is a commutative algebra whose product of two elements A and $B \in L_k^*$ is the shuffle product written $A ⧢ B$. Necklaces are multiplied by forming a sum over all possible ways of interleaving two necklaces to form a new necklace.

Components of an element of $A \in L_k^*$ in a given basis are complex numbers indexed by necklaces $A_{i_1 \dots i_n}$. If $v_i(t)$ is a closed loop parameterised by the variable $t \in [0,1]$ in the original k-dimensional space from which the free Lie algebra was generated then we can use iterated integration to define a mapping $\varphi(A)$ to the complex numbers,

$$\varphi = \oiiint_0^1 \sum_{i_1 \dots i_n} A_{i_1 \dots i_n} v_{i_1}(t_1) \dots v_{i_n}(t_n) dt_1 \dots dt_n$$

Where the sum is over all necklaces and the integral is ordered such that $t_1 < \dots < t_n$

The map φ is then an algebra homomorphism because $\varphi(A ⧢ B) = \varphi(A)\varphi(B)$

The mapping $\varphi(A)$ as a function from closed loops to the complex numbers can be considered a wave function for a state of closed strings in a k-dimensional space. Transitions may be constructed using the free Lie algebra.

My conjecture is that in this way it is possible to construct an algebraic formulation of closed string theory such that the free Lie algebra is the underlying symmetry of the dynamics. Since the Tensor algebra and is corresponding Lie algebra emerge from an analysis of information in a Theory of Theories, this provides a rational for the emergence of physics from first principles.