THE GENERALISED FERMAT EQUATION $Pa^x + Qb^y = Rc^z$ AND RELATED PROBLEMS

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Abstract. The focus of this paper is the generalised Fermat equation, $Pa^x + Qb^y = Rc^z$, considered by Henri Darmon and Andrew Granville. It is closely related to a family of theorems and conjectures including the Fermat-Catalan Conjecture, the Darmon-Granville Theorem, the Beal Conjecture (also known as the Tijdeman-Zagier Conjecture) and Fermat's Last Theorem. We will consider these briefly before offering a proof that no solutions exist even for $P, Q, R > 1$, for cases $x, y, z > 2$, using a new binomial identity for $a^x + b^y$ to an indeterminate power, z. The proof extends to its corollaries the Beal Conjecture and Fermat's Last Theorem.

Introduction

As we consider the generalised Fermat equation, $Pa^x + Qb^y = Rc^z$, there are a number of related theorems and conjectures.

The Fermat-Catalan Conjecture

The Fermat-Catalan Conjecture states[1] that *if* $P, Q, R = 1$ *and* $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1$, (the hyperbolic case), *the equation*

$$a^x + b^y = c^z$$

has only finite solutions.

Only the following ten solutions are currently known:

$$1^7 + 2^3 = 3^2,$$
$$2^5 + 7^2 = 3^4,$$
$$7^3 + 13^2 = 2^9,$$
$$2^7 + 17^3 = 71^2,$$
$$3^5 + 11^4 = 122^2,$$
$$17^7 + 76271^3 = 21063928^2,$$
$$1414^3 + 2213459^2 = 65^7$$
$$9262^3 + 15312283^2 = 113^7,$$
$$43^8 + 96222^3 = 30042907^2,$$
$$33^8 + 1549034^2 = 15613^3.$$

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[1] "Computational Number Theory", in Gowers, Timothy; Barrow-Green, June; Leader, Imre (eds.), The Princeton Companion to Mathematics, Princeton University Press, p.360

The Darmon and Granville Theorem

In 1994, continuing with the hyperbolic case, Henri Darmon and Andrew Granville using Faltings' Theorem proved that *if P, Q, R, a, b, c are fixed positive integers with*

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < 1,$$

then the equation

$$Pa^x + Qb^y = Rc^z$$

has at most finitely many solutions in coprime non-zero integers a, b and c.

The proof is considered very elegant and is outlined briefly by Bennett, Mihailescu and Siksek[2]. However, we wish to go beyond the hyperbolic case to prove that no integer solutions exist even for values $x, y, z > 2$.

The Beal Conjecture (Tijdeman-Zagier Conjecture)

In the mid 1990's, a Texan banker called Andrew Beal noted that the smallest exponent in all ten solutions for the hyperbolic case (where $P, Q, R = 1$) was 2. He therefore conjectured that *for the equation*

$$a^x + b^y = c^z$$

no whole number solutions exist for cases $x, y, z > 2$.

This has become known as the Beal Conjecture (also known as the Tijdeman-Zagier Conjecture)[3].

Fermat's Last Theorem

The famous corollary of this conjecture is Fermat's Last Theorem, for the case $x, y, z = n$. A proof was first discovered by Sir Andrew Wiles in 1993,[4] proving that *for the equation*

$$a^n + b^n = c^n$$

no whole number solutions exist for cases $n > 2$.

An elementary approach

In their paper Bennett, Mihailescu and Siksek survey different approaches to the this family of equations, including, among others, cyclotomic fields, elliptic curves and modular forms, and Galois representations.[5] Here we return to a more elementary approach.

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[2] *The Generalized Fermat Equation* Michael Bennett, Preda Mihailescu and Samir Siksek, https://homepages.warwick.ac.uk/ maseap/papers/bealconj.pdf, p24

[3] See www.bealconjecture.com. Last accessed 14.12.17.

[4] Wiles, Andrew (1995). "Modular elliptic curves and Fermat's Last Theorem". Annals of Mathematics. 141 (3): 443–551.

[5] *The Generalized Fermat Equation* Michael Bennett, Preda Mihailescu and Samir Siksek, https://homepages.warwick.ac.uk/ maseap/papers/bealconj.pdf, p24

Proving that no whole number solutions exist is problematic using elementary methods, since there are an infinite number of cases especially when multiple exponents, x, y, z, are involved. It may be easier to find counterexamples, but becomes harder to prove. Using a 'horizontal' approach, on a case-by-case basis as Fermat and his successors began to do for cases of n, would take forever. A 'vertical' approach, like infinite descent, would seem much better suited. But a 'vertical' approach also has its problems. As Peter Schorer warns, one of the inherent problems of proving the theorem using a 'vertical' approach appears to be that when one assumes that a counterexample exists and then tries to derive a contradiction, the very properties that created the contradiction in the first place appear to belong also to the *non*-counterexample.[6]

However, using binomial theorem we can circumvent these obstacles by expressing both sides of the equation $Pa^x + Qb^y = Rc^z$ in terms of a, b and z, and thereby isolate z from x and y to open the way for a simple proof by contradiction for $z > 2$.

Theorem 0.1. *To demonstrate that for the Fermat equation $Pa^x + Qb^y = Rc^z$, where a, b, c, P, Q, R are square-free integers (of which one of Pa, Qb, Rc at most must be even), and* $\gcd(a, b, c, P, Q, R) = 1$*, no integer solutions exist for the values of $x, y, z > 2$.*

We first observe the following identity for $Pa^x + Qb^y$ as a binomial expansion (where the upper index n is an indeterminate integer):

$$Pa^x + Qb^y = \sum_{k=0}^{n} \binom{n}{k} (a+b)^{n-k}(-ab)^k (Pa^{x-n-k} + Qb^{y-n-k}). \tag{0.1}$$

Note how this new identity includes *standard* factors for a binomial expansion, i.e. $(a+b)^{n-k}(-ab)^k$, but also a *non-standard* factor, i.e. $(Pa^{x-n-k} + Qb^{y-n-k})$.

Note, further, that regardless of the value of n, the right hand side always equals $Pa^x + Qb^y$. This allows us to fix n to any value we choose. So let $n = z$, such that:

$$Pa^x + Qb^y = \sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (Pa^{x-z-k} + Qb^{y-z-k}). \tag{0.2}$$

Proof. We now assume that a solution exists for the equation $Pa^x + Qb^y = Rc^z$ for values of $x, y, z > 2$.

Now let s, t be dependent variables, where $s, t \in \mathbb{Z}$, $gcd(s, ab) = 1$, and $s \neq 0$, such that $[(a+b)s - abt] = c$. From this it follows that:

$$\sum_{k=0}^{z} \binom{z}{k} (a+b)^{z-k}(-ab)^k (Pa^{x-z-k} + Qb^{y-z-k}) = R[(a+b)s - abt]^z. \tag{0.3}$$

[6] Peter Schorer "Is There a "Simple" Proof of Fermat's Last Theorem? Part (1) Introduction and Several New Approaches", 2014, www.occampress.com/fermat.pdf.

Using the binomial theorem we expand (0.3) as:
(0.4)

$$\sum_{k=0}^{z}\binom{z}{k}(a+b)^{z-k}(-ab)^k(Pa^{x-z-k}+Qb^{y-z-k}) = \sum_{k=0}^{z}\binom{z}{k}R(a+b)^{z-k}(-ab)^k s^{z-k}t^k.$$

We know that the right hand side is a power to z since all the components have the correct exponential form for a standard binomial expansion to power z; the left hand side may or may not be. Now, since s and t are dependent on and inseparably tied to the independent variables, $(a + b)$ and ab, and must therefore conform to the standard binomial exponential form, this is the *only* circumstances when the left hand side can be a power to z are when the following equation holds true for every k^{th} term, for any given value of z:

$$(Pa^{x-z-k} + Qb^{y-z-k}) = s^{z-k}t^k. \tag{0.5}$$

Note: Without the independent variables, $(a + b)$ and ab, there could be other circumstances when the left hand side of (0.4) is a power to z. But since we have defined c^z as $[(a + b)s - abt]^z$, this is the only circumstance. In turn, this means that if there exists at least one inequality in (0.5) in any k^{th} term for any given value of z, then the whole of the left hand side cannot be a power to z.

On this basis, we can now complete the proof. Since it is true that:

$$\left(\frac{s^{z-1}t}{s.t^{z-1}}\right)^z = \left(\frac{s^z}{t^z}\right)^{z-2}, \tag{0.6}$$

it follows, from (0.5), that:

$$\left(\frac{Pa^{x-z-1} + Qb^{y-z-1}}{Pa^{x-2z+1} + Qb^{y-2z+1}}\right)^z = \left(\frac{Pa^{x-z} + Qb^{y-z}}{Pa^{x-2z} + Qb^{y-2z}}\right)^{(z-2)}. \tag{0.7}$$

Solutions will exist to this equation
a) *either* if the large fractions on both sides have a value of 1 (since the outer exponents are not equal),
b) *or* if the numerators (to their respective outer exponents) on both sides are equal, *and* simultaneously if the denominators (to their respective outer exponents) on both sides are equal.

Taking these two options in turn (still when $x, y, z > 2$):

a) since $z \neq 2z$, $(Pa^{x-z-1} + Qb^{y-z-1}) \neq (Pa^{x-2z+1} + Qb^{y-2z+1})$ and $(Pa^{x-z} + Qb^{y-z}) \neq (Pa^{x-2z} + Qb^{y-2z})$. So neither fraction in (0.7) has a value of 1, eliminating this option;

b) beginning with denominators, since $(Pa^{x-2z+1} + Qb^{y-2z+1})^z > (Pa^{x-2z} + Qb^{y-2z})^{z-2}$ they are not equal. There is no need to consider the numerators.

Having now eliminated both options it follows that, for all values of $x, y, z > 2$ and all values of k:

$$s^{z-k}t^k \neq (Pa^{x-z-k} + Qb^{y-z-k}). \tag{0.8}$$

However, this contradicts our equation in (0.5). In turn, therefore, the left hand side of the equation in (0.4) cannot be a perfect power (as we assumed it was). And so our initial assumption that solutions exist for the equation $Rc^z = Pa^x + Qb^y$ for values of $x, y, z > 2$ is false. Therefore the conjecture is true. □

What happens for the cases for $z = 1, 2$? Well, from (0.7), when $z = 1$ it follows that:

$$\left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right)^1 = \left(\frac{Pa^{x-1} + Qb^{y-1}}{Pa^{x-2} + Qb^{y-2}}\right)^{-1}, \tag{0.9}$$

$$\Rightarrow \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right) = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-1} + Qb^{y-1}}\right). \tag{0.10}$$

No contradiction.

And again from (0.7), when $z = 2$, it follows that:

$$\left(\frac{Pa^{x-3} + Qb^{y-3}}{Pa^{x-3} + Qb^{y-3}}\right)^2 = \left(\frac{Pa^{x-2} + Qb^{y-2}}{Pa^{x-4} + Qb^{y-4}}\right)^0, \tag{0.11}$$

$$\Rightarrow 1 = 1. \tag{0.12}$$

Again, no contradiction.

So in both cases, when $z = 1$ and when $z = 2$, there is no contradiction. Our non-standard binomial factor, $(Pa^{x-z-k} + Qb^{y-z-k})$ is equal to $(\pm s)^{z-k}(\pm t)^k$ for every value of k, which, in turn, proves both the Beal Conjecture and Fermat's Last Theorem.

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