A proof of the Riemann hypothesis using the two-sided Laplace transform

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1. Introduction

Remark:

We are using $\frac{\partial^n}{\partial z^n}$, $F'(z)$, $F^{(n)}(z)$ and D_z^n as the differential operators and choosing the most suitable notation for the case.

We will begin with the definition of the two-sided Laplace transform.[1] The Laplace transform of a real function $f(t)$ is defined as

$$F(z) \equiv \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt \tag{1}$$

where $z = x + iy$ for x and y real.

We assume $f(t) \geq 0$ for all t and $f(-t) = f(t)$. Further, $f(t)$ is so rapidly decreasing that $F(z)$ is entire. Since we assume that $f(t)$ is an even function, we can write

$$F(z) = \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{zt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cosh(zt)\, dt = 2\int_{0}^{\infty} f(t) \cdot \cosh(zt)\, dt \tag{2}$$

and the power series expansion of $F(z)$

$$F(z) = \sum_{n=0}^{\infty} a_{2n} \cdot z^{2n} = a_0 + a_2 z^2 + a_4 z^4 + \cdots \tag{3}$$

where $a_{2n} = \frac{1}{(2n)!}\int_{-\infty}^{\infty} f(t) \cdot t^{2n} dt$. Note that $f(t)$ is non-negative, hence a_{2n} is strictly positive for all n.Therefore, any coefficient is not missing.

The real and imaginary part of $F(z)$ is

$$F(z) = u(x,y) + iv(x,y) \tag{4}$$

where

$$u(x,y) = \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt) \cdot \cos(yt)\, dt \tag{5}$$

and

$$v(x,y) = \int_{-\infty}^{\infty} f(t) \cdot \sinh(xt) \cdot \sin(yt)\, dt \tag{6}$$

[1] Since we are only dealing with the two-sided Laplace transform, the term “two-sided” will be omitted afterward.

Since x or y is zero, the imaginary part is vanished, rewriting in

$$F(x) = u(x,0) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{xt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cosh(xt)\, dt = \sum_{n=0}^{\infty} a_{2n} \cdot x^{2n} \qquad (7)$$

and

$$F(iy) \equiv F(y) = u(0,y) = \int_{-\infty}^{\infty} f(t) \cdot e^{-iyt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{iyt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cos(yt)\, dt = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot y^{2n} \quad (8)$$

which are $F(z)$ on x-axis and y-axis respectively. Clearly, $F(x)$ and $F(y)$ are even functions and real when x and y real respectively, and since $F(iy)$ is real when y real, we often use $F(y)$ instead of $F(iy)$ for convenience.

To be $F(z)$ entire, the coefficients a_{2n} should be rapidly decreasing. A rough estimation how rapidly the coefficients decrease, we may use the rule of thumb

$$\frac{\sum_{n=1}^{\infty} a_{2n}}{a_0} \qquad (9)$$

and we guess the less the value (9), the more rapidly the coefficients decrease.

Since $F(0) = a_0$ and $F(1) = \sum_{n=0}^{\infty} a_{2n}$, we can write (9) as

$$\frac{F(1)}{F(0)} - 1 \qquad (10)$$

The value (9) should be close to zero. Otherwise $F(z)$ cannot be entire. For example, if $f(t) = e^{-t^2}$, the value (9) is $e^{1/4} - 1 \approx 0.2840$.

2. log- convexity and log-concavity

A function $f(x)$ is log-convex if $ln[f(x)]$ is convex. Similarly, a function $f(x)$ is log-concave if $ln[f(x)]$ is concave[2].

Theorem 1: The log- convexity and log-concavity

1) A function $f(x)$ is log-convex, if and only if

$$f(\lambda x_1 + \mu x_2) \leq [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu} \qquad (11)$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

and

$$f(x) \cdot f''(x) - [f'(x)]^2 \geq 0 \qquad (12)$$

[2] If $F(x) < 0$, then $ln[f(x)]$ is not defined. In this case, we assume that $F(x)$ is log-convex if $F(x) \cdot F''(y) - [F'(y)]^2 \geq 0$ and log-concave if $[F'(y)]^2 - F(x) \cdot F''(y) \geq 0$.

2) A function $f(x)$ is log-concave, if and only if

$$f(\lambda x_1 + \mu x_2) \geq [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu} \tag{13}$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

and

$$f(x) \cdot f''(x) - [f'(x)]^2 \leq 0 \tag{14}$$

Theorem 2: The strictly log- convexity and strictly log-concavity

1) A function $f(x)$ is strictly log-convex, if and only if

$$f(\lambda x_1 + \mu x_2) < [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu} \tag{15}$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

and

$$f(x) \cdot f''(x) - [f'(x)]^2 > 0 \tag{16}$$

2) A function $f(x)$ is strictly log-concave, if and only if

$$f(\lambda x_1 + \mu x_2) > [f(x_1)]^{\lambda} \cdot [f(x_2)]^{\mu} \tag{17}$$

where $\lambda, \mu > 0$ and $\lambda + \mu = 1$.

and

$$f(x) \cdot f''(x) - [f'(x)]^2 < 0 \tag{18}$$

3. The Laguerre inequalities

The necessary but not sufficient conditions of $F(y)$ to have

only real zeros are that $F(y)$ and all the derivatives of $F(y)$ are log-concave, where

$$F(y) = u(0, y) = \int_{-\infty}^{\infty} f(t) \cdot e^{-iyt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{iyt} dt = \int_{-\infty}^{\infty} f(t) \cdot \cos(yt)\, dt$$

Hence, we have the theorem

Theorem 3: The Laguerre inequalities

$F(y)$ belongs to the Laguerre-Pólya class if

$$\left[F^{(n+1)}(y)\right]^2 - F^{(n)}(y) \cdot F^{(n+2)}(y) \geq 0 \tag{19}$$

where $n = 0, 1, 2, 3, \ldots$ and for all $y \in \mathbb{R}$.

It means that if $F(y)$ has only real zeros, then $F(y)$ and all the derivatives of $F(y)$ are log-concave but not conversely.

Proposition 1:

Let $f(x)$ be an even function, then we can express $f(x)$ as polynomial whose powers are all even. So, we can write

$$f(x) = b_0 + b_2 x^2 + \cdots + b_{2n} x^{2n} = \sum_{k=0}^{n} b_{2k} \cdot x^{2k} \tag{20}$$

where b_{2k} is real and can be positive, negative or zero.

Let p(x) be $f(\sqrt{x})$, so, we can write

$$p(x) \equiv f(\sqrt{x}) = b_0 + b_2 x + \cdots + b_{2n} x^n = \sum_{k=0}^{n} b_{2k} \cdot x^k \tag{21}$$

Clearly, if ρ is a root of $f(x)$, i.e., $f(\rho) = 0$ then ρ^2 is a root of $p(x)$. If ρ is real, $p(x)$ has a real root ρ^2. Now, we define another polynomial, namely $q(x) \equiv p(x)|_{x=-x} = p(-x)$, which can be written as

$$q(x) = b_0 - b_2 x + \cdots + b_{2n} x^n = \sum_{k=0}^{n} (-1)^k \cdot b_{2k} \cdot x^k \tag{22}$$

If ρ is a root of $f(x)$, ρ^2 is a root of $p(x)$ and $-\rho^2$ is a root of $q(x)$. Since $f(x)$ is an even function, if ρ is a root of $f(x)$, then -ρ is also a root of $f(x)$. Therefore, if $f(x)$ has $2 \cdot m$ real roots, $p(x)$ and $q(x)$ have m real roots. As mentioned above, if ρ is a real root of $f(x)$, ρ^2 is a real root of p(x) and $-\rho^2$ is a real root of q(x). ρ^2 is non-negative and $-\rho^2$ is non-positive, hence we rewrite the statement above:

If $f(x)$ has $2 \cdot m$ real roots, $p(x)$ has m real roots in the interval $[0, \infty)$ and $q(x)$ has m real roots in the interval $(-\infty, 0]$. Consequently, if $f(x)$ does not have any real root, $p(x)$ has no real root in the interval $[0, \infty)$ and $q(x)$ has no real root in the interval $(-\infty, 0]$. In other words, if $f(x)$ does not have any real root, $f(x)$ does not change the sign at all, hence, $p(x)$ and $q(x)$ do not change the sign in the interval $[0, \infty)$ and $(-\infty, 0]$ respectively.

If $f(x)$ if non-negative or non-positive, i.e., $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in \mathbb{R}$, then $f(x)$ can be zero. Assuming $f(x) \geq 0$ and $f(\rho) = 0$, then since $f(x)$ is non-negative, $f(\rho)$ is a local minimum, thus $f'(\rho) = 0$. The case of $f(x) \leq 0$ is similar, and $f(\rho)$ is a local maximum, thus $f'(\rho) = 0$. Therefore, $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in \mathbb{R}$ and $f(\rho) = 0$, then $f(\rho)$ is an extremum, hence $f'(\rho) = 0$

Since $f(x) = p(x^2)$, we have $f'(\mathrm{x}) = 2\,\mathrm{x} \cdot p'(x^2)$ where $p'(x^2)$ denotes $p'(x)|_{x=x^2}$. Thus, if $f'(\rho) = 0$, then $p'(\rho^2) = 0,$ and therefore $p(\rho^2) = 0$ and $p'(\rho^2) = 0$, which means is that if $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in \mathbb{R}$, $p(x)$ does not change the sign in the interval $[0, \infty)$.

$f(x)$ is an even function and therefore $f'(0) = 0$ but $p'(0) \neq 0$. It is because $f'(\mathrm{x})$ is an odd function. However, $r(x) \equiv f'(\mathrm{x})/x$ is an even function and therefore $f'(\rho) = r(\rho) = 0$. Thus, if $f'(\rho) = r(\rho) = 0$, then $p'(\rho^2) = 0$.

Similarly, if $f(x) \geq 0$ or $f(x) \leq 0$ for all $x \in \mathbb{R}$, then $q(x) = p(-x)$ does not change sign in the interval $(-\infty,\ 0]$.

Now, we define $g(x)$ as follows:

$$g(x) \equiv f(ix) = b_0 - b_2x^2 + \cdots + b_{2n}x^{2n} = \sum_{k=0}^{n}(-1)^k \cdot b_{2k} \cdot x^{2k} \tag{23}$$

then, $q(x) = g(\sqrt{x})$ and naturally, $p(x) = q(-x)$. Therefore, $g(x)$ does not change the sign $\forall x \in \mathbb{R}$, if and only if $q(x)$ and p(x) do not change the sign in the interval $[0,\ \infty)$ and $(-\infty,\ 0]$ respectively.

Consequently, both $f(x)$ and $f(ix)$ do not change the sign $\forall x \in \mathbb{R}$ if and only if either $p(x)$ or $q(x)$ does not change the sign $\forall x \in \mathbb{R}$.

Proposition 2: The log- convexity of $F(x)$

From (7), $F(x)$ is defined as:

$$F(x) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt$$

For $x1$ and $x2$ ($x1 \neq x2$), and $\lambda, \mu > 0$, $\lambda + \mu = 1$

$$F(\lambda x_1 + \mu x_2) = \int_{-\infty}^{\infty} f(t) \cdot e^{-(\lambda x_1 + \mu x_2)t} dt = \int_{-\infty}^{\infty} [f(t) \cdot e^{-x_1 t}]^{\lambda} \cdot [f(t) \cdot e^{-x_2 t}]^{\mu} dt$$

and by the Hölder inequality, we have

$$\int_{-\infty}^{\infty} [f(t) \cdot e^{-x_1 t}]^{\lambda} \cdot [f(t) \cdot e^{-x_2 t}]^{\mu} dt \leq \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-x_1 t} dt\right]^{\lambda} \cdot \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-x_2 t} dt\right]^{\mu}$$

In addition, $[f(t) \cdot e^{-x_1 t}]^{1/\lambda}$ and $[f(t) \cdot e^{-x_2 t}]^{1/\mu}$ are not linearly dependent for $x1 \neq x2$, hence the equality does not hold. So, we have

$$F(\lambda x_1 + \mu x_2) < [F(x_1)]^{\lambda} \cdot [F(x_2)]^{\mu}$$

This means that $F(x)$ is strictly log-convex.

Since $F(x)$ is strictly log-convex, we also have

$$F(x) \cdot F''(x) - [F'(x)]^2 > 0 \tag{24}$$

Now, let $G(x)$ be $F(x) \cdot F''(x) - [F'(x)]^2$, namely

$$G(x) \equiv F(x) \cdot F''(x) - [F'(x)]^2 \tag{25}$$

then $G(x) > 0$ or $G(x) \neq 0$ for all real x. Since $F(x)$, $F''(x)$ and $[F'(x)]^2$ are even functions, $G(x)$ is also an even function. We define another function $p(x) \equiv G(\sqrt{x})$. Since $G(x)$ is an even function, $p(x) \neq 0$ for $0 \leq x < \infty$ and $p(-x) \neq 0$ for $-\infty < x \leq 0$ by the proposition 1.

Since $F(x) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt$,

$$G(x) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^2 \cdot e^{-xt} dt - \left[\int_{-\infty}^{\infty} f(t) \cdot t \cdot e^{-xt} dt\right]^2 \tag{26}$$

and

$$p(x)=\int_{-\infty}^{\infty}f(t)\cdot e^{-\sqrt{x}t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-\sqrt{x}t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-\sqrt{x}t}dt\right]^2 \tag{27}$$

which is non-zero for $x\geq 0$. Moreover, $p(-x)$ is non-zero for $x\leq 0$, we have

$$p(-x)=\int_{-\infty}^{\infty}f(t)\cdot e^{\sqrt{-|x|}t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{\sqrt{-|x|}t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{\sqrt{-|x|}t}dt\right]^2$$

or more intuitively,

$$p(-x)=\int_{-\infty}^{\infty}f(t)\cdot e^{i\sqrt{|x|}t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{i\sqrt{|x|}t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{i\sqrt{|x|}t}dt\right]^2$$

and by changing the variable $t\mapsto -t$ we have

$$p(-x)=\int_{-\infty}^{\infty}f(t)\cdot e^{-i\sqrt{|x|}t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-i\sqrt{|x|}t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-i\sqrt{|x|}t}dt\right]^2 \tag{28}$$

which is non-zero for $|x|\geq 0$. Further, we have $p(-x)=p(i\cdot|x|)$ from Eq. (28).

Eq. (28) is not different than $G(ix)|_{x=\sqrt{|x|}}$ and since $G(ix)$ is real when x real, hence $g(i\cdot|x|)$ is real when x real.

By changing the variable $|x|\mapsto y^2$ in Eq (27), we have

$$p(i\cdot|x|)|_{|x|=y^2}=p(i\cdot y^2)=\int_{-\infty}^{\infty}f(t)\cdot e^{-i\sqrt{y^2}t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-i\sqrt{y^2}t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-i\sqrt{y^2}t}dt\right]^2$$

thus

$$p(i\cdot y^2)=\int_{-\infty}^{\infty}f(t)\cdot e^{-i\cdot|y|t}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-i\cdot|y|t}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-i\cdot|y|t}dt\right]^2$$

or

$$p(i\cdot y^2)=\int_{-\infty}^{\infty}f(t)\cdot e^{-i\cdot yt}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-i\cdot yt}dt-\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-i\cdot yt}dt\right]^2 \tag{29}$$

which is non-zero for $y\geq 0$. Furthermore, since $g(i\cdot y^2)$ is real and an even function, $g(i\cdot y^2)$ is non-zero for all $y\in\mathbb{R}$. Hence, Eq. (29) is nothing but $G(iy)$.

Or more easily, from Eq. (28) $p(-x)$ is non-zero if $|x|\geq 0$, and this means that $p(-x)$ is non-zero $\forall\, x\in\mathbb{R}$. Hence, from proposition 1, both $F(x)$ and $F(iy)$ are non-zero $\forall x\in\mathbb{R}$.

From Eq. (8), we have

$$F(y)=\int_{-\infty}^{\infty}f(t)\cdot e^{-iyt}dt$$

and since $F'(y)=-i\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-iyt}dt$ and $F''(y)=-\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-iyt}dt$

$$F(y)\cdot F''(y)-[F'(y)]^2=-\int_{-\infty}^{\infty}f(t)\cdot e^{-iyt}dt\cdot\int_{-\infty}^{\infty}f(t)\cdot t^2\cdot e^{-iyt}dt+\left[\int_{-\infty}^{\infty}f(t)\cdot t\cdot e^{-iyt}dt\right]^2$$

which is $-G(iy)$. Since $G(iy)$ is non-zero for all $y \in \mathbb{R}$, so is $-G(iy) = F(y) \cdot F''(y) - [F'(y)]^2$.

We know that $F(y) \cdot F''(y) - [F'(y)]^2 \neq 0$ for all $y \in \mathbb{R}$, hence $F(y)$ is either strictly log-convex or strictly log-concave. To determine it, we examine $F(y) \cdot F''(y) - [F'(y)]^2$ at $y = 0$. From Eq. (8), we have

$$F(y) = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot y^{2n}$$

Since $F(0) = a_0$, $F'(0) = 0$ and $F''(0) = -2 \cdot a_2$, we have $F(0) \cdot F''(0) - [F'(0)]^2 = -2a_0 \cdot a_2 < 0$, hence $F(y)$ is strictly log-concave. It is because $F(y) \cdot F''(y) - [F'(y)]^2$ does not change the sign for all y.

Note that we have proved $F(y)$ is log-concave using $p(-x)$ which is log-concave and defined as

$$F(-\sqrt{x}) = \sum_{n=0}^{\infty} (-1)^n \cdot a_{2n} \cdot x^n \quad (x \leq 0) \tag{30}$$

so, $p(0) = a_0$, $p'(0) = -a_2$ and $p''(0) = 2a_4$ and since $p(-x)$ is log-concave, $(a_2)^2 > 2a_0a_4$. Moreover $p(x)$ is positive for $x \geq 0$ and $(a_2)^2 > 2a_0a_4$, $F(\sqrt{x})$ is also log-concave for $x \geq 0$. Further, since $[F'(y)]^2 - F(y) \cdot F''(y) > 0$, $p(-x)$ and $p(x)$ does not change the sign for $x \geq 0$ and $x \leq 0$ respectively. Hence $F(\sqrt{x})$ and $F(-\sqrt{x})$ are log-concave $\forall x \in \mathbb{R}$ because of $(a_2)^2 > 2a_0a_4$.

A more intuitive method to determine $F(y)$ whether log-convex or log-concave from $F(x) \cdot F''(x) - [F'(x)]^2$ is changing x to iy. From $F(x) \cdot F''(x) - [F'(x)]^2 > 0$, we use another notation of derivatives, i.e.

$F(x) \cdot \frac{d^2}{dx^2} F(x) - \left[\frac{d}{dx} F(x)\right]^2 > 0$ and we change x to iy, that is,

$F(iy) \cdot \frac{d^2}{d(iy)^2} F(iy) - \left[\frac{d}{d(iy)} F(iy)\right]^2 = \frac{1}{i^2} F(iy) \cdot \frac{d^2}{dy^2} F(iy) - \left[\frac{1}{i}\frac{d}{dy} F(iy)\right]^2 = -F(iy) \cdot \frac{d^2}{dy^2} F(iy) + \left[\frac{d}{dy} F(iy)\right]^2 > 0$, which derives $F(y) \cdot F''(y) - [F'(y)]^2 < 0$.

This can be explained as follows:

By Eq. (4), $F(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ is the real part and $v(x,y)$ is the imaginary part of $F(z)$. Since

$$F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial (iy)} + i\frac{\partial v}{\partial (iy)}$$

$$F''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial (iy)^2} + i\frac{\partial^2 v}{\partial (iy)^2}$$

and so on.

Since the imaginary part vanishes on x-axis and y-axis, by Eq. (7) and (8), $F(x) = u(x,0)$ and $F(iy) = u(0,y)$. Indeed, $F(x)$ and $F(iy)$, and all their derivatives are generally not same but they are same at the origin where $x = 0$ and $y = 0$. Hence, we can write:

$F(x)|_{x=0} = F(iy)|_{y=0}$ $\quad \frac{\partial}{\partial x} F(x)\Big|_{x=0} = \frac{\partial}{\partial (iy)} F(iy)\Big|_{y=0}$ $\quad \frac{\partial^2}{\partial x^2} F(x)\Big|_{x=0} = \frac{\partial^2 u}{\partial (iy)^2} F(iy)\Big|_{y=0}$ and so on,

hence we have

$$\left(F(x)\cdot\frac{d^2}{dx^2}F(x)-\left[\frac{d}{dx}F(x)\right]^2\right)\bigg|_{x=0}=\left(F(iy)\cdot\frac{d^2}{d(iy)^2}F(iy)-\left[\frac{d}{d(iy)}F(iy)\right]^2\right)\bigg|_{y=0} \quad (31)$$

which leads

$$(F(x)\cdot F''(x)-[F'(x)]^2)|_{x=0}=(-F(y)\cdot F''(y)+[F'(y)]^2)|_{y=0} \quad (32)$$

Therefore, if both $F(x)\cdot F''(x)-[F'(x)]^2$ and $F(iy)\cdot F''(iy)-[F'(iy)]^2$ do not change the sign $\forall x,y\in\mathbb{R}$, the sign of $F(x)\cdot\frac{d^2}{dx^2}F(x)-\left[\frac{d}{dx}F(x)\right]^2$ and $F(iy)\cdot\frac{d^2}{d(iy)^2}F(iy)-\left[\frac{d}{d(iy)}F(iy)\right]^2$ does not change $\forall x,y\in\mathbb{R}$ and the sign is same as $x=0$ and $y=0$.

Furthermore, if $G(x)$ is the sum of products of two derivatives of $F(x)$, and $G(x)$ and $G(iy)$ do not change the sign $\forall x,y\in\mathbb{R}$, then the sign $G(x)|_{x=iy}$ is same as of $G(x)$ as long as $G(iy)$ is real.

We have proved the first step of the Laguerre inequalities. From (19), if n is even, the n^{th} derivative of $F(z)$ is as follows:

$$F^{(2k)}(z)=\int_{-\infty}^{\infty}f(t)\cdot t^{2k}\cdot e^{-zt}dt \quad (33)$$

where $n=2k$.

We define $f_{2k}(t)\equiv f(t)\cdot t^{2k}$. Since both $f(t)$ and t^{2k} are non-negative and even, $f_{2k}(t)$ is also a non-negative even function. So, we can write

$$F^{(2k)}(z)\equiv F_{2k}(z)=\int_{-\infty}^{\infty}f_{2k}(t)\cdot e^{-zt}dt \quad (34)$$

where k = 0, 1, 2, …

We have proved that the Laplace transform of any non-negative even function holds the Laguerre inequalities. That is, $F_{2k}(iy)\equiv F^{(2k)}(iy)$ is log-concave for all $k\geq 0$.[3]

Now, we will prove the Laguerre inequalities for odd n. Let n be $2k+1$ for k = 0, 1, 2, … , then form (7), the $(2k+1)^{th}$ derivative of $F(x)$, i.e., $F^{(2k+1)}(x)$ is as follows:

$$F^{(2k+1)}(x)=\int_{-\infty}^{\infty}f(t)\cdot t^{2k+1}\cdot e^{-zt}dt=x\cdot\int_{-\infty}^{\infty}f(t)\cdot t^{2k+2}\cdot\frac{e^{-zt}}{xt}dt \quad (35)$$

and since

$$\int_{-\infty}^{-1}e^{zt\tau}d\tau=\frac{e^{-zt}}{xt}$$

we have

$$F^{(2k+1)}(x)=x\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{-1}f(t)\cdot t^{2k+2}\cdot e^{zt\tau}\,d\tau\cdot dt \quad (36)$$

We define $G_{2k}(x)$ as

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[3]Indeed, $F^{(2k)}(iy)=(-1)^k\cdot F^{(2k)}(x)\big|_{x=iy}$, but the sign of $F^{(2k)}(iy)\cdot F^{(2k+2)}(iy)$ and $\left[F^{(2k+1)}(iy)\right]^2$ is same for all k.

$$G_{2k}(x) \equiv \frac{1}{x} \cdot F^{(2k+1)}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{-1} f(t) \cdot t^{2k+2} \cdot e^{xt\tau} \, d\tau \cdot dt \tag{37}$$

For $x1$ and $x2$ ($x1 \neq x2$), and $\lambda, \mu > 0$, $\lambda + \mu = 1$,

$$G_{2k}(\lambda x_1 + \mu x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{-1} f(t) \cdot t^{2k+2} \cdot e^{(\lambda x_1 + \mu x_2) \cdot t\tau} \, d\tau \cdot dt$$

or

$$G_{2k}(\lambda x_1 + \mu x_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{-1} [f(t) \cdot t^{2k+2} \cdot e^{x_1 t\tau}]^{\lambda} \cdot [f(t) \cdot t^{2k+2} \cdot e^{x_2 t\tau}]^{\mu} \, d\tau \cdot dt$$

and by the Hölder inequality of double integral. We have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{-1} [f(t) \cdot t^{2k+2} \cdot e^{x_1 t\tau}]^{\lambda} \cdot [f(t) \cdot t^{2k+2} \cdot e^{x_2 t\tau}]^{\mu} \, d\tau \cdot dt$$

$$< \left[\int_{-\infty}^{\infty} \int_{-\infty}^{-1} f(t) \cdot t^{2k+2} \cdot e^{x_1 t\tau} \, d\tau \cdot dt\right]^{\lambda} \cdot \left[\int_{-\infty}^{\infty} \int_{-\infty}^{-1} f(t) \cdot t^{2k+2} \cdot e^{x_2 t\tau} \, d\tau \cdot dt\right]^{\mu}$$

hence

$$G_{2k}(\lambda x_1 + \mu x_2) < [G_{2k}(x_1)]^{\lambda} \cdot [G_{2k}(x_2)]^{\mu} \tag{38}$$

which means $G_{2k}(x)$ is log-convex.

From (35) and (37), the inequality (38) can be written:

$\int_{-\infty}^{\infty} f(t) \cdot t^{2k+2} \cdot \mathrm{sinhc}(\lambda x_1 t + \mu x_2 t) \, dt$

$$< \left[\int_{-\infty}^{\infty} f(t) \cdot t^{2k+2} \cdot \mathrm{sinhc}(x_1 t) \, dt\right]^{\lambda} \cdot \left[\int_{-\infty}^{\infty} f(t) \cdot t^{2k+2} \cdot \mathrm{sinhc}(x_2 t) \, dt\right]^{\mu}$$

where $\mathrm{sinhc}(xt) = \frac{\sinh(xt)}{xt}$. Note that $\mathrm{sinhc}(xt) \geq 1$ and even.

With the same manner we used before, it can be shown that

$$G_{2k}(iy) = (-1)^k \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2k+2} \cdot \mathrm{sinc}(yt) \, dt$$

is strictly log-concave. The function $\mathrm{sinc}(yt)$ is defined as $\frac{\sin(xt)}{xt}$.

From (36),

$$F^{(2k+1)}(x) = x \cdot G_{2k}(x) \tag{39}$$

and

$$F^{(2k+1)}(iy) = (-1)^k \cdot y \cdot G_{2k}(iy) \tag{40}$$

$G_{2k}(x)$ is log-convex but x is log-concave, therefore $F^{(2k+1)}(x)$ is not log-convex for all $x \in \mathbb{R}$. $F^{(2k+1)}(iy)$, however, is log-concave $\forall y \in \mathbb{R}$ because both $\pm y$ and $G_{2k}(iy)$ are log-concave $\forall y \in \mathbb{R}$.

Since $F(iy)$ is nothing but Fourier Transform of $f(t)$, we have shown that the Fourier transform of a non-negative even function satisfies the Laguerre inequalities.

4. The generalized Laguerre inequalities

From (1), the Laplace transform is defined as

$$F(z) \equiv \int_{-\infty}^{\infty} f(t) \cdot e^{-zt} dt$$

and therefore

$$|F(z)|^2 = F(z) \cdot F^*(z) = F(x+iy) \cdot F(x-iy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t_1) \cdot f(t_2) \cdot e^{-x\cdot(t_1+t_2)} \cdot e^{-iy(t_1-t_2)}\, dt_1 \cdot dt_2 \quad (41)$$

By changing the variables $\mathrm{t} = t_1 + t_2$ and $\tau = t_1$, we have

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-xt} \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)}\, d\tau\, dt \quad (42)$$

Letting $\tau \mapsto -\tau$, and assuming $f(t)$ is even, we have

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{iy\tau} \cdot e^{iy(t+\tau)} d\tau\right] e^{-xt}\, dt \quad (43)$$

or simply,

$$|F(x+iy)|^2 = \int_{-\infty}^{\infty} r_y(t) \cdot e^{-xt}\, dt \quad (44)$$

where

$$r_y(t) = \int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} d\tau = \int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{iy\tau} \cdot e^{iy(t+\tau)} d\tau \quad (45)$$

The conjugate of $r_y(t)$

$$r_y^*(t) = \int_{-\infty}^{\infty} f(\tau) \cdot f(t+\tau) \cdot e^{-iy\tau} \cdot e^{-iy(t+\tau)} d\tau \quad (46)$$

and by substitution $\tau \mapsto \tau - t$, and assuming $f(t)$ is even, we have

$$r_y^*(t) = \int_{-\infty}^{\infty} f(\tau - t) \cdot f(\tau) \cdot e^{-iy(\tau-t)} \cdot e^{-iy\tau} d\tau = \int_{-\infty}^{\infty} f(\tau) \cdot f(t-\tau) \cdot e^{-iy\tau} \cdot e^{iy(t-\tau)} \cdot d\tau$$

which is the equation (34), therefore, $r_y(t)$ is real. Moreover, $r_y(t)$ is an even function which can be easily proved. The function $r_y(t)$ is real and even but does not hold the positivity, namely, It can be negative.

Since $|F(x+iy)|^2$ is even for x, the Eq. (44) can be written as

$$|F(x+iy)|^2 = \sum_{n=0}^{\infty} A_{2n} \cdot x^{2n} \quad (47)$$

where

$$A_{2n} = \frac{1}{(2n)!}\int_{-\infty}^{\infty} r_y(t)\cdot t^{2n}\,dt \tag{48}$$

Imagine $|F(x+iy)|^2$ on the horizontal line where y is constant. If $A_{2n} \geq 0$ for all n, then we have a unique global minimum at $x=0$ and $|F(x+iy)|^2$ is increasing while $|x|$ increasing. Hence If A_{2n} is non-negative for all n, zeros of $|F(x+iy)|^2$ can exist only at $x=0$, i.e., iy-axis.

By reforming (45), so that

$$r_y(t) = \int_{-\infty}^{\infty} f(\tau)\cdot e^{iy\tau}\cdot f(t+\tau)\cdot e^{iy(t+\tau)}d\tau \tag{49}$$

and by letting $g(\tau) = f(\tau)\cdot e^{-iy\tau}$, $r_y(t)$ is the cross-correlation function of $g(\tau)$ and $g^*(\tau)$ where $g^*(\tau) = f(\tau)\cdot e^{iy\tau}$. Let $F(\omega)$ be the Fourier transform of $f(\tau)$, then the Fourier transform of $g(\tau)$ is $F(\omega-y)$ and the Fourier transform of $g^*(\tau)$ is $F(\omega+y)$. By the cross-correlation theorem, we have

$$r_y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega-y)\cdot F(\omega+y)\cdot e^{it\omega}d\omega$$

and since $F(y)$ is even, we have

$$r_y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(y-\omega)\cdot F(y+\omega)\cdot e^{it\omega}d\omega \tag{50}$$

which is similar to the Wigner-Ville distribution function. By changing variable $x = i\theta$, from (44), we have

$$|F(i\theta+iy)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(y-\omega)\cdot F(y+\omega)\cdot e^{i\omega t}\cdot e^{-i\theta t}\,d\omega\,dt \tag{51}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(y-\omega)\cdot F(y+\omega)\cdot e^{i\omega t}\cdot e^{-i\theta t}\,d\omega\,dt = \int_{-\infty}^{\infty} F(y-\omega)\cdot F(y-\omega)\cdot\left[\int_{-\infty}^{\infty} e^{i\omega t}\cdot e^{-i\theta t}dt\right]d\omega$$

and

$$\int_{-\infty}^{\infty} e^{i\omega t}\cdot e^{-i\theta t}dt = 2\pi\cdot\delta(\theta-\omega)$$

thus, we have

$$|F(i\theta+iy)|^2 = \int_{-\infty}^{\infty} F(y-\omega)\cdot F(y+\omega)\cdot\delta(\omega-\theta)\,d\omega$$

and by omitting i for convenience, we have,

$$|F(\theta+y)|^2 = F(y-\theta)\cdot F(y+\theta) \tag{52}$$

which is the characteristic equation of $|F(x+iy)|^2$ where $\mathrm{x} = i\theta$, hence, from (44)

$$|F(\theta+y)|^2 = \int_{-\infty}^{\infty} r_y(t)\cdot e^{-i\theta t}\,dt \tag{53}$$

The n^{th} moment of $|F(x+iy)|^2$, which is denoted as M_n, is defined as follows

$$M_n(y) = \int_{-\infty}^{\infty} t^n \cdot r_y(t)\, dt \tag{54}$$

or

$$M_n(y) = (-1)^n \cdot D_x^n |F(x+iy)|^2|_{x=0} \tag{55}$$

Another method to get $M_n(y)$ is differentiating (53), that is,

$$M_n(y) = \frac{1}{(-i)^n} \cdot D_\theta^n |F(\theta + y)|^2|_{\theta=0}$$

or by (52), $M_n(y)$ is $\frac{1}{(-i)^n} \cdot D_\theta^n [F(\theta - y) \cdot F(\theta + y)]_{\theta=0}$ which can be computed using the Leibniz rule, that is,

$$M_n(y) = \frac{1}{(-i)^n} \cdot D_\theta^n [F(y-\theta) \cdot F(y+\theta)]_{\theta=0} = \frac{1}{(-i)^n} \cdot \sum_{k=0}^{n} (-1)^k \cdot \binom{n}{k} \cdot F^{(k)}(y) \cdot F^{(n-k)}(y) \tag{56}$$

However, since $r_y(t)$ is an even function, $M_n(y)$ vanishes when n is odd and we need to compute only for even n, hence,

$$M_{2n}(y) = D_\theta^{2n} [F(y-\theta) \cdot F(y+\theta)]_{\theta=0} = (-1)^n \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \tag{57}$$

and we have

$$|F(x+iy)|^2 = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \cdot M_{2n}(y) \cdot x^{2n} = \sum_{n=0}^{\infty} L_n(y) \cdot x^{2n} \tag{58}$$

where

$$L_n(y) = (-1)^n \cdot \frac{1}{(2n)!} \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \tag{59}$$

Theorem 4: The generalized Laguerre inequalities

The zeros of $F(z)$ locate only on the iy-axis if and only if $L_n(y) \geq 0$ for any y and n.

Definition: The copositive-definite function

A function $f(x)$ is copositive-definite if and only if

$$\sum_{n=1}^{N} \sum_{k=1}^{N} c_n c_k^* \, f(x_n + \mathrm{x}_k) \geq 0 \tag{60}$$

for any complex values c_n, real values x_n and non-negative integer $N > 0$.

Some properties of the copositive-definite function

1. If $f(x)$ is copositive-definite, $f(0) \geq 0$.

2. If $f(x)$ is copositive-definite, its $(2n)^{th}$ derivatives, i.e., $f^{(2n)}(x)$ is also copositive-definite.
3. If $f(x)$ and $g(x)$ are copositive-definite, $f(x) \cdot g(x)$ is also copositive-definite.

Indeed, $F(y+\theta) = F(iy+i\theta) = F[i(y+\theta)]$, i.e., this function lies on the iy-axis. $F(y-\theta)$ is the same. We will map $F(y-\theta) \cdot F(y+\theta)$ on x-axis, i.e., $F(x-\theta) \cdot F(x+\theta)$, and we have

$$F(x-\theta) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{\theta t} dt \tag{61}$$

and

$$F(x+\theta) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot e^{-\theta t} dt \tag{62}$$

$F(x-\theta)$ is copositive-definite for θ, because

$$\sum_{n=1}^{N} \sum_{k}^{N} c_n c_k^* F(x-(\theta_n+\theta_k)) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot \sum_{n=1}^{N} \sum_{k}^{N} c_n c_k^* \, e^{(\theta_n+\theta_k)t} \, dt$$
$$= \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} \cdot \left| \sum_{n=1}^{N} c_n e^{\theta_n t} \right|^2 dt \geq 0$$

In the same way, $F(x+\theta)$ is also copositive-definite for θ, thus $F(x-\theta) \cdot F(x+\theta)$ is copositive-definite for θ by the property 3.

Since $F(x-\theta) \cdot F(x+\theta)$ is copositive-definite for θ, $D_\theta^{2n}[F(x-\theta) \cdot F(x+\theta)]$ is copositive-definite by the property 3. Also, $D_\theta^{2n}[F(x-\theta) \cdot F(x+\theta)]_{\theta=0} \geq 0$ and we have

$$M_{2n}(x) = D_\theta^{2n}[F(x-\theta) \cdot F(x+\theta)]_{\theta=0} = \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(x) \cdot F^{(2n-k)}(x) \geq 0 \tag{63}$$

and

$$L_n(x) = \frac{1}{(2n)!} \cdot M_{2n}(y)$$

$F^{(k)}(x) \cdot F^{(2n-k)}(x)$ is an even function, and therefore $M_{2n}(x)$ is an even function, hence the power series of $M_{2n}(x)$ has only even powers of x.Thus by the proposition 1, $P_{2n}(x) \equiv M_{2n}(\sqrt{x})$ does not change the sign for $x \geq 0$, and $Q_{2n}(x) \equiv P_{2n}(-x)$ does not change the sign for $x \leq 0$.

From (7), we have

$$F(x) = u(x,0) = \int_{-\infty}^{\infty} f(t) \cdot e^{-xt} dt = \int_{-\infty}^{\infty} f(t) \cdot e^{xt} dt$$

and

$$F^{(k)}(x) \cdot F^{(2n-k)}(x) = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{xt} \, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{xt} \, dt \tag{64}$$

Let $p_k(x) = F^{(k)}(x) \cdot F^{(2n-k)}(x)\big|_{x=\sqrt{x}} = F^{(k)}(\sqrt{x}) \cdot F^{(2n-k)}(\sqrt{x})$, and $q_k(x) = p_k(-x)$, then

$$p_k(-x) = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{-\sqrt{x}\cdot t}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{-\sqrt{x}\cdot t}\, dt$$

If $x \leq 0$, then

$$p_k(-x) = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{-i\sqrt{|x|}\cdot t}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{-i\sqrt{|x|}\cdot t}\, dt = p_k(-i \cdot |x|) \tag{65}$$

By changing variable $|x| \mapsto y^2$, we have

$$p_k(-i \cdot |x|)\big|_{|x|=y^2} = p_k(-i \cdot y^2) = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{-i\cdot|y|\cdot t}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{-i\cdot|y|\cdot t}\, dt$$

and since

$$\int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{i\cdot|y|\cdot t}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{i\cdot|y|\cdot t}\, dt = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{-i\cdot|y|\cdot t}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{-i\cdot|y|\cdot t}\, dt$$

we have

$$p_k(-i \cdot y^2) = \int_{-\infty}^{\infty} f(t) \cdot t^k \cdot e^{-i\cdot yt}\, dt \cdot \int_{-\infty}^{\infty} f(t) \cdot t^{2n-k} \cdot e^{-iy\cdot t}\, dt$$

which is nothing but

$$p_k(-i \cdot |x|)\big|_{|x|=y^2} = (-1)^n \cdot F^{(k)}(i \cdot y) \cdot F^{(2n-k)}(i \cdot y) \tag{66}$$

From (63) we can write

$$Q_{2n}(x) \equiv P_{2n}(-x) = \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot p_k(-x) \cdot p_{2n-k}(-x) \tag{67}$$

where $Q_{2n}(x)$ does not change the sign for $x \leq 0$, hence

$$M_{2n}(y) = D_\theta^{2n}[F(y-\theta) \cdot F(y+\theta)]_{\theta=0} = (-1)^n \cdot \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(iy) \cdot F^{(2n-k)}(iy)$$

or since $F^{(k)}(iy) \cdot F^{(2n-k)}(iy)$ is real, i can be omitted. Thus

$$M_{2n}(y) = (-1)^n \cdot \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \tag{68}$$

does not change sign for all $y \in \mathbb{R}$ and $n \geq 0$, which yields $M_{2n}(y) \geq 0$ or $M_{2n}(y) \leq 0$ for all $y \in \mathbb{R}$ and $n \geq 0$.

To determine the sign of $M_{2n}(y)$, we simply substitute $n = 0$ and $y = 0$, because the sign of $M_{2n}(y)$ does not change for all $y \in \mathbb{R}$ and $n \geq 0$. Hence

$$M_0(0) = [f(0)]^2 = a_0^2 > 0$$

where a_0 is defined in (3). Therefore $M_{2n}(y) \geq 0$ for all $y \in \mathbb{R}$ and $n \geq 0$.

Another way to determine the sign of $M_{2n}(y)$ is the substitution x to iy. Since

$$F^{(k)}(x) \cdot F^{(2n-k)}(x)\Big|_{x=0} = \frac{d^k}{d(iy)^k} F(y) \cdot \frac{d^{2n-k}}{d(iy)^{2n-k}} F(y)\Bigg|_{y=0}$$

and $M_{2n}(x)$ and $M_{2n}(iy)$ do not change the sign, from (63) we have

$$M_{2n}(iy) = \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot \frac{d^k}{d(iy)^k} F(y) \cdot \frac{d^{2n-k}}{d(iy)^{2n-k}} F(y) \geq 0$$

which is

$$M_{2n}(iy) = \frac{1}{i^k \cdot i^{2n-k}} \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot \frac{d^k}{d(iy)^k} F(y) \cdot \frac{d^{2n-k}}{d(iy)^{2n-k}} F(y) \geq 0$$

Hence

$$M_{2n}(iy) = M_{2n}(y) = (-1)^n \cdot \sum_{k=0}^{2n} (-1)^k \cdot \binom{2n}{k} \cdot F^{(k)}(y) \cdot F^{(2n-k)}(y) \geq 0$$

Since $L_n(x) = \frac{1}{(2n)!} \cdot M_{2n}(y) \geq 0$, the generalized Laguerre inequalities are valid for a two-sided Laplace transform $F(z)$ of a non-negative even function $f(t)$ as long as $F(z)$ converses.

5. The Riemann hypothesis

The Riemann zeta function $\zeta(s)$ is defined

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

where $s = \sigma + i\omega$.

It is known that the zeros of $\zeta(s)$ are located only on the strip $0 < \sigma < 1$. Riemann conjectured that all the zeros of $\zeta(s)$ are located on the line $\sigma = \frac{1}{2}$, so-called “Riemann hypothesis”.

Using the Riemann’s functional equation, an entire and symmetric function can be obtained which is called the xi function $\xi(s)$ where

$$\xi(s) = \frac{1}{2} \pi^{-\frac{s}{2}} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s) \tag{69}$$

The function $\xi(s)$ can be written as

$$\xi(s) = \int_{-\infty}^{\infty} \varphi(t) \cdot e^{-\left(s-\frac{1}{2}\right)t} dt \tag{70}$$

where

$$\varphi(t) = 2\pi \sum_{n=1}^{\infty} n^2 \cdot e^{-\pi n^2 e^{2t}} \cdot \left(2\pi n^2 e^{\frac{9}{2}t} - 3e^{\frac{5}{2}t}\right) \tag{71}$$

and it can be shown that $\varphi(t) > 0$ for all t and an even function.

By substitution $z = s - \frac{1}{2}$ where $z = x + iy$, and $\varphi(t)$ is even, we have

$$\Phi(z) = \int_{-\infty}^{\infty} \varphi(\mathrm{t}) \cdot e^{-zt} dt = \int_{-\infty}^{\infty} \varphi(\mathrm{t}) \cdot \cosh(zt)\, dt \tag{74}$$

and since $\Phi(\mathrm{z})$ is a shifted function by $\frac{1}{2}$ of $\xi(\mathrm{s})$, $\Phi(\mathrm{z})$ is entire and the zeros of $\Phi(\mathrm{z})$ should be located on the strip $-\frac{1}{2} < x < \frac{1}{2}$. From (69), we have

$$\Phi(z) = \frac{1}{2} \pi^{-\frac{1}{4}} \cdot \pi^{-\frac{z}{2}} \cdot \left(z^2 - \frac{1}{4}\right) \cdot \Gamma\left(\frac{z}{2} + \frac{1}{4}\right) \cdot \zeta\left(z + \frac{1}{2}\right) \tag{75}$$

which is Riemann's original definition of xi-function.

We consider the function $\varphi(\mathrm{t})$ defined in (71). It is positive and even. Moreover, it is decreasing very rapidly[4], thus $\Phi(iy)$ belongs to the Laguerre-Pólya class and has only real zeros. It means that all the zeros of $\Phi(\mathrm{z})$ are located at $\mathrm{x} = 0$, and hence, all the zeros of $\xi(\mathrm{s})$ and $\zeta(s)$ are located at $\sigma = \frac{1}{2}$. Thus, the Riemann hypothesis is true.

Another popular definition $\Phi(z)$ is:

$$\Phi(iz) = \int_{-\infty}^{\infty} \varphi(\mathrm{t}) \cdot \cos(zt)\, dt = 2 \int_{0}^{\infty} \varphi(\mathrm{t}) \cdot \cos(zt)\, dt$$

and by substitution $\mathrm{t} \mapsto 2\mathrm{t}$, we have

$$\Phi(iz) = 4 \int_{0}^{\infty} \varphi(2\mathrm{t}) \cdot \cos(2zt)\, dt \tag{76}$$

We define $\emptyset(\mathrm{t})$ as

$$\emptyset(\mathrm{t}) = \pi \sum_{n=1}^{\infty} n^2 \cdot e^{-\pi n^2 e^{2t}} \cdot (2\pi n^2 e^{9t} - 3e^{5t})$$

then $\emptyset(\mathrm{t}) = \frac{1}{2}\, \varphi(2\mathrm{t})$ and eq. (74) will be

$$\Phi(iz) = 8 \int_{0}^{\infty} \emptyset(\mathrm{t}) \cdot \cos(2zt)\, dt$$

and by defining $\Xi(\mathrm{z}) = \frac{1}{8}\, \Phi(\mathrm{iz}/2)$, we have

$$\Xi(\mathrm{z}) = \int_{0}^{\infty} \emptyset(\mathrm{t}) \cdot \cos(zt)\, dt \tag{77}$$

or simply,

$$\Xi(z) = 2\, \Phi(iz) \tag{78}$$

This function is called the big-xi or upper-case xi function and used to prove the Riemann hypothesis and to find the location of zeros in most literatures. Since $\Phi(iz)$ is only the rotation of $\Phi(z)$ by 90^o, the zeros of $\Xi(z)$ locate only on the x-axis.

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[4] According to rule of thumb (10), $\frac{\Phi(1)}{\Phi(0)} - 1 \approx 0.0233$, which is much smaller than $f(t) = e^{-t^2}$