On propagation of light-ray and Sagnac effect in Kerr-Newman-NUT spacetime[1]

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Abstract

The paper explores the light-ray propagation and Sagnac effect in Kerr-Newman-NUT spacetime. It has been analyzed the spacetime curvature structure of these solutions and represented that the Kerr-Newman-NUT spacetime is one of the exact analytical solution for rotating regular black hole. The area of the horizon and ergosphere of the black hole has been explicitly derived. The electromagnetic feature of the Kerr-Newman-NUT black hole has been also discussed. The effect of the NUT parameter in capture cross section of photon by the black hole so called the shadow black hole has been analyzed. Finally, the Sagnac effect in the Kerr-Newman-NUT space has explicitly discussed.

Keywords: Kerr-Newman-NUT spacetime, Sagnac effect, time delay, black hole shadow

1. Introduction

The Sagnac effect is quit-well known phenomenon, not only in general relativity (GR) but also in special relativity (SR), related to the propagation of the light-ray near rotating system. It is that in a rotating ring interferometer one counter-propagating wave acquires a phase shift relative to another counter-propagating wave, which is directly proportional to the angular velocity of rotation, the area covered by the interferometer, and the wave frequency. The Sagnac effect, along with the Michelson-Morley experiments [1, 2] is one of the fundamental experiments of the theory relativity. This effect is well known and thoroughly studied in [3]. Currently, the Sagnac effect has been recorded (in addition to the optical range) for radio waves and X-rays [4, 5], as well as for waves of non-electromagnetic nature, de-Broglie waves of material particles.

It is known that the Sagnac effect also finds an adequate explanation within the framework of general theory of relativity (GR) (see, for example, [6]). In this case, the metric tensor in the frame of reference accompanying the rotation of the interferometer is used for calculations. As well as have been input general relativistic corrections to the Sagnac effect in the Kerr spacetime [7]. Also calculated the upper limit for the NUT charge [8] and correlation between the Mach principle and the Sagnac effect [9, 10] has been investigated using that method. General relativistic quantum interference effects in the slowly rotating NUT space-time as the Sagnac effect and the phase shift effect of interfering particle in neutron interferometer are considered in [11]. In work [12], it has been calculated the phase shift for charged particle interference experiment in a more electrovac Plebyanski-Demanski black hole spacetime.

The Sagnac effect also can be considered as a consequence of different time dilation in rotating frames of reference accompanying the motion of the phase fronts of counter -propagating waves, which is due to the difference in the

[1] It can be also found that in some of literatures, Kerr-Newman-NUT spacetime is also called as Kerr-Newman-Taub-NUT spacetime.
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phase velocities of counter propagating waves relative to the inertial frame of reference during the rotation of the ring interferometer. The same result can be obtained proceeding from the difference between the Newtonian (non-relativistic) scalar gravitational potential of centrifugal forces for counter propagating waves in the frames of reference considered above, which is a consequence of the equivalence principle.

Novel feature of the Kerr-Newman-NUT spacetime has been discussed in Ref. [13, 14]. Null-geodesic motion has been studied in [15, 16, 17]. The collision of two particles near the horizon of Kerr-Newman-NUT black hole has been studied in [18]. Photon-region and Shadow of the black hole has been studied in [19]. Thermodynamic properties of black hole and Hawking radiation in Kerr-Newman-NUT space has been discussed [20, 21, 22, 23, 24, 25]. In Ref. [26] the effects associated with the gravitomagnetic monopole moment of the source on the motion of test particles and electromagnetic waves has been studied. [27]. The detailed analysis of charged test particle motion in the equatorial plane in Kerr-Newman-Taub-NUT spacetime has been investigated in [28, 29, 30, 19]. The various optical properties of black holes like the black hole shadow has been discussed in Refs. [31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 17].

In this paper, we are interested in demonstrating the Sagnac effect in the Kerr-Newman-NUT spacetime. The paper is organized in the following way. In Sec. 2, we provide in very detailed analysis of the properties of the Kerr-Newman-NUT spacetime, namely, structure of horizon and ergosphere, the curvature invariants and the effective mass of black hole. The electromagnetic fields in the vicinity of the black hole is considered. Section 3 is devoted to study the Sagnac effect. Finally, in Sec. 4, we summarize obtained results and give future outlook related to the present work.

2. Kerr-Newman-NUT spacetime

In Boyer-Lindquist coordinates $x^\alpha = (t, r, \theta, \phi)$, the Kerr-Newman-NUT spacetime can be expressed as [2]

$$ds^2 = \frac{\Delta}{\Sigma}(dt - \chi d\phi)^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2 + l^2)d\phi - adt\right]^2 , \tag{1}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2 - l^2$, $\Sigma = r^2 + (l + a\cos\theta)^2$ and $\chi = a\sin^2\theta - 2l\cos\theta$, with three free parameters: the total mass M, specific spin a, charge Q and gravitomagnetic monopole moment l of the black hole, respectively. The components of the associated vector potential of the electromagnetic field are given by

$$A_\alpha = \frac{rQ}{\Sigma}(-1, 0, 0, \chi) . \tag{2}$$

The outer spacelike horizon of the black hole is determined as

$$r_+ = M + \sqrt{M^2 + l^2 - Q^2 - a^2} , \tag{3}$$

which is larger than the Schwarzschild radius ($r_g = 2M$) at $l^2 > a^2 + Q^2$, smaller at $l^2 < a^2 + Q^2$ and equal when $l^2 = a^2 + Q^2$, respectively. Note that in order to be exists horizon, the black hole's parameters should satisfy the following condition $M^2 + l^2 \geq a^2 + Q^2$. One has to emphasise that rotating black hole also is characterized by another type of radius, so-called ergosphere which has a form:

$$r_{\rm erg} = M + \sqrt{M^2 + l^2 - Q^2 - a^2\cos^2\theta} \geq r_+ . \tag{4}$$

The surfaces area of the horizon and ergosphere of the black hole, i.e. $A_+ = A(r_+)$ and $A_{\rm erg} = A(r_{\rm erg})$, can be determined as follows:

$$A(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{g_{\theta\theta}(r,\theta) g_{\phi\phi}(r,\theta)} . \tag{5}$$

[2]Throughout the paper, we use a spacelike signature $(+, -, -, -)$, a system of units in which $G = c = 1$. Greek indices are taken to run from 0 to 3, and Latin indices from 1 to 3.

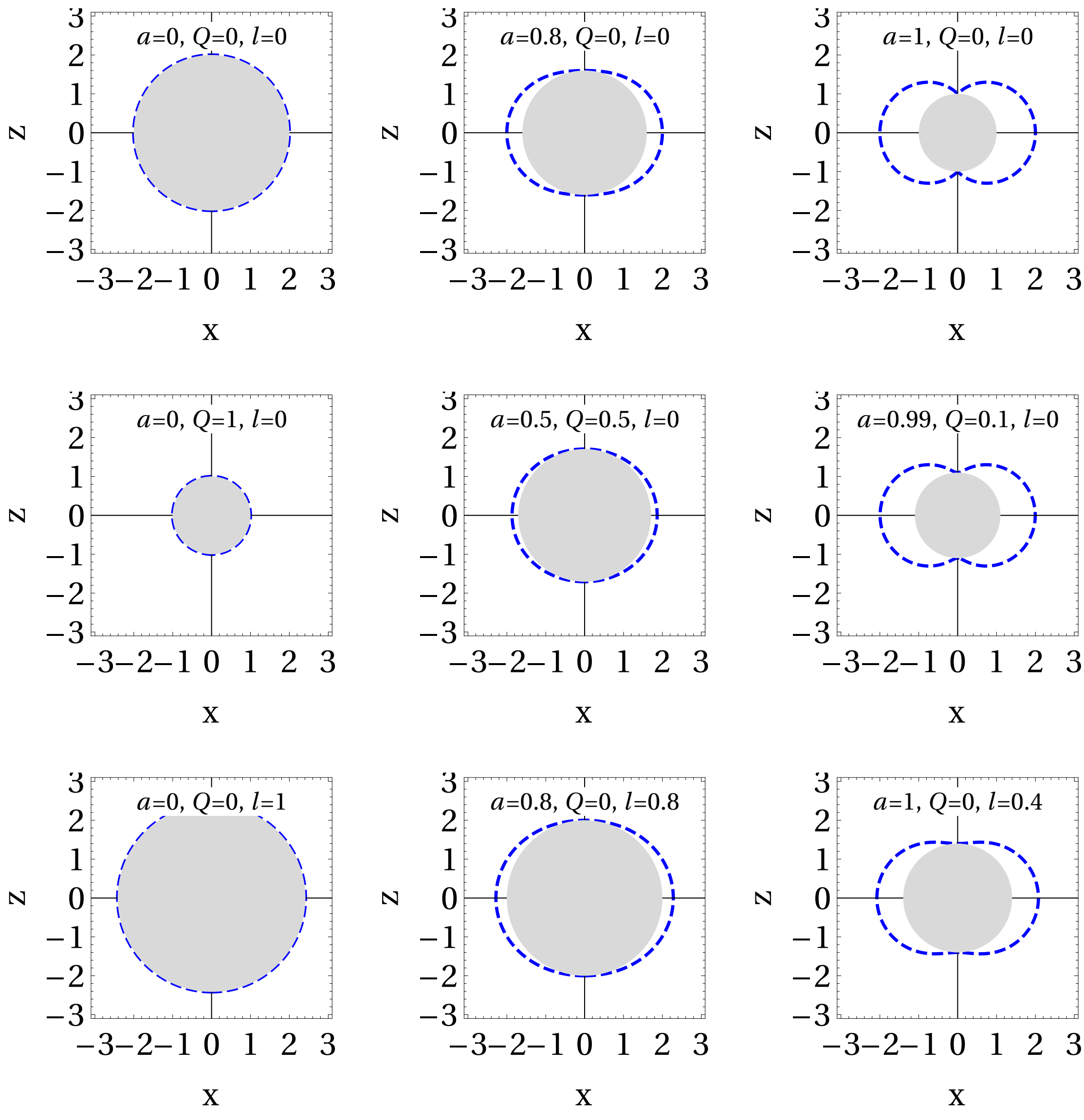

Figure 1: The horizon and ergosphere are illustrated for different values of the black hole's parameters in $(x-z)$ plane. The shaved regions represent horizon, while dashed lines correspond to ergosphere of the black hole .

Hereafter introducing new variable $x = \cos\theta$ in equation (5) and performing simple algebraic manipulations, one can find that the area of horizon is calculated by

$$A_+ = 2\pi \int_{-1}^{1} dx \sqrt{(r_+^2 + a^2)^2 + \Delta_+ a^2 (x^2 - 1)} = 4\pi(r_+^2 + a^2)\ , \tag{6}$$

while the area of the ergosphere takes the following integral form:

$$\begin{aligned} A_{\rm erg} &= 2\pi \int_{-1}^{1} dx \sqrt{(r_{\rm erg}^2 + a^2)^2 + a^4(x^2-1)^2} \\ &= 2\pi \int_{-1}^{1} dx \sqrt{\left[\left(M + \sqrt{M^2 + l^2 - Q^2 - a^2x^2}\right)^2 + a^2\right]^2 + a^4(x^2-1)^2}\ . \end{aligned} \tag{7}$$

Unfortunately, it is impossible to evaluate the integral in (7), however, for leading order of spin parameter, one can have

$$A_{\rm erg} = 2\pi\left[r_+^2 + 2M^2 + l^2 - Q^2 + \frac{7a^2}{3} + \frac{2aM}{\eta^2}\sin^{-1}\eta\right]\ , \tag{8}$$

where $\eta = \frac{a}{\sqrt{M^2+l^2-Q^2}}$.

Figure 1 shows the plots of the horizon and ergosphere of the black hole in $(x - z)$ plane for different values of the parameters a, Q and l. One can see from Fig. 1 that difference of shape of the horizon and ergosphere is strongly depends on the spin parameter, while other parameters (Q, l) change size f the horizon and ergosphere of the Kerr-Newman-NUT black hole.

2.1. *Effective mass*

It is well known that there is no way to accurately determine the energy of systems in general relativity (GR). The difficulty arises from the fact that the energy-momentum tensor does not include contributions from gravitational field energy and gravitational waves. However, there is exist in literature the efficient way defining the total energy of an isolated physical system. In Ref [48], the definition for the effective mass of the rotating black hole has been nicely discussed. It is also interesting to present the effective mass for the Kerr-Newman-NUT black hole, which can be determined as [3]

$$M_{\rm eff} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(\frac{\Delta'}{2} - \frac{\Delta}{2}\frac{g'_{\phi\phi}}{g_{\phi\phi}}\right)\ , \tag{9}$$

where prime denotes the derivative with respect to radial coordinate r. The explicit form of the effective mass in equation (9) for Kerr-Newman-NUT metric is found as

$$\begin{aligned} M_{\rm eff} = \frac{1}{2}\int_{-1}^{1} dx &\left\{\frac{r\Delta}{r^2 + (l + ax)^2} - \left(r - M - \frac{2r\Delta}{r^2 + a^2 + l^2}\right)\right. \\ &\left.\times\left[1 + \frac{\Delta}{(r^2 + a^2 + l^2)^2}\left((ax + 2l)^2 - a^2 + \frac{4l^2}{x^2 - 1}\right)\right]^{-1}\right\}\ . \end{aligned} \tag{10}$$

where $x = \cos\theta$ is already defined in previous calculations. It is easy to see from equation (10) that because of NUT parameter l, evaluating of the above integral is quite tricky. However, absence of NUT parameter, i.e. $l = 0$, with

[3] The effective mass of the black hole is also defined as

$$M_{\rm eff} = \frac{\Delta}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left(\frac{\Delta'}{\Delta} - \frac{g'_{\phi\phi}}{g_{\phi\phi}}\right) = \frac{\Delta}{4}\int_{-1}^{1} dx \left(\ln\frac{\Delta}{g_{\phi\phi}}\right)'\ .$$

$\Delta = r^2 - 2Mr + a^2 + Q^2$, one can obtain

$$M_{\rm eff} = \frac{\Delta}{a}\tan^{-1}\left(\frac{a}{r}\right) + \frac{\left(r^2+a^2\right)\left[\left(r^2+a^2\right)(r-M) - 2r\Delta\right]}{a\sqrt{\Delta\left[\left(r^2+a^2\right)^2 - a^2\Delta\right]}}\tan^{-1}\left(\frac{a\sqrt{\Delta}}{\sqrt{\left(r^2+a^2\right)^2 - a^2\Delta}}\right), \tag{11}$$

which coincides with Ref. [48], while in the leading order of the black hole parameters, the expression for the effective mass reads

$$M_{\rm eff} = M + \frac{l^2}{r} - \frac{Q^2}{r} - \frac{a^2}{r}\ . \tag{12}$$

The presence of spin a, charge Q and gravitomagnetic charge l reduces the effective mass of the hole. This reduction is maximum at the horizon. From the definition of effective mass, it is clear that it is the measure of gravitational charge contained inside the sphere of radius r at which it is being evaluated. When evaluated at the horizon, it represents the gravitational mass contained inside the horizon, which is related to the surface gravity (temperature) of the hole by. Thus, the temperature and the gravitational charge contained in the horizon, as defined here, are very intimately related. Since the charge contained in the horizon goes to zero for maximal rotation (charge), so does the temperature of the hole. That is, if there is no charge inside, the hole should have zero temperature.

2.2. Curvature invariants

It is worth notice that the Kerr-Newman-NUT metric describes the exact analytical solution of regular rotating black hole's spacetime. In order to get a better idea about spacetime structure, one has to analyses the curvature invariants, described by Ricci scalar R, Ricci square $R_{\alpha\beta}R^{\alpha\beta}$ [4] and Kretschman scalar $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. It is well-known that the Kerr-Newman-NUT spacetime belongs Ricci flat solution of the Einstein-Maxwell equations, which means Ricci scalar vanises i.e. $R = 0$. However, other two curvature scalars do not vanish. The explicit form of the Ricci square and Kretchman scalar are not presented due to cumbersome expressions. However, at the origin $r = 0$ and $\theta = \pi/2$, the curvature invariants take the form

$$\lim_{r\to 0}\left(\lim_{\theta\to\pi/2} R_{\alpha\beta}R^{\alpha\beta}\right) = \frac{4Q^4}{l^8}\ , \tag{13}$$

$$\lim_{r\to 0}\left(\lim_{\theta\to\pi/2} K\right) = \frac{48}{l^4}\left(1 - \frac{M^2+2Q^2}{l^2} + \frac{7Q^4}{6l^4}\right)\ , \tag{14}$$

which are finite. It means that Kerr-Newman-NUT spacetime can be a candidate for exact analytical rotating regular black hole solution. It is also easy to check that even absence of the black hole mass the expressions for the curvature invariants yield

$$\lim_{M\to 0,\theta\to\pi/2} K = \frac{8\left(6l^8 - 6l^6\left(2Q^2+15r^2\right) + l^4\left(7Q^4+120Q^2r^2+90r^4\right) - 2l^2\left(17Q^4r^2+30Q^2r^4+3r^6\right) + 7Q^4r^4\right)}{\left(l^2+r^2\right)^6}\ , \tag{15}$$

In fact that absence of black hole's charge, i.e. $Q = 0$, the Kerr-Newman-NUT solution reduces to the Kerr-NUT metric which is one of the well-known vacuum solution (i.e. $T_{\alpha\beta} = 0$) of Einstein field equations. In this case, one can immediately show that the Ricci square vanishes i.e., $R_{\alpha\beta}R^{\alpha\beta} = 0$, while other scalar invariants yield

$$\lim_{r\to 0}\left(\lim_{\theta\to\pi/2} K\right) = \frac{48}{l^4}\left(1 - \frac{M^2}{l^2}\right)\ . \tag{16}$$

[4]The Ricci square is realated to the electromagnetic field tensor as

$$R_{\alpha\beta}R^{\alpha\beta} = 4F_{\alpha\mu}F^{\alpha\nu}F_{\beta}^{\ \mu}F^{\beta}_{\ \nu} - F_{\alpha\beta}F^{\alpha\beta}\ .$$

which is again finite at the origin. It is easy to show that even for $M = 0$ the Kretschmann scalar is nonzero

$$\lim_{M\to 0}\lim_{\theta\to\pi/2} K = \frac{48l^2\left(l^6 - 15l^4r^2 + 15l^2r^4 - r^6\right)}{(l^2+r^2)^6} . \tag{17}$$

Note that the same result can be obtained in Taub-NUT spacetime which is described by metric (43) in the following limiting cases $a = 0$ and $Q = 0$. Finally, from all facts above one can conclude that Kerr-Taub-NUT, as well as, Taub-NUT spacetimes can be candidate for exact analytical rotating and static regular black hole solutions.

2.3. *Electromagnetic fields configuration*

Another interesting feature of Kerr-Newman-NUT spacetime is the electromagnetic field configuration near the black hole. To analysis it, one can determine the components of the electromagnetic field tensor (i.e. $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$). Using the expression (2), one can determine the component of the the electromagnetic fields tensor as

$$F_{rt} = Q\frac{r^2 - (l + a\cos\theta)^2}{\Sigma^2} , \qquad F_{r\phi} = -\chi F_{rt} , \tag{18}$$

$$F_{\theta t} = -\frac{2Qra\sin\theta(l + a\cos\theta)}{\Sigma^2} , \qquad F_{\theta\phi} = -\frac{a^2 + l^2 + r^2}{a} F_{\theta t} , \tag{19}$$

while the components of the electromagnetic fields $(E^{\hat{i}}, B^{\hat{i}})$ measured by zero angular momentum (ZAMO) frame can be expressed as [5]

$$E^{\hat{r}} = \frac{Q}{\Sigma^2}\left(2r^2 - \Sigma\right)\sqrt{1 + \frac{a\chi}{\Sigma}} , \tag{20}$$

$$E^{\hat{\theta}} = \frac{2Qr\sqrt{a\chi}}{\Sigma^{5/2}}(l + a\cos\theta) , \tag{21}$$

$$E^{\hat{\phi}} = 0 , \tag{22}$$

$$B^{\hat{r}} = \frac{2Q}{\Sigma^3}\sqrt{1 - \frac{a^2\sin^2\theta}{\Delta}}\left[2r(\Sigma + a\chi)(l + a\cos\theta) + \frac{\chi}{\sin\theta}\sqrt{\Delta}\left(\Sigma - 2r^2\right)\right] , \tag{23}$$

$$B^{\hat{\theta}} = \frac{4Qr(\Sigma + a\chi)(l + a\cos\theta)}{\Sigma^3}\left[\sqrt{1 - \frac{a^2\sin^2\theta}{\Delta}} - \sqrt{\frac{a^2\chi^2}{\Sigma + a\chi} - \frac{a^2\sin^2\theta}{\Delta}}\right] , \tag{24}$$

$$B^{\hat{\phi}} = 0 , \tag{25}$$

Notice that the components of the electric field are produced by black hole charge itself, however, the components of the induced magnetic field produced by rotation of the black hole charge. Absence of the rotation of the black hole, i.e. $a = 0$, the radial component of the electric field remain, while the magnetic field should vanish, however, it does not, because of the NUT parameter. In case when $Q \neq 0$, $l \neq 0$ and $a = 0$, the components of the electromagnetic field can be rewritten as

$$E^{\hat{r}} = Q\frac{r^2 - l^2}{(r^2 + l^2)^2} , \quad E^{\hat{\theta}} = 0 , \quad E^{\hat{\phi}} = 0 , \tag{26}$$

$$B^{\hat{r}} = \frac{4Ql}{(r^2 + l^2)^3}\left[r(r^2 + l^2) + \sqrt{\Delta}\left(r^2 - l^2\right)\cot\theta\right] , \quad B^{\hat{\theta}} = \frac{4Qrl}{(r^2 + l^2)^2} , \quad B^{\hat{\phi}} = 0 , \tag{27}$$

One can easily see from equation (27) that even non-rotating black hole can produce the magnetic field due to the NUT parameter. In previous subsection, we mentioned that the NUT parameter plays role of mass when we calculate curvature invariants, however, here we show that it has a rotational feature of the spacetime. In order to have an idea, it is important to produce magnetic field line around black hole.

[5] The components of the electromagnetic fields measured by a local observer as

$$E^{\hat{i}} = \sqrt{-g^{ii}F_{it}\left(g^{tt}F_{it} + g^{t\phi}F_{i\phi}\right)} , \qquad B^{\hat{i}} = \frac{1}{2}\epsilon_{ijk}\sqrt{g^{jj}g^{kk}}F_{jk} .$$

2.4. Shadow of black hole

One of the hot topic is investigating the optical processes (gravitational lensing, shadow, ect...) around gravitational compact object due to the possible observation of the event horizon of the supermassive black hole in the center of our galaxy or AGN (active galactic nuclei). In fact that the light-ray (photon) propagates following the null-geodesic line in the vicinity gravitational compact objects. However, there are several method to describe photon motion in curved space. For convenience, we will use Hamilton-Jacobi formalism. The Hamilton-Jacobi equation for the propagating the light ray in curved space is given by

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0\,, \tag{28}$$

where S is the action. Using the conserved quantities, namely, the energy $\mathcal{E}$ and angular momentum $\mathcal{L}$ of the photon, the solution of the equation (28) can be decomposed as follows:

$$S = -\mathcal{E}t + \mathcal{L}\phi + S_r(r) + S_\theta(\theta)\,. \tag{29}$$

where $S_r(r)$ and $S_\theta(\theta)$ are radial and angular functions. Now by inserting the expression (29) into (28), one can obtain

$$\Delta\left(\frac{\partial S_r}{\partial r}\right)^2 + \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + \left(\frac{\mathcal{L}-\chi\mathcal{E}}{\sin\theta}\right)^2 - \frac{1}{\Delta}\left[\left(r^2+a^2+l^2\right)\mathcal{E} - a\mathcal{L}\right]^2 = 0\,. \tag{30}$$

Finally, hereafter performing simple algebraic manipulations, equation motion for photon can be expressed as

$$\Sigma\dot{t} = \mathcal{L}\left(\frac{\chi}{\sin^2\theta} - a\frac{r^2+a^2+l^2}{\Delta}\right) + \mathcal{E}\left(\frac{\left(r^2+a^2+l^2\right)^2}{\Delta} - \frac{\chi^2}{\sin^2\theta}\right), \tag{31}$$

$$\Sigma\dot{\phi} = \mathcal{L}\left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right) + \mathcal{E}\left(a\frac{r^2+a^2+l^2}{\Delta} - \frac{\chi}{\sin^2\theta}\right), \tag{32}$$

$$\Sigma\dot{r} = \sqrt{R(r)}\,, \qquad R(r) \geq 0\,, \tag{33}$$

$$\Sigma\dot{\theta} = \sqrt{T(\theta)}\,, \qquad T(\theta) \geq 0\,, \tag{34}$$

where dot donates derivative with respect to an affine parameter λ and the functions $R(r)$ and $T(\theta)$ are defined as

$$R(r) = \left[\left(r^2+a^2+l^2\right)\mathcal{E} - a\mathcal{L}\right]^2 - \left[\mathcal{K} + (\mathcal{L}-a\mathcal{E})^2\right]\Delta\,, \tag{35}$$

$$T(\theta) = \mathcal{K} + (\mathcal{L}-a\mathcal{E})^2 - \left(\frac{\mathcal{L}-\chi\mathcal{E}}{\sin\theta}\right)^2, \tag{36}$$

where $\mathcal{K}$ is the Carter constant of the motion.

When the light passes near the massive object, some part of the light will be captured by the massive object and then will move along spherical orbits defining the boundary of the shadow of the lensing object. If the photon moves along the spherical geodesics, it satisfies to the following conditions, i.e. $R(r) = 0$ and $R'(r) = 0$, or more precisely:

$$\left[\left(r^2+a^2+l^2\right)\mathcal{E} - a\mathcal{L}\right]^2 - \left[\mathcal{K} + (\mathcal{L}-a\mathcal{E})^2\right]\Delta = 0\,, \tag{37}$$

$$4r\mathcal{E}\left[\left(r^2+a^2+l^2\right)\mathcal{E} - a\mathcal{L}\right] - \left[\mathcal{K} + (\mathcal{L}-a\mathcal{E})^2\right]\Delta' = 0\,, \tag{38}$$

here prime denotes the derivation with respect to radial coordinate r. The equation (37) is responsible for the turning point of the light-ray and the equation (38) represents the radius of the spherical orbit. To determine the unstable circular orbit of the photon, we introduce the new definitions as follows $\xi = \mathcal{L}/\mathcal{E}$ and $\eta = \mathcal{K}/\mathcal{E}^2$. From the equations (37) and (38), one can find

$$\xi = \frac{r^2+a^2+l^2}{a} - \frac{4r\Delta}{a\Delta'}\,, \tag{39}$$

$$\eta = \frac{8r\Delta\left(l^2+r^2\right)}{a^2\Delta'} + \frac{16r^2\Delta\left(a^2-\Delta\right)}{a^2\Delta'^2} - \frac{\left(r^2+l^2\right)^2}{a^2}\,. \tag{40}$$

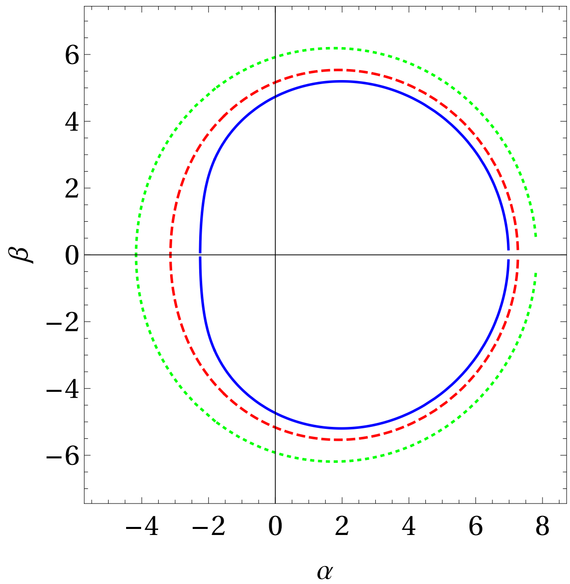

Figure 2: Dependence of a shadow of black hole on NUT parameter. The solid line represents maximally rotating black hole with $l = 0$, while the dashed line is responsible for $l = 0.5$ and the dotted line responsible for $l = 0.9$.

Now we focus on constructing the apparent shape of the Kerr-Newman-NUT black hole shadow. Before go on further, we need to introduce celestial coordinates α and β, in order to find the location of the shadow (screen) for a better visualization. The coordinate α represents the apparent perpendicular distance of image as seen from the axis of symmetry, while the coordinate β is the apparent perpendicular distance of the image from its projection on the equatorial plane. The explicit form of the celestial coordinates α and β can be introduced as

$$\alpha = \lim_{r_0 \to \infty} \left(-r_0^2 \sin\theta_0 \frac{d\phi}{dr} \right) , \tag{41}$$

$$\beta = \lim_{r_0 \to \infty} \left(-r_0^2 \frac{d\theta}{dr} \right) . \tag{42}$$

As a result we present shadow of a maximally rotating neutral Kerr-Newman-NUT black hole. Figure 2 draws the shadow of the black hole and it is shown that by increasing the NUT parameter the black hole's shadow gets larger. Because, the square of the NUT parameter stands with negative sign in the lapse function which leads to make gravitational field stronger near the black hole. Here, it is worth noting that all type of the characteristic radii such as, horizon, ISCO, photonsphere etc. increase due to the NUT parameter.

3. Sagnac effect in Kerr-Newman-NUT spacetime

The main target of the present research note is to explore the Sagnac effect in the Kerr-Newman-NUT spacetime. As we already mention that Kerr-Newman-NUT spacetime is characterized by the four free parameters, (M, a, l, Q), and from the mathematical point of view it will be bit tricky to analyses. However, in order to get correct answer for Sagnac time delay in Kerr-Newman-NUT spacetime, we will produce it in Kerr and Kerr-Newman, as well Kerr-NUT spaces, by setting $Q = l = 0$, $l = 0$ and $Q = 0$, which are previously obtained by other authors.

3.1. Equatorial orbit

First we assume that the rotation of the black hole is uniform, so that the rotation angle of the source or observer is defined as $\phi_0 = \omega_0 t$. In the case of motion in equatorial plane $\theta = \pi/2$ with constant radius $r = R =$ const, the metric

(1) takes a form:

$$d\tau^2 = \frac{dt^2}{R^2+l^2}\left\{\Delta_R\,(1-a\omega_0)^2 - \left[(R^2+a^2+l^2)\omega_0 - a\right]^2\right\} . \tag{43}$$

where $\Delta_R = \Delta(r=R)$. For light-ray moving along the same circular path, the proper must be vanishes, i.e. $d\tau = 0$, in this case one can have

$$\Delta\,(1-a\Omega)^2 - \left[(R^2+a^2+l^2)\Omega - a\right]^2 = 0 , \tag{44}$$

where $\Omega = d\phi/dt$. Now solving equation (44) for Ω, one can obtain two different roots in the form [7]

$$\Omega_\pm = \frac{a\left(2MR+2l^2-Q^2\right) \pm \left(l^2+R^2\right)\sqrt{R^2-2MR+a^2+Q^2-l^2}}{a^2\,(2MR+2l^2-Q^2)+(R^2+l^2)\,(R^2+l^2+a^2)} . \tag{45}$$

The rotation angles for light-ray are then $\phi_\pm = \Omega_\pm t$, hereafter eliminating t, one can have

$$\phi_\pm = \frac{\Omega_\pm}{\omega_0}\phi_0 . \tag{46}$$

The first intersection of the world lines of the two light-ray with the one of the orbiting observer after the emission at time, $t = 0$, is when

$$\phi_\pm = \phi_0 \pm 2\pi = \frac{\Omega_\pm}{\omega_0}\phi_0 , \tag{47}$$

or

$$\phi_{0\pm} = \mp\frac{2\pi\omega_0}{\Omega_\pm - \omega_0} = \frac{2\pi\omega_0}{\frac{a\left(2MR+2l^2-Q^2\right)\pm\left(l^2+R^2\right)\sqrt{R^2-2MR+a^2+Q^2-l^2}}{a^2\left(2MR+2l^2-Q^2\right)+\left(R^2+l^2\right)\left(R^2+l^2+a^2\right)} - \omega_0} , \tag{48}$$

From equation (43), one can easily obtain

$$d\tau = \frac{d\phi_0}{\omega_0}\sqrt{\frac{\Delta\,(1-a\omega_0)^2 - [(R^2+a^2+l^2)\omega_0 - a]^2}{R^2+l^2}} . \tag{49}$$

Hereafter integrating equation (49) between ϕ_{0-} and ϕ_{0+}, and after simple algebraic manipulations, one can get the expression for Sagnac delay in the Kerr-Newman-NUT space in the form:

$$\delta\tau = 4\pi\frac{\zeta a(1-a\omega_0) - \omega_0\left(R^2+l^2+a^2\right)}{\sqrt{1-\zeta(1-a\omega_0)^2 - \omega_0^2\,(R^2+l^2+a^2)}} , \tag{50}$$

where $\zeta = (2MR+2l^2-Q^2)/(R^2+l^2)$, while in Kerr spacetime Sagnac delay takes the form [7]:

$$\delta\tau = \frac{4\pi}{R}\frac{2aM - \omega_0\left(R^3+a^2(2M+R)\right)}{\sqrt{1-\frac{2M}{R}(1-a\omega_0)^2 - \omega_0^2\,(R^2+a^2)}} . \tag{51}$$

From equation (50) one can see that the Sagnac delay to be zero in the case when

$$\omega_0 = \frac{a\zeta}{R^2+l^2+a^2(1+\zeta)} , \tag{52}$$

for nonzero value of the spin parameter of the black hole. An interesting fact is that even for $\omega_0 = 0$, Sagnac delay exists and it can be found as, $\delta\tau = 4\pi a\zeta/\sqrt{1-\zeta}$, while in the weak field approximation in which $\beta = \omega_0 R << 1$, the expression for Sagnac delay reads

$$\begin{aligned}\delta\tau = 4\pi\Bigg\{ & a\left[\frac{2M}{R}\left(1+\frac{l^2}{R^2}-\frac{Q^2}{R^2}\right)+\frac{2l^2}{R^2}-\frac{Q^2}{R^2}\right] \\ & -R\beta\left[1+\frac{M}{R}\left(1+\frac{3a^2}{R^2}+\frac{3l^2}{R^2}-\frac{3Q^2}{2R^2}\right)+\frac{a^2}{R^2}+\frac{l^2}{R^2}-\frac{Q^2}{2R^2}\right] \\ & +3a\beta^2\left[\frac{M}{R}\left(1+\frac{6l^2}{R^3}-\frac{3Q^2}{R^3}\right)+\frac{l^2}{R^2}-\frac{Q^2}{2R^2}\right]\Bigg\} . \end{aligned} \tag{53}$$

3.2. Polar orbit

Now we focus on the propagation of the light-ray in polar orbit with, $r = R = \text{const}$, $\phi = \text{const}$ and $\theta_0 = \omega_0 t$. In this case the proper time can be expressed as

$$\begin{aligned} d\tau^2 &= \frac{\Delta_R - a^2\sin^2\theta}{\Sigma_R}dt^2 - \Sigma_R d\theta^2 \\ &= \left[\frac{\Delta_R - a^2\sin^2\omega_0 t}{R^2+(l+a\cos\omega_0 t)^2} - (R^2+(l+a\cos\omega_0 t)^2)\omega_0^2\right]dt^2 . \end{aligned} \tag{54}$$

Similarly, in the case when $ds = 0$, we will have

$$\frac{d\theta}{dt} = \frac{\sqrt{\Delta_R - a^2\sin^2\theta}}{\Sigma_R} , \tag{55}$$

Hereafter performing algebraic manipulate, one can get

$$\begin{aligned} t &= \int\frac{\Sigma_R d\theta}{\sqrt{\Delta_R - a^2\sin^2\theta}} = \int d\theta\frac{R^2+(l+a\cos\theta)^2}{\sqrt{\Delta_R - a^2\sin^2\theta}} \\ &= \sqrt{\Delta_R}E\left(\theta\middle|\frac{a^2}{\Delta_R}\right) + \frac{\zeta(R^2+l^2)}{\sqrt{\Delta_R}}F\left(\theta\middle|\frac{a^2}{\Delta_R}\right) + 2l\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{\Delta_R - a^2\sin^2\theta}}\right) , \end{aligned} \tag{56}$$

where $E(\theta|k)$ and $F(\theta|k)$ are, respectively, the elliptic integrals of the first and second kinds defined as

$$F(\theta|k) = \int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} , \qquad E(\theta|k) = \int_0^\theta d\theta\sqrt{1-k^2\sin^2\theta} . \tag{57}$$

Since the range of the polar angle is given as $0 < \theta < \pi$, the elliptic integrals satisfy the following properties $E(\theta|k) = E(\theta \pm \pi|k)$ and $F(\theta|k) = F(\theta \pm \pi|k)$. However, in this case it is impossible to get analytical expression for θ in terms of time t. For simplicity, we assume that rotation parameter is quite small in comparison with other parameters.

3.3. Null-geodesic motion

It is also interesting task to consider Sagnac time delay which arises from fundamental frequencies of the light-ray orbiting around black hole. One of such frequencies, so-called Keplerian frequency Ω, can be determined by consider geodesic motion in Kerr-Newman-NUT spacetime as

$$\frac{du^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0 , \qquad \Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}\left(\partial_\beta g_{\gamma\lambda} + \partial_\gamma g_{\beta\lambda} - \partial_\lambda g_{\beta\gamma}\right) , \tag{58}$$

where $u^\alpha = dx^\alpha/d\lambda$ is four-velocity of the light-ray, λ is an affine parameter. The conserved quantities, namely, the energy and angular momentum of the light-ray, can be found as

$$g_{tt}u^t + g_{t\phi}u^\phi = -\mathcal{E} , \qquad g_{t\phi}u^t + g_{\phi\phi}u^\phi = \mathcal{L} . \tag{59}$$

On the other hand the four-velocity of the light-ray satisfies the following condition: $g_{\alpha\beta}u^\alpha u^\beta = 0$, which leads

$$g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + V(r,\theta) = 0\,, \tag{60}$$

where

$$V(r,\theta) = \frac{g_{tt}\mathcal{L}^2 + 2g_{t\phi}\mathcal{E}\mathcal{L} + g_{\phi\phi}\mathcal{E}^2}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\,. \tag{61}$$

Keplerean frequency: Now we consider propagation of the light-ray in circular orbit with the four-velocity of $u^\alpha = (u^t, 0, 0, u^\phi)$, then hereafter using the equation (58), one can obtain

$$\Omega_K = -\frac{\partial_r g_{t\phi}}{\partial_r g_{\phi\phi}} \pm \sqrt{\left(\frac{\partial_r g_{t\phi}}{\partial_r g_{\phi\phi}}\right)^2 - \frac{\partial_r g_{tt}}{\partial_r g_{\phi\phi}}}\,, \tag{62}$$

which takes a form [6]

$$\Omega_{K\pm} = \frac{1}{\frac{l^2+R^2}{\sqrt{MR-Q^2+l^2(2-M/R)}} \pm a}\,. \tag{63}$$

Recalling equations (46)-(50), one can easily get the expression for Sagnac time delay in the form:

$$\delta\tau = 4\pi\frac{\sqrt{(R^2+l^2)\Lambda} - \omega_0(R^2+l^2-a^2\Lambda)}{\Lambda - 2\omega_0\sqrt{(R^2+l^2)\Lambda} + \omega_0^2(R^2+l^2-a^2\Lambda)} \times\sqrt{\frac{\Delta_R\,(1-a\omega_0)^2 - \left[(R^2+a^2+l^2)\omega_0 - a\right]^2}{R^2+l^2}}\,, \tag{64}$$

where Λ is defined as

$$\Lambda = \frac{MR - Q^2 + l^2(2-M/R)}{R^2+l^2}\,, \tag{65}$$

Absence of ω_0, Sagnac delay takes a form

$$\delta\tau = 4\pi\sqrt{\frac{(R^2+l^2)(R^2-2MR+Q^2+l^2)}{MR-Q^2+l^2(2-M/R)}}. \tag{66}$$

In (3) figure shows that the fractional Sagnac time delay depends on different parameters of the Kerr-Newman-NUT space time. In the case of Kerr spacetime the expression (64) reads

$$\delta\tau = 4\pi\frac{\sqrt{MR} - \omega_0(R^2-a^2\Lambda)}{M - 2\omega_0 R\sqrt{MR} + \omega_0^2(R^3-a^2M)}\sqrt{\Delta_R\,(1-a\omega_0)^2 - \left[(R^2+a^2)\omega_0 - a\right]^2}\,, \tag{67}$$

[6] It is well know that Keplerian frequency in Kerr spacetime expressed as

$$\Omega_{K\pm} = \frac{1}{R\sqrt{\frac{R}{M}} \pm a}\,,$$

however, in Ref. [7], it is presented as

$$\Omega_{K\pm} = \frac{2aM \pm \sqrt{3a^2M^2 + MR^3}}{Ma^2 - R^3}\,,$$

that does not give correct result.

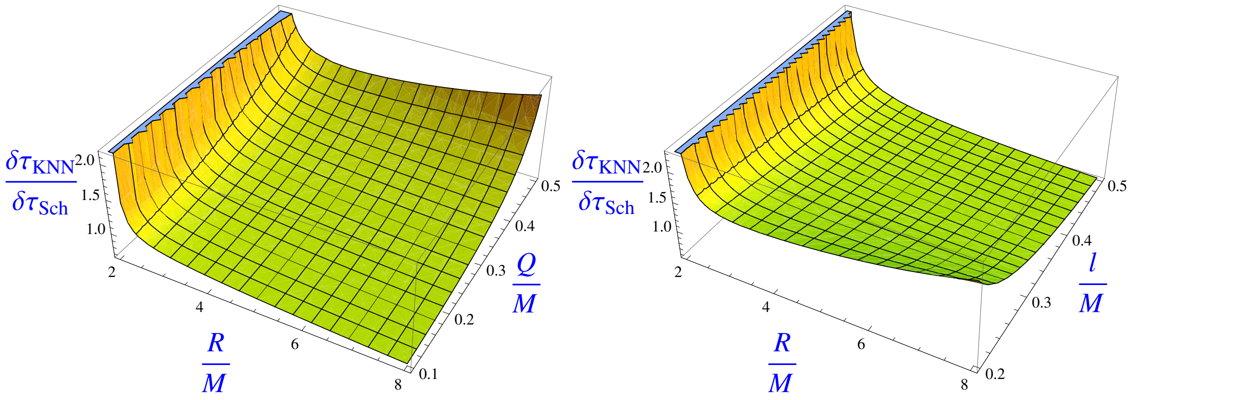

Figure 3: Left panel: Dependence of fractional time delay of Kerr-Newman-NUT black hole (in equation (66)) from radius of Sagnac interferometer and charge of black hole for the value $l/M = 0.3$ of NUT charge. Right panel: Dependence of fractional time delay of black hole from radius of Sagnac interferometer and NUT charge of black hole for the value $Q/M = 0.4$ of charge black hole.

Gyroscopic frequency: There do exist another type of orbital frequency arises from angular part of geodesic equation:

$$\Omega_G = -\frac{\partial_\theta g_{t\phi}}{\partial_\theta g_{\phi\phi}} \pm \sqrt{\left(\frac{\partial_\theta g_{t\phi}}{\partial_\theta g_{\phi\phi}}\right)^2 - \frac{\partial_\theta g_{tt}}{\partial_\theta g_{\phi\phi}}}\,, \tag{68}$$

Notice that this frequency occurs due to rotation of the central object. In Kerr-Newman-NUT spacetime, it takes a form:

$$\Omega_{\mathrm{G}\pm} = \frac{\pm a}{(R^2 + l^2)\left(1 \pm \sqrt{\frac{\Delta_R}{\Delta_R - a^2}}\right) + a^2}\,, \tag{69}$$

and the corresponding time delay is

$$\delta\tau = -4\pi \frac{S(R^2+l^2)\sqrt{1+\frac{a^2}{S(R^2+l^2)}} - a\omega_0(R^2 - 4MR + 2Q^2 - 3l^2) + a^3\omega_0 S}{2\omega_0 S(R^2+l^2)\sqrt{1+\frac{a^2}{S(R^2+l^2)}} + a\omega_0^2(R^2+l^2)(1-2S) + aS(a^2\omega_0^2 - 1)} \times \sqrt{\frac{\Delta(1-a\omega)^2 - [(R^2+a^2+l^2)\omega_0 - a]^2}{R^2+l^2}}\,, \tag{70}$$

where S is defined as

$$S = \frac{R^2 - 2MR + Q^2 - l^2}{R^2 + l^2}\,. \tag{71}$$

Starting from the exact results for a Kerr-Newman-NUT metric and considering suitable approximations of them we have obtained the corrections to the Sagnac effect that the mass and angular momentum of a rotating object introduce. These are conceptually important, evidencing and strengthening the analogy between the Sagnac effect and the Bohm-Aharonov effect: particularly relevant to this purpose. Unfortunately, when considering the Earth as the source of the gravitational field the corrections are indeed very tiny, but per se in the range of what current optical interference measurements allow, provided a convenient zero "pure" Sagnac term is experimentally fixed. When considering devices such as ring lasers, where standing oppositely propagating waves form, the Sagnac time difference is automatically converted into a frequency shift and in general a fractional frequency shift may well be easier to measure than the equivalent fringe shift. Of course, here the difficulty is in stabilizing standing electromagnetic waves around the Earth, either in space or on the surface of the planet. However, what is hard for light might not be so using radio waves, provided their Sagnac effect was not reduced too much. So far we consider only visible electromagnetic field to analyse the Sagnac effect.

4. Conclusions

In the present paper, we discussed the Sagnac effect in the Kerr-Newman-NUT spacetime. All findings are summarized as follows:

- In order to investigate the Kerr-Newman-NUT spacetime properties, we first determined the curvature invariants such as Ricci square, and Kretschmann scalar. Our calculations have shown that for the non-zero value of the NUT parameter $l \neq 0$, the curvature invariants are always finite in any points of the spacetime, which concludes that the Kerr-Newman-NUT metric is one of the exact analytical rotating regular (asymptotically not flat) black hole solution. On the other hand, it is also shown that the NUT parameter is equivalent to the mass of the central object.
- We have investigated the structure of the horizon and ergosphere of the black hole in the Kerr-Newman-NUT spacetime. It has been shown that for slowly rotating spacetime, the horizon and ergosphere are indistinguishable, however, for rapidly rotating spacetime there is a big difference between these two areas. On top of that, we also determine the surface area of the horizon and ergosphere of the black hole, explicitly.
- In fact that in both theoretical and astrophysical point of view the electromagnetic fields play very important role in the vicinity of the compact object. For this reason, we have discussed the electromagnetic field configuration around the Kerr-Newman-NUT spacetime and presented the components of the electromagnetic fields. It is shown that the NUT parameter plays an important role to produce the induced magnetic field even around non-rotating black hole.
- We construct shadow of the black hole, on the other hand it is also called capture cross section of black hole. Here, we meanly focus on the shadow of the maximally rotating neutral black hole for the different values of the NUT parameter. Our results show that size of the black hole's shadow gets large due the NUT parameter.
- Finally, we have investigated the Sagnac effect in the Kerr-Newman-NUT spacetime in order to test NUT parameter on it. We consider three different cases for determining of the Sagnac time delay in Kerr-Newman-NUT spacetime, (i) motion in equatorial orbit, (ii) motion in polar orbit, (iii) motion in Keplerean orbit. The detailed analyses show that the Sagnac time delay decreases due the NUT parameter.

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