A Noncommutative Spacetime Realization of Quantum Black Holes, Regge Trajectories and Holography

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Abstract

It is shown that the radial spectrum associated with a fuzzy sphere in a *noncommutative* phase space characterized by the Yang algebra, leads *exactly* to a Regge-like spectrum $GM_l^2 = l = 1, 2, 3, \ldots$, for *all* positive values of l, and which is consistent with the extremal quantum Kerr black hole solution that occurs when the outer and inner horizon radius coincide $r_+ = r_- = GM$. The condition $GM_l^2 = l$ is tantamount to the mass-angular momentum relation $M_l^2 = lM_p^2$ implying that the (extremal) horizon area is quantized in multiples of the minimal Planck area. Another important feature is the holographic nature of these results that are based in recasting the Yang algebra associated with an $8D$ noncommuting phase space, involving $\mathbf{x}_\mu, \mathbf{p}_\nu, \mu, \nu = 0, 1, 2, 3$, in terms of the $undeformed$ realizations of the Lorentz algebra generators J_{AB} corresponding to a $6D$-spacetime, and associated to a $12D$-phase-space with coordinates $X_A, P_A; A = 0, 1, 2, \ldots, 5$. We hope that the findings in this work, relating the Regge-like spectrum $l = GM^2$ and the quantized area of black hole horizons in Planck bits, via the Yang algebra in Noncommutative phase spaces, will help us elucidate some of the impending issues pertaining the black hole information paradox and the role that string theory and quantum information will play in its resolution.

Keywords: Snyder, Yang algebra; Noncommutative Spacetimes; Black Holes; Regge Trajectories; Strings; Holography.

The idea of a Quantum Spacetime where the spacetime coordinates do not commute was proposed early on by Heisenberg and Ivanenko as a way to eliminate infinities from Quantum Field Theory. Snyder published the first concrete example [1] of a noncommutative algebra involving the spacetime coordinates, and it was generalized shortly after by Yang [2], to include noncommuting momentum variables as well. We learnt from General Relativity that the Poincare algebra cannot be implemented on a curved spacetime, but only on its flat tangent space (Minkowski spacetime). The momentum operators don't commute on a curved spacetime. And vice versa, by Born's principle of reciprocity [10], the coordinate operators do not commute on a curved *momentum* space. This prompted the formulation of Quantum Mechanics and Quantum Field Theory in Noncommutative spacetimes (also called Noncommutative QFT), and which might cast some light in the formulation of Quantum Gravity by encoding both key aspects of a curved and a noncommuting spacetime (a curved noncommuting spacetime).

In [11] we suggested that Born's Reciprocal Relativity Theory in Phase spaces is the arena to implement a space-time-matter unification. More precisely : quantum matter curves noncommuting spacetime, and vice versa, noncommuting spacetime curves quantum matter (quantum momentum space) as a result of the back-reaction of quantum spacetime on quantum matter. We believe that it is this Born's reciprocity principle that holds important clues to quantize gravity (geometry) in curved phase spaces within the context of Finsler geometry.

Most recently, Nonassociative structures arising from recent developments in Quantum Mechanics with magnetic monopoles, in string theory and M-theory with non-geometric fluxes, and in M-theory with non-geometric Kaluza-Klein monopoles, have risen to prominence [3] and paving the way towards the construction of Noncommutative and Nonassociative gravity. In this work we shall mainly focus on the role that the Yang algebra has in black hole physics. Because the references on Noncommutative Geometry, Quantum Groups, Noncommutative QFT, Fuzzy spaces, Fractal spacetimes, curved κ-Minkowski spacetimes, Poisson-Lie algebras, ... are vast, we refer to the most recent work by [4] for a list of some of the relevant references.

The Yang algebra in $4D$ is given by the following commutators (in $c = 1$ units) in terms of the generators $\mathbf{x}_\mu, \mathbf{p}_\nu, J_{\mu\nu}, \mathcal{N}$, and the two scales $L_p, \mathcal{L}$,

$$[\mathbf{x}_\mu,\ \mathbf{x}_\nu] \ = \ -\,i\ L_p^2\ J_{\mu\nu},\ \ [\mathbf{p}_\mu,\ \mathbf{p}_\nu] \ = \ -\,i\ (\frac{\hbar}{\mathcal{L}})^2\ J_{\mu\nu}, \tag{1}$$

$$[\mathbf{x}_\mu,\ J_{\nu\rho}] \ = \ i\ (\ \eta_{\mu\rho}\ \mathbf{x}_\nu\ -\ \eta_{\mu\nu}\ \mathbf{x}_\rho\ ) \tag{2}$$

$$[\mathbf{p}_\mu,\ J_{\nu\rho}] \ = \ i\ (\ \eta_{\mu\rho}\ \mathbf{p}_\nu\ -\ \eta_{\mu\nu}\ \mathbf{p}_\rho\ ) \tag{3}$$

$$[\mathbf{x}_\mu,\ \mathbf{p}_\nu] \ = \ i\hbar\ \eta_{\mu\nu}\ \mathcal{N},\ \ [\mathbf{x}_\mu,\ \mathcal{N}] \ = \ i\ \frac{L_p^2}{\hbar}\ \mathbf{p}_\mu,\ \ [\mathbf{p}_\mu,\ \mathcal{N}] \ = \ -\ i\ \frac{\hbar}{\mathcal{L}^2}\ \mathbf{x}_\mu \tag{4}$$

and where the $[J_{\mu\nu}, J_{\rho\sigma}]$ commutators are the same as the ones of the $so(3,1)$ Lorentz algebra in $4D$ and corresponding to boosts and rotations. They are of the form $\eta_{\mu\sigma} J_{\nu\rho} \pm$ permutations, and $[J_{\mu\nu}, \mathcal{N}] = 0$. The spacetime metric is $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

Starting with a flat $6D$ spacetime $X_A = \{X_0, X_I\} = \{X_0, X_1, X_2, X_3, X_4, X_5\}$, with signature $(-, +, +, \ldots, +)$, the above Yang algebra can be realized in terms of the following angular momentum/boost operators $J_{\mu\nu}, J_{\mu 4}, J_{\mu 5}, J_{45}$, associated to a $so(5,1)$ Lorentz algebra in $6D$, and which are defined by

$$J_{\mu\nu} = X_{[\mu} \; P_{\nu]}; \; J_{\mu 4} = X_{[\mu} \; P_{4]}; \; J_{\mu 5} = X_{[\mu} \; P_{5]}; \; J_{45} = X_{[4} \; P_{5]} \qquad (5a)$$

where X_A, P_B obey the standard commutation relations of the Weyl-Heisenberg algebra associated with the ordinary QM defined over a *classical* $6D$ spacetime.

$$[X_I, \; X_J] = 0, \;\; [P_I, \; P_J] = 0, \;\; [X_I, \; P_J] = i\hbar \; \delta_{IJ} \qquad (5b)$$

$$[X_0, \; X_I] = 0, \;\; [P_0, \; P_I] = 0; \;\; [X_0, \; P_0] = -i\hbar, \;\; [X_0, \; P_I] = 0 \qquad (5c)$$

By establishing the following one-to-one correspondence

$$\mathbf{x}_\mu \leftrightarrow \frac{L_p}{\hbar} J_{\mu 4}, \quad \mathbf{p}_\mu \leftrightarrow \frac{1}{\mathcal{L}} J_{\mu 5}, \quad \mathcal{N} \leftrightarrow \frac{L_p}{\mathcal{L}} \; J_{45}, \quad \mu, \nu = 0, 1, 2, 3 \qquad (6)$$

which requires the introduction of an ultra-violet cutoff scale L_p given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant), one can verify that the Yang algebra is recovered. Therefore, the Yang algebra captures both physics in the ultra-violet (small scales) and in the infra-red (large scales), a key property that a successful theory of Quantum Gravity must have.

There are crucial *differences* between this present work and previous work by other authors [5], [7], for example, in their evaluation of the areal spectrum in Snyder Space. The work of [5] is based on a $so(3) \oplus so(3)$ algebra using two sets of $so(3)$ generators J_i and $J_{i4}, \; i, j = 1, 2, 3$. After performing the linear combinations $J_{i4} \pm J_i$, with $\hat{x}_i \leftrightarrow J_i$, using the addition rules of angular momentum $|l_1 - l_2| \leq l \leq l_1 + l_2$, in the limiting case $l = l_1 + l_2 = 2l_1$ when $l_1 = l_2$, it turns out that $\hat{x}_i \hat{x}^i = 2l_1(l_1 + 1) + 2l_2(l_2 + 1) - l(l + 1)$, and it leads to the exact *cancellation* of the quadratic terms $4l_1^2 - l^2 = 0$, leaving only the linear terms $2l_1 = l$ in the angular momentum, which is the desired goal to show that the areas $4\pi r^2 = 4\pi \hat{x}_i \hat{x}^i$ are proportional to l (an integer), in order to prove that the areas are quantized in integer units of the Planck area.

However this construction based on two fuzzy spheres (involving the superposition of angular momentum) is problematic for the following reason. Recurring to the isomorphim of the Lie algebra $so(4) = su(2)_L \oplus su(2)_R$, one has that $\mathcal{L}_i, \mathcal{R}_i$ are the respective generators of the $su(2)_L, su(2)_R$ algebras, and $[\mathcal{L}_i, \mathcal{R}_j] = 0$. The linear combination $\mathcal{M}_i = \mathcal{L}_i + \mathcal{R}_i$ does behave like an angular momentum (Pauli spin) algebra. But the other linear combination $\mathcal{N}_i = \mathcal{L}_i - \mathcal{R}_i$ does *not*. Namely, the commutators $[\mathcal{M}_i, \mathcal{M}_j] = i\epsilon_{ijk} \mathcal{M}_k$ do close, but the commutators $[\mathcal{N}_i, \mathcal{N}_j] = i\epsilon_{ijk} \mathcal{M}_k$ do not. And because the generators $\mathcal{N}_i$ do *not* behave like true angular momentum generators, the spectrum of $(\mathcal{N}_i)^2$ is *no* longer given by $l_2(l_2 + 1)$. Therefore, one can *no* longer use the key expression

$2l_1(l_1+1)+2l_2(l_2+1)-l(l+1)$ in order to cancel out the quadratic terms in l because $\langle l_2|(\mathcal{N}_i)^2|l_2\rangle \neq l_2(l_2+1)$.

For these reasons we shall focus solely on the J_{i4} generator and one, and only one, fuzzy sphere, and follow a very *different* procedure here based on the above Yang algebra [2]. Upon using the realization of the *noncommuting* coordinates $\mathbf{x}_i \leftrightarrow \frac{L_p}{\hbar} J_{i4}$ in terms of the angular momentum operator variables $J_{i4} \equiv (X_i P_4 - X_4 P_i)$, and expressed in terms of the commuting coordinates X_i, whose momenta operators are given as usual by $P_j = -i\hbar \frac{\partial}{\partial X_j}$, $P_4 = -i\hbar \frac{\partial}{\partial X_4}$, the operator $\mathbf{r}^2$ can be written (in $\hbar = c = 1$ units) as

$$\mathbf{r}^2 \ = \ \mathbf{x}_i \ \mathbf{x}_i \ = \ \mathbf{J}_{i4} \ \mathbf{J}_{i4} \ =$$
$$-\ L_p^2 \left(\ - X_i \frac{\partial}{\partial X_4} \ + \ X_4 \frac{\partial}{\partial X_i} \right) \left(\ - X_i \frac{\partial}{\partial X_4} \ + \ X_4 \frac{\partial}{\partial X_i} \right) \ =$$
$$-\ L_p^2 \ \left(X_i^2 \frac{\partial^2}{\partial X_4^2} - X_i \frac{\partial}{\partial X_i} - 2 X_4 X_i \frac{\partial^2}{\partial X_i \partial X_4} - X_4 \frac{\partial}{\partial X_4} + X_4^2 \frac{\partial^2}{\partial X_i^2} \right) \quad (7)$$

where $i = 1, 2, 3$.

Using the definition (following the Einstein summation convention) $X_i X_i = R^2$, the chain rule yields

$$\frac{\partial}{\partial X_i} \ = \ \frac{\partial R}{\partial X_i} \ \frac{\partial}{\partial R} \ = \ \frac{X_i}{R} \ \frac{\partial}{\partial R} \ \Rightarrow \ X_i \ \frac{\partial}{\partial X_i} \ = \ R \ \frac{\partial}{\partial R} \quad (8)$$

which allows to express some of the derivatives in eq-(7) explicitly in terms of $\frac{\partial}{\partial R}$ giving

$$\mathbf{r}^2 \ \Psi \ = \ - \ L_p^2 \ \left(\ R^2 \ \frac{\partial^2}{\partial X_4^2} \ - R \ \frac{\partial}{\partial R} \ - 2 \ (X_4 \frac{\partial}{\partial X_4}) \ (R \ \frac{\partial}{\partial R}) - X_4 \frac{\partial}{\partial X_4} + X_4^2 \ \nabla^2 \right) \ \Psi \quad (9)$$

The eigenvalue equation that one wants to solve is given by

$$\mathbf{r}^2 \ \Psi(R, X_4, \theta, \phi) \ = \ r^2 \ \Psi(R, X_4, \theta, \phi) \quad (10)$$

where r^2 is the eigenvalue associated with the $\mathbf{r}^2$ **operator**. As usual, we separate variables by writing

$$\Psi(R, X_4, \theta, \phi) \ = \ \Phi(R) \ V(X_4) \ Y_{lm}(\theta, \varphi) \quad (11)$$

where the angular variables θ, φ correspond to the classical *commuting* coordinates X_1, X_2, X_3 and **not** to the non-commuting ones $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$. Namely one has

$$X_1 \ = \ R \ sin(\theta) \ cos(\varphi), \quad X_2 \ = \ R \ sin(\theta) \ sin(\varphi), \quad X_3 \ = \ R \ cos(\theta), \quad (12)$$

with $X_1^2 + X_2^2 + X_3^2 = R^2$ and $Y_{lm}(\theta, \varphi)$ are the spherical harmonics. l is the angular momentum quantum number associated with the angular momentum variables $J_{ij} = X_i P_j - X_j P_i$ $(i,j = 1,2,3)$ and which must *not* be *confused* with the J_{i4} angular momentum ones. After inserting Ψ given by (11) into eq-(9), and dividing by $\Phi(R)V(X_4)$, leads to the differential equation

$$- L_p^2 \left(- \frac{R}{\Phi} \frac{d\Phi}{dR} - \frac{X_4}{V} \frac{dV}{dX_4} - 2 \left(\frac{R}{\Phi} \frac{d\Phi}{dR}\right) \left(\frac{X_4}{V} \frac{dV}{dX_4}\right) + \frac{R^2}{V} \frac{d^2 V}{dX_4^2} \right) -$$

$$L_p^2 \, X_4^2 \left(\frac{1}{\Phi R^2} \frac{d}{dR} R^2 \frac{d\Phi}{dR} - \frac{l(l+1)}{R^2} \right) - r^2 = 0 \quad (13)$$

As usual, the angular dependence decouples by recurring to the expression of the $d = 3$ Laplace operator ∇^2 in spherical coordinates and whose action on the function $\Phi(R)V(X_4)Y_{lm}(\theta,\varphi)$ gives

$$Y_{lm}(\theta,\varphi) \left(\frac{1}{R^2} \frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R} - \frac{l(l+1)}{R^2} \right) \Phi(R) \, V(X_4) \quad (14a)$$

In higher spatial dimensions than $d = 3$ the radial piece of the Laplace operator is

$$\frac{1}{R^{d-1}} \frac{\partial}{\partial R} R^{d-1} \frac{\partial}{\partial R} - \frac{l(l+d-2)}{R^2} \quad (14b)$$

and one could generalize the eigenvalue problem to higher dimensional horizons [7].

Introducing the simple ansatz $V(X_4) = X_4$ in eq-(13) leads to

$$3 \, L_p^2 \, \frac{R}{\Phi} \frac{d\Phi}{dR} + L_p^2 - L_p^2 \, X_4^2 \left(\frac{1}{\Phi R^2} \frac{d}{dR} R^2 \frac{d\Phi}{dR} - \frac{l(l+1)}{R^2} \right) - r^2 = 0 \quad (15)$$

By defining $\lambda^2 \equiv (\frac{r}{L_p})^2$ and plugging the trial function $\Phi(R) = r^\alpha$ into eq-(15) gives, after some straightforward algebra, the following relation

$$\frac{X_4^2}{R^2} \left(\alpha(\alpha+1) - l(l+1) \right) - 3\alpha + (\lambda^2 - 1) = 0 \quad (16)$$

From which one infers

$$\alpha(\alpha+1) - l(l+1) = 0; \quad \lambda^2 - 1 - 3\alpha = 0 \quad (17)$$

The first relation above yields two roots

$$\alpha = \frac{1 \pm \sqrt{1 + 4l(l+1)}}{2} = \frac{1 \pm \sqrt{(2l+1)^2}}{2} = \frac{1 \pm (2l+1)}{2} \Rightarrow \alpha_\pm = l; -l-1 \quad (18)$$

The second relation in (17) gives

$$\lambda^2 - 1 = \left(\frac{r}{L_p}\right)^2 - 1 = 3 \, \alpha \quad (19)$$

the root $\alpha_- = -l-1$ is discarded because it furnishes unphysical $r^2 < 0$ solutions in eq-(19). Hence one learns from eqs-(18,19) that

$$\alpha_+ \ = \ l \ \Rightarrow \ \lambda^2 \ = \ (\frac{r}{L_p})^2 \ = \ 3l \ + 1 \ \Rightarrow \ r_l^2 \ = \ (3l+1) \ L_p^2 \qquad (20)$$

Eq-(20) is the one which provides the (square of the) radial spectrum in terms of the angular momentum quantum number $l = 0, 1, 2, \ldots$. We may notice how the angular momentum l appears *linearly* in (20) as desired. There was no need to combine two sets of angular momentum operators, J_{i4} alone suffices. Area quantization in terms of angular momentum (spin) $A \sim L_p^2 \sqrt{j(j+1)}$ has also been found in Loop Quantum Gravity using Penrose Spin Networks.

Let us for the moment use the expression for the Schwarzschild black hole horizon radius $r = r_H = 2GM$, with $G = L_p^2$ in eq-(20), and afterwards, we shall focus more appropriately and correctly on the horizon radii of the rotating Kerr black hole solution with mass M *and* angular momentum J (with $\hbar = 1$). Using the radius-angular momentum relation found in eq-(20) and $r = r_H = 2GM$ one arrives at

$$\frac{4G^2M^2}{L_p^2} \ = \ 4L_p^2M^2 \ = \ 3l \ + 1 \ \Rightarrow \ l \ = \ \frac{4L_p^2}{3} \ M^2 \ - \frac{1}{3} \qquad (21)$$

Eq-(21) has the same Regge behavior of a bosonic string $J = \alpha' M^2 + \alpha_0$, the only difference is that the Regge intercept is *negative* $-\frac{1}{3}$; the slope is $\frac{4}{3}$ in $L_p^2 = 1$ units, and the angular momentum takes *discrete* values $J = l = 0, 1, 2, \ldots$. The most salient feature of the Regge trajectory result of eq-(21) is that it is *valid* for *all* values of $J = l$. An analogous Regge type behavior, but only valid in the very large l limit, with a negative intercept, was obtained more recently by [9] using very *different* methods than the ones presented in this work in their study of horizons and wave functions of Planckian Quantum black holes.

A Kerr black hole is described in term of its mass M and non-vanishing $J \neq 0$. Thus, the discrete values of $J = l$ are now given by positive integers $l = 1, 2, 3, \ldots$, where the $l = 0$ value is excluded. The radius of the inner and outer horizons of a Kerr black hole are given by ($c = 1$)

$$r_\pm \ = \ \frac{2GM \ \pm \ \sqrt{(2GM)^2 - 4(\frac{J}{M})^2}}{2} \qquad (22)$$

Inserting the expression for the outer horizon r_+ (22), for discrete values of $J = l = 1, 2, 3, \ldots$, and corresponding to discrete values of the mass M_l, directly into eq-(20) gives now

$$\left(\frac{2GM_l \ + \ \sqrt{(2GM_l)^2 - 4(\frac{l}{M_l})^2}}{2L_p} \right)^2 \ = \ 3l \ + 1 \qquad (23)$$

Eq-(23) is the one which renders the spectrum. Namely, it is the one which provides the discrete mass-angular momentum relation M_l vs l, after *eliminating* one variable in terms of the other. Eliminating l in terms of M_l from eq-(23) leads to a very complicated *quartic* order algebraic equation in l, and whose four roots determine the functional forms of $l = l(M_l)$, furnishing a nonlinear generalization of the Regge trajectories.

Since this is a very complicated task we shall adopt another strategy and focus on the extremal Kerr black hole solution that occurs when the outer and inner horizon radius coincide $r_+ = r_- = GM$. In other words, when $GM_l^2 = l$ the square root terms of eq-(22) vanish and yields $r_\pm = GM$. A careful inspection reveals that the extremal condition $GM_l^2 = l$ can only be implemented at the expense of replacing the fixed Planck scale L_p for a running scale $L(M)$ not unlike it occurs in the Renormalization Group program. Now one has *two* very *simple* equations, instead of one very complicated eq-(23), of the form

$$(\frac{GM_l}{L})^2 \ = \ 3l \ +1, \ \ GM_l^2 \ = \ l, \ \ (G = L_p^2) \tag{24}$$

From which one obtains the $L = L(M_l)$ relation

$$L^2 \ = \ L_p^2 \ \frac{GM_l^2}{3GM_l^2+1} \ = \ L_p^2 \ \frac{l}{3l+1} \tag{25}$$

In the $M_l, l \to \infty$ limit one ends up with $L(\infty) = \frac{L_p}{\sqrt{3}} < L_p$. Eq-(25) can be interpreted as the relation determining the spectral length $L = L(M^2)$, where now L is a running scale like it occurs in the Renormalization Group program.

Another route that one can take is by *redefining* the angular momentum by simply writing $3l + 1 = l' = J'$ and restricting the discrete values of J to the set $J' = 1, 4, 7, 10, \ldots$. In this way one would have retained the Planck scale L_p as the minimal one and mantained the extremality condition $GM^2 = J'$ for those *restricted* values of J'. In this case, the mass spectrum $M_{n'}^2 = n' M_p^2$ is just *truncated* to the values with $J' = n' = 1, 4, 7, 10, \ldots$.

Redefining the angular momentum by writing $3l + 1 = l' = J'$ is not so farfetched. The quantum notion of the general relativistic angular momentum in asymptotically flat spacetimes is a very subtle one [6]. Angular momentum at null infinity has a supertranslation *ambiguity* from the lack of a preferred Poincare group and a similar ambiguity when the center-of-mass position changes as linear momentum is radiated. Quantizing angular momentum requires a supertranslation-invariant angular momentum in the center-of-mass frame. The authors [6] have recently proposed one such definition of angular momentum involving nonlocal quantities on the 2-sphere, which could be used to define a quantum notion of general-relativistic angular momentum.

Proceeding with eq-(25), because l cannot be zero for a Kerr black hole, one cannot have $L = 0$ in eq-(25). Having $L = 0$ would have lead to *commuting* spacetime coordinates $[x_i, x_j] = iL^2 J_{ij} = 0$ and contradicting the basic premise of this work based on the noncommutativity of spacetime. An $l = 0$ value also gives a zero mass $M_{l=0} = 0$ in the spectrum and one would not have had a black hole. Hence, in this model of the quantum Kerr black hole, a nonzero mass M_l is correlated to a nonzero angular momentum l, and in turn, is directly correlated to a cutoff scale $L(M_l) \neq 0$ that measures the strength of the noncommutativity of spacetime $[x_i, x_j] = iL^2 J_{ij} \neq 0$.

The theory of Relativity achieved a space-time unification. A theory of quantum gravity must achieve a space-time-matter unification. This is realized very naturally in string theory. The embedding coordinates of the string's world-sheet into a background spacetime are scalar fields (matter) from the world-sheet point of view. A more thorough discussion on this space-time-matter unification within the context of Born's Reciprocal Relativity Theory in Phase Spaces can be found in [11].

Since the minimum value of l is now $l = 1$, there is a *lower* bound to the mass $GM_{l=1}^2 = l = 1 \Rightarrow M_{l=1} = M_p$ given precisely by the Planck mass. In this case the value of L turns out to be $L(M_p) = \frac{L_p}{2} < L_p$. In all cases, L and L_p have the *same* order of magnitude and obey $L < L_p$. In string theory, for example, the string length does not coincide with the Planck scale.

To sum up, by choosing a running ultraviolet cutoff $L = L(M_l) < L_p$ displayed by eq-(25), the radial spectrum associated with a fuzzy sphere in a Noncommutative spacetime (phase space) characterized by the Yang algebra, leads *exactly* to the Regge-like spectrum $GM_l^2 = l = 1, 2, 3, \ldots$, for *all* positive values of l, and which is consistent with the extremal quantum Kerr black hole solution that occurs when the outer and inner horizon radius coincide $r_+ = r_- = GM$. The condition $GM_l^2 = l$ ($G = L_p^2 = M_p^{-2}$) is tantamount to the mass-angular momentum relation $M_l^2 = lM_p^2$ and which has precisely the same form as $M_n^2 = nM_p^2$, with $n = 1, 2, 3, \ldots$, implying then that the (extremal) horizon area is quantized in multiples of the minimal Planck area. This result spans microscopic (small values of n) and macroscopic (large values of n) Kerr black holes. We also found another route by redefining $3l + 1 = l' = J'$ such that the mass spectrum $M_{n'}^2 = n'M_p^2$ is now truncated to the values with $J' = n' = 1, 4, 7, 10, \ldots$, and the Planck scale is maintained as the minimal one.

One should emphasize that one cannot forget the original and more complicated eq-(23) involving *only* the Planck scale L_p, and the outer horizon radius $r_+ > GM$, leading to a *nonlinear* generalization of the Regge spectrum. The discrete values of J are constrained now by the *domain* $1 \leq J \leq GM^2$ instead of being given by the extremality condition $GM_l^2 = J = l \geq 1$. For example, setting $l = 0$ in eq-(23) yields automatically $M_{l=0} = \frac{1}{2}M_p$. Setting $l = 1$ leads to a cubic equation for $M_{l=1} \equiv M_1$ given by

$$4L_p^3 M_1^3 \ - \ 4L_p^2 M_1^2 \ - 1 \ = \ 0 \tag{26}$$

and whose solution for M_1 lies in the interval $M_p < M_1 < 2M_p$. And, in general, one would have to find the four roots of a quartic algebraic equation in $J = l$, and sort out which one of those four roots corresponds to a meaningful physical trajectory $J = l = l(M_l)$ such that l increases with M_l. There may be complex roots appearing in complex conjugate pairs which must be discarded.

Having analyzed the spectrum, let us turn to the eigenfunctions (wave functions)

$$\Psi_{lm}(R, X_4, \theta, \varphi) \;=\; R^l \; Y_{lm}(\theta, \varphi) \; X_4 \tag{27}$$

The solution converges at $R = 0$ since $\alpha = l \geq 0$, but diverges at $R = X_4 = \infty$ leading to a non-normalizable wave function (not square-integrable). This is precisely where the introduction of the *infrared* cutoff scale $\mathcal{L}$ for R and X_4 associated with the Yang algebra becomes important. Choosing a finite segment interval for X_4 lying in $[-\mathcal{L}, +\mathcal{L}]$, and a finite radius $R = \mathcal{L}$ for the classical sphere described by the classical commuting coordinates X_1, X_2, X_3, one can then properly normalize the wave function, as it occurs with the standard plane wave solutions $\Psi = e^{ipx}$ in QM. The latter are not square-integrable unless we introduce an infrared cutoff and place the free particle in a box of finite size. The normalization factor N_{lm} is obtained from the condition

$$N^2_{lm} \int_0^{\mathcal{L}} R^{2l} \; dR \int \; Y^2_{lm}(\theta, \varphi) \; sin(\theta) \; d\theta \; d\varphi \int_{-\mathcal{L}}^{\mathcal{L}} \; (X_4)^2 \; dX_4 \;=\; 1 \tag{28}$$

To find the most general solutions to the eigenvalue equation (10) is very difficult. In this work we chose the simplest separation of variables possible leading to satisfactory physical results.

In the work by [7] an explicit representation of the Snyder algebra was used in terms of the compact *momentum* ρ_i variable as follows

$$x_i \;=\; i\hbar \; \sqrt{1 - \kappa^2 \rho^2} \; \frac{\partial}{\partial \rho_i}, \quad p_i \;=\; \frac{\rho_i}{\sqrt{1 - \kappa^2 \rho^2}}, \quad 0 \;<\; \rho^2 \;<\; \frac{1}{\kappa^2} \tag{29}$$

when $\kappa^2 = 0$ one recovers the standard representation of the $\{x_i \sim \frac{\partial}{\partial p_i}, p_i\}$; $\{p_i \sim \frac{\partial}{\partial x_i}, x_i\}$ operators of QM in commutative spacetimes $[x_i, x_j] = [p_i, p_j] = 0$.

The representation (29) used by [7] can be easily extended to the generic case of the sphere S^d in higher dimensions in order to be able to understand the structure of the eigenvalues and of the eigenfunctions in an exhaustive way. The eigenfunctions turned out to be given in terms of hypergeometric series, which upon *truncation*, lead to the Jacobi polynomials and generated the following areal spectrum (in $d = 3$) $r^2_{N,l} = L^2_p[N(N+2) - l(l+1)]$, where N is the main quantum number given by $N = 2n + l$. When $N = l$, Valtancoli obtained the Bekenstein quantization condition for the area of the event horizon $4\pi r^2_N = 4\pi N L^2_p$ [7].

A Quantum-Mechanical model of the Kerr-Newman black hole was studied a while back by [8]. The classical Hamiltonian written in terms of mass M, the electric charge Q and angular momentum J of the black hole variables, and their conjugate momenta, is replaced by the corresponding self-adjoint Hamiltonian operator and an eigenvalue equation for the Arnowitt-Deser-Misner (ADM) mass of the hole, from the point of view of a distant observer at rest, is obtained. In a certain very restricted sense, this eigenvalue equation may be viewed as a sort of "Schrodinger equation of black holes". Their "Schrodinger equation" implies that the ADM mass, electric charge and angular momentum spectra of black holes are discrete, and the mass spectrum is bounded from below. Moreover, the spectrum of the quantities $M, Q, a = \frac{J}{M}$ is strictly positive when an appropriate self-adjoint extension is chosen. The WKB analysis yields the result that the *large* eigenvalues of M, Q and a are of the form $\sqrt{2n}$, n is an integer. It turns out that this result is closely related to Bekenstein's proposal on the discrete horizon area spectrum of the black hole. The Kerr-Newman black hole solutions with the outer/inner horizons $r_\pm(M, Q, J)$ was not discussed in the present work and warrants further investigation.

In this work we have arrived at similar results as those in [5], [7], [8], [9], and *more*, but directly from the Yang's algebra of noncommutative phase space by studying the spectrum of the quantum black hole horizon's radius-squared operator $\mathbf{r}^2$. This is also consistent with the basic idea of [9] that the event horizon of a quantum black hole undergoes quantum oscillations.

Are there transitions among the states of different mass (different horizon areas) ? And if so, are they mediated by photons mimicking the analog of Hawking radiation ? In the $3D$ spherically symmetric QM oscillator the expectation value of r^2 is proportional to the *energy* as result of the quantum virial theorem, beacuse r^2 is just proportional to the potential $V(r)$ that is an intrinsic part of the Hamiltonian. Thus, QM transitions among the energy eigenstates will occur by the emission/absorption of photons.

However, the operator $\mathbf{r}^2$ (the *areal* operator) for the Noncommuting spacetime case studied here is not part of a Hamiltonian. And since the wave-function Ψ_{lm} of eq-(27) does not correspond to an energy eigenstate, it is not clear if the emission/absorption of photons occurs. On the other hand, the study of a QM oscillator (and a Dirac oscillator) based on a Hamiltonian operator defined over a Noncommuting spacetime, and described by the Snyder algebra using the same representation as (29), can be found in [7].

Valtancoli [7] discussed in detail that the quantization of d isotropic oscillators in a Noncommutative Snyder geometry gives rise to *two* relevant quantum numbers, from which one can deduce that the residual degeneracy of the states is reduced to $d-2$ degrees of freedom. The spectrum contains, besides a *linear* term in the main quantum number N (that in the commutative limit is the sum of the single particle quantum numbers n_i), a *quadratic* term dependent also on a secondary quantum number k, such as $N-k$ is an even positive integer number. In this case, an emission/absorption of photons should occur resulting in transitions between these energy eigenstates.

The modified uncertainty relations due to the Yang algebra are obtained from the Robertson-Schrodinger inequalities (we are dropping the bold face font in the x, p coordinates)

$$\Delta x_i \ \Delta p_j \ \geq \ \frac{1}{2} \ |\langle \ [x_i, \ p_j] \ \rangle| \ \Rightarrow \ \Delta x_i \ \Delta p_j \ \geq \ \frac{L_p}{2\mathcal{L}} \ \delta_{ij} \ |\langle \ J_{45} \ \rangle| \ = \ \frac{|\tilde{m}|\hbar L_p}{2\mathcal{L}} \ \delta_{ij}$$

$$\Delta x_o \ \Delta p_o \ \geq \ \frac{|\tilde{m}|\hbar L_p}{2\mathcal{L}} \tag{30}$$

after evaluating the expectation values with respect to the normalized eigenfunctions of J_{45} given by $\frac{1}{\sqrt{2\pi}} e^{i\tilde{m}\phi}$.

In general one can write the modifications of the Weyl-Heisenberg algebra as [12] $[\mathbf{x}_\mu, \mathbf{p}_\nu] = i\hbar g_{\mu\nu}(\mathbf{x}, \mathbf{p})$ where $g_{\mu\nu}$ is a 4×4 matrix whose entries are comprised of operator-valued quantities. The operator-valued metric is associated with a curved phase space and whose geometry is best represented by Finsler geometry instead of a Riemannian one [12]. When the operator-valued entries of $g_{\mu\nu}$ are given by polynomials in the $\mathbf{x}, \mathbf{p}$ operators, one must perform a judicious Weyl-ordering in order to ensure the Hermiticity of the operator-valued metric. A key example of a modified Weyl-Heisenberg algebra, respecting the Born's reciprocity symmetry $x \leftrightarrow p$, and leading to yet another example of a generalized uncertainty relation, is

$$[\mathbf{x}_\mu, \mathbf{p}_\nu] \ = \ i\hbar \ \left(\ \eta_{\mu\nu} \ + \ \frac{\mathbf{x}_{(\mu}\mathbf{x}_{\nu)}}{\mathcal{L}^2} \ + \ L_p^2 \ \mathbf{p}_{(\mu}\mathbf{p}_{\nu)} \ + \ \frac{L_p}{\mathcal{L}} \ (\mathbf{x}_\nu \mathbf{p}_\mu + \mathbf{p}_\mu \mathbf{x}_\nu) \ \right) \quad (31)$$

with $\mathbf{x}_{(\mu}\mathbf{x}_{\nu)} \equiv \frac{1}{2}(\mathbf{x}_\mu\mathbf{x}_\nu + \mathbf{x}_\nu\mathbf{x}_\mu), \ldots .....$ to ensure that the whole expression inside the parenthesis in the right-hand-side of (31) is Hermitian $(\ldots)^\dagger = (\ldots)$. Eq-(31) also displays an ultraviolet/infrared entanglement in the mixed $\mathbf{xp} + \mathbf{px}$ terms involving the ratio of the two scales $\frac{L_p}{\mathcal{L}}$. As a result of the modified Weyl-Heisenberg algebra (31), the *realization* of the Lorentz generators are also modified

$$J_{\mu\nu} \ = \ \frac{1}{2} \ (\mathbf{x}_\mu \ \mathbf{p}_\nu \ - \ \mathbf{x}_\nu \ \mathbf{p}_\mu \ + \ \mathbf{p}_\nu \ \mathbf{x}_\mu \ - \ \mathbf{p}_\mu \ \mathbf{x}_\nu) \quad (32)$$

due to the Weyl-ordering prescription. The most salient feature of eqs-(31,32) is that despite there are *modifications* of the *realization* of the Lorentz generators (32), the Lorentz algebra itself remains *unmodified*, *undeformed*. Consequently, there is no need to abandon Lorentz invariance.

Another important feature is the issue of "holography" and "hidden" dimensions. Our findings in this work are based in recasting Yang's Noncommutative algebra, associated with an $8D$ noncommuting phase space (involving $\mathbf{x}_\mu, \mathbf{p}_\nu, \mu, \nu = 0, 1, 2, 3$) in terms of the standard *undeformed* realizations of the Lorentz algebra generators J_{AB} in *higher* dimensions. The coordinates describing the $6D$ spacetime ($12D$ phase space) are X_A, P_B, with $A, B = 0, 1, 2, \ldots, 5$.

The number of J_{AB} generators of the $so(5,1)$ Lorentz algebra in $6D$ is 15, which is the same as the number of generators of the *conformal* algebra $so(4,2)$ in $4D$. This interplay between the conformal algebra in $4D$ and the Lorentz algebra in higher $6D$, is reminiscent of holography and the gauge/gravity, AdS/CFT correspondence[14]. And more importantly, it is also reminiscent of an inherent Classical/Quantum duality. In [12] we found examples where Bohm's quantum potential in QM has a one-to-one correspondence with the classical Newtonian gravitational potential. For an early discussion of holography and the Yang algebra see [13].

We hope that the findings in this work, relating the Regge-like spectrum $l = GM^2$ and the quantized area of black hole horizons in Planck bits, via the Yang algebra in Noncommutative phase spaces, will help us elucidate some of the impending issues pertaining the black hole information paradox and the role that string theory and quantum information will play in its resolution. Concluding, Noncommutative and Nonassociative Gravity appear to be very appealing avenues of research in the future of Quantum Gravity.

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