Principles for the Hypercomplex Electrodynamics

Vyacheslav Telnin

Abstract

In classical electrodynamics, four-vectors with four real numbers t, x, y, z are used. It is noted that the same result can be achieved with the help of quaternions with three real numbers x, y, z and one imaginary $i \cdot t$. This suggests that we can go further, and consider all four numbers t, x, y, z complex. And deal with quaternions with complex coefficients: $(a + i \cdot t)$, $(x + i \cdot b)$, $(y + i \cdot c)$, $(z + i \cdot d)$. These objects, for the sake of brevity, we call octads. But you can go even further, and work with quaternions, where all four numbers t, x, y, z also quaternionic. For the sake of brevity, we call these objects Q2 numbers. All next text deals with the translation of classical electrodynamics into the languages of octads and Q2 numbers.

Content

1.Quaternions.

From the set of hypercomplex numbers, quaternions are distinguished. These are numbers of the form:

$$i \cdot i = j \cdot j = k \cdot k = i \cdot j \cdot k = -1$$

$$i_1 = 1;\ \ i_2 = i;\ \ i_3 = j;\ \ i_4 = k;$$

$$i_n \cdot i_k = i_m \cdot \varphi^m{}_{n\,k}$$

$$x = i_n \cdot x^n$$

$$(n = 1, 2, 3, 4)\ (or\ n = a, x, y, z\ )$$

The multiplication table for the quaternions:

$$i_n \cdot i_k = i_m \cdot \varphi^m{}_{n\,k}$$

	i_1	i_2	i_3	i_4	i_k
i_1	i_1	i_2	i_3	i_4	
i_2	i_2	$-i_1$	i_4	$-i_3$	
i_3	i_3	$-i_4$	$-i_1$	i_2	
i_4	i_4	i_3	$-i_2$	$-i_1$	
i_n					$i_m \cdot \varphi^m{}_{n\,k}$

2.Cartesian product of quaternions.

We can consider the Cartesian product of several quaternions:

$$Y1 \times Y2 \times Y3 \times \ldots \times YN = QN;$$

The numbers QN are multiplied with one another by the corresponding quaternions.

Let's consider the simple case N = 2:

$$Q2 = Y1 \ \times Y2;$$

Let Q be one of the numbers Q2:

$$Q = i_{\mu} \times i_{n} \cdot x^{\mu} \cdot x^{n}$$

$$(\mu = 1, 2, 3, 4) \ \ (\text{или } \mu = a, x, y, z\,)$$

$$(n = 1, 2, 3, 4) \ (\text{или } n = a, x, y, z\,)$$

It can be seen that Q is a 16-dimensional vector:

$$Q = m_k \cdot Q^k$$

k = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16)

	i_1	i_2	i_3	i_4	i_n
i_1	1	2	9	10	
i_2	3	4	11	12	
i_3	5	6	13	14	
i_4	7	8	15	16	
i_μ					k

Q^k = (a, t, x, b, y, c, z, d, г, ц, ж, ш, и, щ, л, ъ)

Q^k	i_1	i_2	i_3	i_4	i_n
i_a	a	t	г	ц	
i_x	x	b	ж	ш	
i_y	y	c	и	щ	
i_z	z	d	л	ъ	
i_μ					$x^{\mu} \cdot x^n$

а б в г д е ё ж з и й

к л м н о п р с т у ф

х ц ч ш щ ъ ы ь э ю я (Russian alphabet)

In general terms:

$$m_k = i_{\mu} \times i_{n} \cdot m^{\mu\, n}{}_{k}$$

$$Q^k = Q_{\nu\, m}{}^{k} \cdot x^{\nu} \cdot x^{m}$$

The algebraic tensors for quaternions and Q2 numbers are defined as:

$$i_{n} \cdot i_k = i_{m} \cdot \varphi^{m}{}_{n\, k}$$

$$m_{n} \cdot m_{k} = m_{r} \cdot F^{r}{}_{n\, k}$$

We express $F^{r}{}_{n\, k}$ in terms of $\varphi^{m}{}_{n\, k}$:

$$m_{n} \cdot m_{k} = (n_{\nu} \times i_{p} \cdot m^{\nu\, p}{}_{n}) \cdot (n_{\mu} \times i_{q} \cdot m^{\mu\, q}{}_{k}) =$$

$$= (n_{\nu} \cdot n_{\mu}) \times (i_{p} \cdot i_{q}) \cdot m^{\nu\, p}{}_{n} \cdot m^{\mu\, q}{}_{k} =$$

$$= (n_{\lambda} \cdot \varphi^{\lambda}{}_{\nu\mu}) \times (i_h \cdot \varphi^{h}{}_{p\, q}) \cdot m^{\nu\, p}{}_{n} \cdot m^{\mu\, q}{}_{k} =$$

$$= (n_{\lambda} \times i_h) \cdot \varphi^{\lambda}{}_{\nu\mu} \cdot \varphi^{h}{}_{p\,q} \cdot m^{\nu\,p}{}_{n} \cdot m^{\mu\,q}{}_{k} =$$

$$= m_r \cdot F^{r}{}_{n\,k} = (n_{\lambda} \times i_h) \cdot m^{\lambda\,h}{}_{r} \cdot F^{r}{}_{n\,k}$$

From here:

$$m^{\lambda\,h}{}_{r} \cdot F^{r}{}_{n\,k} = \varphi^{\lambda}{}_{\nu\mu} \cdot \varphi^{h}{}_{p\,q} \cdot m^{\nu\,p}{}_{n} \cdot m^{\mu\,q}{}_{k}$$

We introduce for $m^{\lambda\,h}{}_{r}$ the inverse element $ш^{r}{}_{\rho\,\omega}$ as:

$$ш^{r}{}_{\rho\,\omega} \cdot\ m^{\lambda\,h}{}_{r} = \delta^{\lambda}_{\rho} \cdot \delta^{h}_{\omega}$$

And then we get

$$F^{r}{}_{n\,k} = ш^{r}{}_{\rho\,\omega} \cdot \varphi^{\rho}{}_{\nu\mu} \cdot \varphi^{\omega}{}_{p\,q} \cdot m^{\nu\,p}{}_{n} \cdot m^{\mu\,q}{}_{k}$$

So we expressed the algebraic tensor $F^{r}{}_{n\,k}$ for Q2 numbers in terms of the algebraic tensor $\varphi^{\rho}{}_{\nu\mu}$ for quaternions.

The numbers Q2 can be called quaternions with quaternion coefficients. And they require 16 real numbers for their description. For the simplicity of the derivation the basic equations, we consider only 8 real numbers: (a, t, x, b, y, c, z, d). These will be quaternions with complex coefficients. Let's call them octads.

So, if q is an octad, then:

$$q = i_{\mu} \times i_{n} \cdot x^{\mu} \cdot x^{n}$$

$$(\mu = 1, 2, 3, 4)\ \ (or\ \mu = a, x, y, z\ )$$

$$(n = 1, 2)$$

$$q = n_{\nu} \cdot q^{\nu}\ \ \ (\nu = a, t, x, b, y, c, z, d)$$

We use the following form ($i_1 = 1$; $i_2 = i$):

$$q^{\nu} = (a, t, x, b, y, c, z, d)$$

q^{ν}	1	i	i_n
i_a	a	t	
i_x	x	b	
i_y	y	c	
i_z	z	d	
i_{μ}			$x^{\mu} \cdot x^n$

$$q = i_a \times (a + i \cdot t) + i_x \times (x + i \cdot b) + i_y \times (y + i \cdot c) +$$
$$+ i_z \times (z + i \cdot d)$$

$$q = n_a \cdot a + n_t \cdot t + n_x \cdot x + n_b \cdot b + n_y \cdot y + n_c \cdot c +$$
$$+ n_z \cdot z + n_d \cdot d$$

$$n_t = i_a \times i; \quad n_b = i \times i; \quad n_c = i_y \times i; \quad n_d = i_z \times i;$$

To multiply octades, we build the following table:

$$n_{\mu} \cdot n_{\nu} = n_{\lambda} \cdot f^{\lambda}{}_{\mu\nu}$$

	a	**t**	**x**	**b**	**y**	**c**	**z**	**d**	**v**
a	a	t	x	b	y	c	z	d	
t	t	-a	b	-x	c	-y	d	-z	
x	x	b	-a	-t	z	d	-y	-c	
b	b	-x	-t	a	d	-z	-c	y	
y	y	c	-z	-d	-a	-t	x	b	
c	c	-y	-d	z	-t	a	b	-x	
z	z	d	y	c	-x	-b	-a	-t	
d	d	-z	c	-y	-b	x	-t	a	
μ									$f^{\lambda}{}_{\mu\nu} \cdot \lambda$

Here $f^{\lambda}{}_{\mu\nu}$ is an algebraic tensor for octads.

3.Algebraic and metric tensors

The multiplication of the vectors n_{μ} sets the algebraic tensor

$$n_{\mu} \cdot n_{\nu} = n_{\lambda} \cdot f^{\lambda}{}_{\mu\nu}$$

In a non-curved space $n_{a} = 1$ and one can define the metric tensor $\eta_{\mu\nu}$ via the algebraic tensor $f^{\lambda}{}_{\mu\nu}$ like this:

$$(n_{\mu}, n_{\nu}) = \eta_{\mu\nu} = n_{a} \cdot f^{a}{}_{\mu\nu} = 1 \cdot f^{a}{}_{\mu\nu} = f^{a}{}_{\mu\nu}$$

4.Conjugation

For quaternions $1^* = 1 \quad i^* = -i \quad j^* = -j \quad k^* = -k$

$$i_1 = 1; \quad i_2 = i; \quad i_3 = j; \quad i_4 = k;$$

$$i_1{}^* = i_1; \quad i_2{}^* = -i_2; \quad i_3{}^* = -i_3; \quad i_4{}^* = -i_4$$

$$i_n{}^* = i_k \cdot d^k{}_n$$

$$(i_n \cdot i_k)^* = i_k{}^* \cdot i_n{}^*$$

For Q2 numbers:

$$Q = i_\mu \times i_n \cdot x^\mu \cdot x^n$$

$$(\mu = 1,2,3,4) \; (n = 1,2,3,4)$$

You can conjugate Q in different ways: by the left Cartesian factor, by the right one, or by both at once. To distinguish between these conjugations, we introduce a dash sign over Q. If it exists, then it must be conjugated by the left Cartesian factor. If Q has an asterisk, then conjugate it by the right Cartesian factor. If there is both a dash and an asterisk, then it is necessary to conjugate both Cartesian factors.

$$\overline{Q} = i_\mu{}^* \times i_n \cdot x^\mu \cdot x^n \qquad Q^* = i_\mu \times i_n{}^* \cdot x^\mu \cdot x^n$$

$$\overline{Q}^* = i_\mu{}^* \times i_n{}^* \cdot x^\mu \cdot x^n$$

$$Q = m_\mu \cdot Q^\mu \qquad \overline{m_\mu} = m_\rho \cdot C^\rho{}_\mu \quad m_\mu{}^* = m_\rho \cdot H^\rho{}_\mu$$

If the product of Q2 numbers is conjugated, then a permutation of Q2 factors is added:

$$\overline{(Q \cdot R)} = \overline{R} \cdot \overline{Q} \qquad (Q \cdot R)^* = R^* \cdot Q^*$$

For octads, the conjugations behave the same as for Q2 numbers (consideration is made for Q2 numbers, and octads are a part of them).

$$q = n_{\mu} \cdot q^{\mu} \qquad \overline{n_\mu} = n_\rho \cdot c^{\rho}{}_{\mu} \qquad n_\mu{}^* = n_\rho \cdot h^{\rho}{}_{\mu}$$

$$c^{\mu}{}_{\mu} = (1, 1, -1, -1, -1, -1, -1, -1) \qquad \mu = (a, t, x, b, y, c, z, d)$$

$$h^{\mu}{}_{\mu} = (1, -1, 1, -1, 1, -1, 1, -1) \qquad \mu = (a, t, x, b, y, c, z, d)$$

We will call the tensors $d^{k}{}_{n}$, $C^{\rho}{}_{\mu}$, $H^{\rho}{}_{\mu}$, $c^{\rho}{}_{\mu}$, $h^{\rho}{}_{\mu}$ by the conjugation tensors.

5.Indexless expressions

For the octad, there are three types of its representation in formulas:

$$\begin{gathered} n_a \times (q^a + i \cdot q^t) + n_x \times \left(q^x + i \cdot q^b\right) + \\ + n_y \times (q^y + i \cdot q^c) + n_z \times \left(q^z + i \cdot q^d\right); \end{gathered}$$

II $n_{\mu} \cdot q^{\mu} \qquad (\mu = a, t, x, b, y, c, z, d)$

III q

The latter method – indexless – is the most compact. Let's look, for example, at generalization of Maxwell's equations, written in an indexless form for Q2 numbers (also for octads) (the output of this formula will be given later):

$$\overline{D} \cdot F = -\, 4\pi \cdot j$$

The mathematical language of *classic* electrodynamics uses three real coordinates (x, y, z) and one imaginary ($i \cdot t$). And, accordingly, three real components of the potential A^x, A^y, A^z and one imaginary component ($i \cdot A^t$).

The *octad* mathematical language uses 4 *complex* coordinates. That is, 4 real coordinates (a, x, y, z) and 4 imaginary coordinates ($i \cdot t,\ i \cdot b,\ i \cdot c,\ i \cdot d$). You can take the *indexless* action functions for *classical* electrodynamics and build by them the *indexless octad* action functions. Then vary these *indexless octad* action functions by the corresponding *indexless octad* variables. And we obtain equations for *index-free octad* electrodynamics. Then go to their *index octad* view. These equations will connect the 8 fields $F^a, F^t, F^x, F^b, F^y, F^c, F^z, F^d$ with the eight coordinates $q^{\,a}, q^{\,t}, q^{\,x}, q^{\,b}, q^{\,y}, q^{\,c}, q^{\,z}, q^{\,d}$. And you get the *index octad* electrodynamics.

If in the *indexless octad* equations you will consider all variables to be the Q2 numbers, then you get the *indexless Q2* electrodynamics. Then you should go to the *index Q2* form of equations. These equations will connect the sixteen fields F^μ ($\mu = a, t\ x, b, y, c, z, d,$ г, ц, ж, ш, и, щ , л, ъ) with the sixteen coordinates Q^μ. And we get the equations for *index Q2* electrodynamics.

6.The first indexless octad action function

In classical electrodynamics, the action function for a particle of mass m is:

$$S = -\,mc \cdot \int ds \qquad ds \cdot ds = (dx, dx)$$

$$dx - vector, \qquad ds, S - scalars$$

Consider the indexless octad action function for a particle of mass m:

$${}_1S = -\,mc \cdot \int ds \qquad ds \cdot ds = d\,\overline{q} \cdot dq$$

$$dq - octad \qquad ds,\, {}_1S - scalars$$

We vary the indexless S_1 by the indexless δq:

$$\delta\ _1 S = - mc \cdot \int \delta\, ds = - mc \cdot \int \frac{\delta(d\overline{q} \cdot dq)}{2 \cdot ds} =$$

$$= - mc \cdot \int \frac{(d\ \delta\overline{q}) \cdot dq + d\overline{q} \cdot (d\ \delta q)}{2 \cdot ds} =$$

$$= - \frac{1}{2} \cdot \int [(d\ \delta\overline{q}) \cdot p + \overline{p} \cdot (d\ \delta q)] =$$

$$= \frac{1}{2} \cdot \int (\delta\ \overline{q} \cdot dp + d\ \overline{p} \cdot \delta q) \qquad p = mc \cdot \frac{dq}{ds}$$

For a free particle at any $\delta\ \overline{q}$ and δq, there must be

$$\delta\ _1 S = 0$$

From here we have: $dp = 0 \qquad d\ \overline{p} = 0$

That is, the momentum p is conserved.

7.Indexless connection of fields F with potentials A

For the potentials A, we define the derivative by the coordinates q as:

$$dA = dq \cdot \frac{dA}{dq} = dq \cdot F$$

$$D = n_\nu \cdot \partial^\nu \qquad A = n_\mu \cdot A^\mu \qquad F = D \cdot A$$

8.Index octad connection of the fields F with potentials A.

Indexless view of fields: $F = D \cdot A$

And the index octad view is as follows:

$$F^{\lambda} = f^{\lambda}{}_{\nu\,\mu} \cdot \partial^{\nu} A^{\mu}$$

(Here it is taken into account that for octads, the algebraic tensor is $f^{\lambda}{}_{\nu\,\mu}$. For Q2 numbers, use the algebraic tensor $F^{\lambda}{}_{\nu\,\mu}$.)

For octads, all eight F^{λ} are:

$$F^{a} = \partial^{a}A^{a} - \partial^{\mathrm{t}}\mathrm{A}^{\mathrm{t}} - \partial^{x}A^{x} + \partial^{b}A^{b} -$$

$$-\partial^{y}A^{y} + \partial^{c}A^{c} - \partial^{z}A^{z} + \partial^{d}A^{d}$$

$$F^{t} = \partial^{a}A^{t} + \partial^{c}A^{a} - \partial^{x}A^{b} - \partial^{b}A^{x} -$$

$$-\partial^{y}A^{c} - \partial^{c}A^{y} - \partial^{z}A^{d} - \partial^{d}A^{z}$$

$$F^{x} = \partial^{a}A^{x} - \partial^{t}A^{b} + \partial^{x}A^{a} - \partial^{b}A^{t} +$$

$$+\partial^{y}A^{z} - \partial^{c}A^{d} - \partial^{z}A^{y} + \partial^{d}A^{c}$$

$$F^{b} = \partial^{a}A^{b} + \partial^{t}A^{x} + \partial^{x}A^{t} + \partial^{b}A^{a} +$$

$$+\partial^{y}A^{d} + \partial^{c}A^{z} - \partial^{z}A^{c} - \partial^{d}A^{y}$$

$$F^{y} = \partial^{a}A^{y} - \partial^{t}A^{c} - \partial^{x}A^{z} + \partial^{b}A^{d} +$$

$$+\partial^{y}A^{a} - \partial^{c}A^{t} + \partial^{z}A^{x} - \partial^{d}A^{b}$$

$$F^{c} = \partial^{a}A^{c} + \partial^{t}A^{y} - \partial^{x}A^{d} - \partial^{b}A^{z} +$$

$$+\partial^{y}A^{t} + \partial^{c}A^{a} + \partial^{z}A^{b} + \partial^{d}A^{x}$$

$$F^{z} = \partial^{a}A^{z} - \partial^{t}A^{d} + \partial^{x}A^{y} - \partial^{b}A^{c} -$$

$$-\partial^{y}A^{x} + \partial^{c}A^{b} + \partial^{z}A^{a} - \partial^{d}A^{t}$$

$$F^{d} = \partial^{a}A^{d} + \partial^{t}A^{z} + \partial^{x}A^{c} + \partial^{b}A^{y} -$$

$$-\partial^{y}A^{b} - \partial^{c}A^{x} + \partial^{z}A^{t} + \partial^{d}A^{a}$$

9.F fields in classical electrodynamics

To move from the octad electrodynamics to the classical one, you should put:

$$A^a = A^b = A^c = A^d = 0$$

And the remaining non-zero A^t, A^x, A^y, A^z should not depend on q^a, q^b, q^c, q^d. That is, the four derivatives:

$$\partial^a, \partial^b, \partial^c, \partial^d$$

from them should give zero. Here is what remains after this transition:

$$F^{a} = -\partial^{t}A^{t} - \partial^{x}A^{x} - \partial^{y}A^{y} - \partial^{z}A^{z} =$$

$$= \partial_{t}A^{t} + \partial_{x}A^{x} + \partial_{y}A^{y} + \partial_{z}A^{z} = 0$$

(This is zero by the Lorentz calibration.) Further

$$F^t = 0$$

$$F^x = \partial^y A^z - \partial^z A^y = -\partial_y A^z + \partial_z A^y = -H^x$$

$$F^b = \partial^t A^x + \partial^x A^t = -\partial_t A^x - \partial_x A^t = E^x$$

$$F^y = -\partial^x A^z + \partial^z A^x = \partial_x A^z - \partial_z A^x = -H^y$$

$$F^c = \partial^t A^y + \partial^y A^t = -\partial_t A^y - \partial_y A^t = E^y$$

$$F^z = \partial^x A^y - \partial^y A^x = -\partial_x A^y + \partial_y A^x = -H^z$$

$$F^d = \partial^t A^z + \partial^z A^t = -\partial_t A^z - \partial_z A^t = E^z$$

And we have:

$$F^a = 0 \quad F^t = 0 \quad F^x = -H^x \quad F^b = E^x$$

$$F^y = -H^y \quad F^c = E^y \quad F^z = -H^z \quad F^d = E^z$$

10.The second octad indexless action function

In classical electrodynamics, the action function for a particle with an electric charge e in an electromagnetic field is:

$$S = -\, e \cdot \int (A\,, dx)$$

$$dx \text{ и } A - vectors \qquad S - scalar$$

For octadic electrodynamics, as an analog, we construct

$${}_2S = -\, e \cdot \int \frac{1}{2} \cdot [\overline{A} \cdot dq + d\overline{q} \cdot A]$$

$$dq\,, A\ - octads,\ {}_2S - scalar$$

11.The third octad indexless action function

In classical electrodynamics, the action function for the

field

$$S = -\, \frac{1}{16\pi} \cdot \int F_{i\,k} \cdot F^{\,i\,k} \cdot d\Omega$$

$$d\Omega = dt \cdot dx \cdot dy \cdot dz$$

$F_{i\,k}$ – tensor   S – scalar

In three-dimensional form:

$$S = \frac{1}{8\pi} \cdot \int (E^2 - H^2) \cdot d\Omega$$

For octad electrodynamics, as an analog, we form:

$${}_3S = \frac{1}{8\pi} \cdot \int \overline{F} \cdot F \cdot d\Omega \qquad F = D \cdot A$$

$$d\Omega = da \cdot dt \cdot dx \cdot db \cdot dy \cdot dc \cdot dz \cdot dd$$

$$F - octad \qquad {}_3S - scalar$$

12.Variation by *δA*

$$\delta\ {}_3S = \frac{1}{8\pi} \cdot \int [\left(\overline{D \cdot \delta A}\right) \cdot F + \overline{F} \cdot (D \cdot \delta A)] \cdot d\Omega =$$

$$= \frac{1}{8\pi} \cdot \int [\left(\overline{\delta A} \cdot \overleftarrow{\overline{D}}\right) \cdot F + \overline{F} \cdot (D \cdot \delta A)] \cdot d\Omega =$$

$$= \frac{1}{8\pi} \cdot \int [-\delta\overline{A} \cdot \left(\overline{D} \cdot F\right) - (\overline{F} \cdot \overleftarrow{D}) \cdot \delta A]\ \cdot d\Omega$$

$$e = \int \rho \cdot \frac{d\Omega}{dt}$$

$$\delta\ {}_2S = -\int \rho \cdot \frac{d\Omega}{dt} \cdot \int \frac{1}{2} \cdot \left[\delta\overline{A} \cdot dq + d\overline{q} \cdot \delta A\right] =$$

$$= -\int \rho \cdot \frac{1}{2} \cdot \left[\delta\overline{A} \cdot \frac{dq}{dt} + \frac{d\overline{q}}{dt} \cdot \delta A\right] \cdot d\Omega =$$

$$= \int \frac{1}{2} \cdot \left[-\delta\overline{A} \cdot j - \overline{j} \cdot \delta A\right] \cdot d\Omega$$

$$j = \rho \cdot \frac{dq}{dt}$$

$$\delta\ {}_2S + \delta\ {}_3S = 0$$

$$\delta\overline{A} \cdot \left(-\frac{1}{2} \cdot j - \frac{1}{8\pi} \cdot \left(\overline{D} \cdot F\right)\right) = 0$$

$$\left(-\frac{1}{2} \cdot \overline{j} - \frac{1}{8\pi} \cdot \left(\overline{F} \cdot \overleftarrow{D}\right)\right) \cdot \delta A = 0$$

The last two equations coincide when conjugated. And give the following equation:

$$\overline{D} \cdot F = -4\,\pi \cdot j$$

This is an index-free equation for the fields of F. In the transition to classical electrodynamics, it passes into Maxwell's equations. Let's show it.

13.Index view of equations for fields F

Substitute in the indexless equation

$$\overline{D} \cdot F = -4\,\pi \cdot j$$

definitions for D, F, J:

$$D = n_{\lambda} \cdot \partial^{\lambda} \qquad \overline{n_{\lambda}} = n_{\mu} \cdot c^{\mu}{}_{\lambda} \qquad F = n_{\sigma} \cdot F^{\sigma} \qquad j = n_{\nu} \cdot j^{\nu}$$

We obtain an index formula suitable for octads:

$$f^{\,\nu}{}_{\mu\,\sigma} \cdot c^{\mu}{}_{\lambda} \cdot \partial^{\lambda} F^{\sigma} = -4\pi \cdot j^{\nu}$$

(Here it is taken into account that for octads, the algebraic tensor is $f^{\,\nu}{}_{\mu\,\sigma}$ and the conjugation tensor is $c^{\mu}{}_{\lambda}$. For Q2 numbers, use the algebraic tensor $F^{\,\nu}{}_{\mu\,\sigma}$ and the conjugation tensor $C^{\mu}{}_{\lambda}$.)

The case of Q2, due to the large length of the index equations, we will not prescribe here. We will limit ourselves to the octad variant.

In the case of the octads will be 8 equations:

$$-4\pi \cdot j^{a} = \partial^{a} F^{a} - \partial^{t} F^{t} + \partial^{x} F^{x} - \partial^{b} F^{b} +$$

$$+ \partial^{y} F^{y} - \partial^{c} F^{c} + \partial^{z} F^{z} - \partial^{d} F^{d}$$

$$-4\pi \cdot j^t = \partial^a F^t + \partial^t F^a + \partial^x F^b + \partial^b F^x +$$

$$+ \partial^y F^c + \partial^c F^y + \partial^z F^d + \partial^d F^z$$

$$-4\pi \cdot j^x = \partial^a F^x - \partial^t F^b - \partial^x F^a + \partial^b F^t -$$

$$- \partial^y F^z + \partial^c F^d + \partial^z F^y - \partial^d F^c$$

$$-4\pi \cdot j^b = \partial^a F^b + \partial^t F^x - \partial^x F^t - \partial^b F^a -$$

$$- \partial^y F^d - \partial^c F^z + \partial^z F^c + \partial^d F^y$$

$$-4\pi \cdot j^y = \partial^a F^y - \partial^t F^c + \partial^x F^z - \partial^b F^d -$$

$$- \partial^y F^a + \partial^c F^t - \partial^z F^x + \partial^d F^b$$

$$-4\pi \cdot j^c = \partial^a F^c + \partial^t F^y + \partial^x F^d + \partial^b F^z -$$

$$- \partial^y F^t - \partial^c F^a - \partial^z F^b - \partial^d F^x$$

$$-4\pi \cdot j^z = \partial^a F^z - \partial^t F^d - \partial^x F^y + \partial^b F^c +$$

$$+ \partial^y F^x - \partial^c F^b - \partial^z F^a + \partial^d F^t$$

$$-4\pi \cdot j^d = \partial^a F^d + \partial^t F^z - \partial^x F^c - \partial^b F^y +$$

$$+ \partial^y F^b + \partial^c F^x - \partial^z F^t - \partial^d F^a$$

To move from the octadic electrodynamics to the classical one, you should put:

$$F^a = 0 \quad F^t = 0 \quad F^x = -H^x \quad F^b = E^x$$

$$F^y = -H^y \quad F^c = E^y \quad F^z = -H^z \quad F^d = E^z$$

$$j^a = j^b = j^c = j^d = 0$$

And non-zero $F^x, F^b, F^y, F^c, F^z, F^d$ should not depend on q^a, q^b, q^c, q^d. That is, the four derivatives

$$\partial^a, \partial^b, \partial^c, \partial^d$$

from them should give zero. Here is what remains after this transition:

$$0 = -\,4\pi \cdot j^{a} = \partial^{x} F^{x} + \partial^{y} F^{y} + \partial^{z} F^{z} =$$

$$= -\,\partial_{x} F^{x} - \partial_{y} F^{y} - \partial_{z} F^{z} =$$

$$= \partial_{x} H^{x} + \partial_{y} H^{y} + \partial_{z} H^{z} = div\,\vec{H}$$

$$div\,\vec{H} = 0$$

$$-\,4\pi \cdot j^{t} = -\,4\pi \cdot \rho =$$

$$= \partial^{x} F^{b} + \partial^{y} F^{c} + \partial^{z} F^{d} =$$

$$= -\partial_{x} F^{b} - \partial_{y} F^{c} - \partial_{z} F^{d} =$$

$$= -\partial_{x} E^{x} - \partial_{y} E^{y} - \partial_{z} E^{z} = -div\,\vec{E}$$

$$div\,\vec{E} = 4\pi \cdot \rho$$

$$-\,4\pi \cdot j^{x} = -\partial^{t} F^{b} - \partial^{y} F^{z} + \partial^{z} F^{y} =$$

$$= \partial_{t} F^{b} + \partial_{y} F^{z} - \partial_{z} F^{y} =$$

$$= \partial_{t} E^{x} - \partial_{y} H^{z} + \partial_{z} H^{y} =$$

$$= \partial_{t} E^{x} - [rot\,\vec{H}]^{x} = -\,4\pi \cdot j^{x}$$

$$-\,4\pi \cdot j^{b} = \partial^{t} F^{x} - \partial^{y} F^{d} + \partial^{z} F^{c} =$$

$$= -\partial_{t} F^{x} + \partial_{y} F^{d} - \partial_{z} F^{c} =$$

$$= \partial_{t} H^{x} + \partial_{y} E^{z} - \partial_{z} E^{y} =$$

$$= \partial_t H^x + [rot\,\vec{E}]^x = -4\pi \cdot j^b = 0$$

$$-4\pi \cdot j^y = -\partial^t F^c + \partial^x F^z - \partial^z F^x =$$

$$= \partial_t F^c - \partial_x F^z + \partial_z F^x =$$

$$= \partial_t E^y + \partial_x H^z - \partial_z H^x =$$

$$= \partial_t E^y - [rot\,\vec{H}]^y = -4\pi \cdot j^y$$

$$-4\pi \cdot j^c = \partial^t F^y + \partial^x F^d - \partial^z F^b =$$

$$= -\partial_t F^y - \partial_x F^d + \partial_z F^b =$$

$$= \partial_t H^y - \partial_x E^z + \partial_z E^x =$$

$$= \partial_t H^y + [rot\,\vec{E}]^y = -4\pi \cdot j^c = 0$$

$$-4\pi \cdot j^z = -\partial^t F^d - \partial^x F^y + \partial^y F^x =$$

$$= \partial_t F^d + \partial_x F^y - \partial_y F^x =$$

$$= \partial_t E^z - \partial_x H^y + \partial_y H^x =$$

$$= \partial_t E^z - [rot\,\vec{H}]^z = -4\pi \cdot j^z$$

$$-4\pi \cdot j^d = \partial^t F^z - \partial^x F^c + \partial^y F^b =$$

$$= -\partial_t F^z + \partial_x F^c - \partial_y F^b =$$

$$= \partial_t H^z + \partial_x E^y - \partial_y E^x =$$

$$= \partial_t H^z + [rot\,\vec{E}]^z = -4\pi \cdot j^d = 0$$

We got four equations:

$$div\,\vec{H} = 0$$

$$div\,\vec{E} = 4\pi \cdot \rho$$

$$\partial_t \vec{E} - [rot\,\vec{H}] = -\,4\pi \cdot \vec{j}$$

$$\partial_t \vec{H} + [rot\,\vec{E}] = 0$$

And these are the Maxwell equations.

That is, in the transition from octadic electrodynamics to classical electrodynamics, the equations for the fields F pass into the Maxwell equations for the fields $\vec{E}$ and $\vec{H}$.

14.Motion equations for a massive charged particle in the field F.

The action function for a particle of mass m and charge e in an octad field F with potential A is:

$$S = {}_1S + {}_2S$$

$${}_1S = -\,mc \cdot \int ds \qquad ds \cdot ds = d\,\overline{q} \cdot dq$$

$${}_2S = -\,e \cdot \int \frac{1}{2} \cdot [\overline{A} \cdot dq + d\overline{q} \cdot A]$$

To describe the motion of this particle in the field F, we vary S by δq and $\delta\,\overline{q}$:

$$\delta\ {}_1S = \frac{1}{2} \cdot \int (\delta\,\overline{q} \cdot dp + d\,\overline{p} \cdot \delta q) \quad p = mc \cdot \frac{dq}{ds}$$

$$\delta\ {}_2S = -\,e \cdot \int \frac{1}{2} \cdot \left[\overline{A} \cdot d\ \delta q + d\ \delta\overline{q} \cdot A\right] =$$

$$= -e \cdot \int \frac{1}{2} \cdot \left[- d\overline{A} \cdot \delta q - \delta\overline{q} \cdot dA\right]$$

$$\delta S = \delta\ _1 S + \delta\ _2 S =$$

$$= \frac{1}{2} \cdot \int \{\delta\, \overline{q} \cdot (dp + e \cdot dA) +$$

$$+(d\, \overline{p} + e \cdot\ d\overline{A}) \cdot \delta q\}$$

And we equate δS to zero: $\delta S = 0$

In order for this to be performed for any δq and $\delta\, \overline{q}$, you need:

$$dp + e \cdot dA = 0 \quad\quad and \quad\quad d\, \overline{p} + e \cdot\ d\overline{A} = 0$$

The second equation reduces to the first equation. And we have the equation:

$$dp = -e \cdot dA = -e \cdot dq \cdot F$$

$$\frac{dp}{dt} = -e \cdot \frac{dq}{dt} \cdot F = -e \cdot V \cdot F \quad\quad V = \frac{dq}{dt}$$

$$\frac{dp}{dt} = -e \cdot V \cdot F$$

Let's move on to the index octadic view:

$$n_\lambda \cdot \frac{d\, p^\lambda}{dt} = -e \cdot n_\lambda \cdot f^{\lambda}{}_{\mu\nu} \cdot V^{\,\mu} \cdot F^\nu$$

$$\frac{d\, p^\lambda}{dt} = -e \cdot f^{\lambda}{}_{\mu\nu} \cdot V^{\,\mu} \cdot F^\nu$$

(Here it is taken into account that for octads, the algebraic tensor is $f^{\,\nu}{}_{\mu\,\sigma}$. For Q2 numbers, use the algebraic tensor $F^{\,\nu}{}_{\mu\,\sigma}$.)

Let's check this formula for octads:

$$\frac{dp^a}{dt} = -e \cdot (V^a \cdot F^a - V^t \cdot F^t - V^x \cdot F^x + V^b \cdot F^b -$$

$$- V^y \cdot F^y + V^c \cdot F^c - V^z \cdot F^z + V^d \cdot F^d)$$

$$\frac{dp^t}{dt} = -e \cdot (V^a \cdot F^t + V^t \cdot F^a - V^x \cdot F^b - V^b \cdot F^x -$$

$$- V^y \cdot F^c - V^c \cdot F^y - V^z \cdot F^d - V^d \cdot F^z)$$

$$\frac{dp^x}{dt} = -e \cdot (V^a \cdot F^x - V^t \cdot F^b + V^x \cdot F^a - V^b \cdot F^t +$$

$$+ V^y \cdot F^z - V^c \cdot F^d - V^z \cdot F^y + V^d \cdot F^c)$$

$$\frac{dp^b}{dt} = -e \cdot (V^a \cdot F^b + V^t \cdot F^x + V^x \cdot F^t + V^b \cdot F^a +$$

$$+ V^y \cdot F^d + V^c \cdot F^z - V^z \cdot F^c - V^d \cdot F^y)$$

$$\frac{dp^y}{dt} = -e \cdot (V^a \cdot F^y - V^t \cdot F^c - V^x \cdot F^z + V^b \cdot F^d +$$

$$+ V^y \cdot F^a - V^c \cdot F^t + V^z \cdot F^x - V^d \cdot F^b)$$

$$\frac{dp^c}{dt} = -e \cdot (V^a \cdot F^c + V^t \cdot F^y - V^x \cdot F^d - V^b \cdot F^z +$$

$$+ V^y \cdot F^t + V^c \cdot F^a + V^z \cdot F^b + V^d \cdot F^x)$$

$$\frac{dp^z}{dt} = -e \cdot (V^a \cdot F^z - V^t \cdot F^d + V^x \cdot F^y - V^b \cdot F^c -$$

$$- V^y \cdot F^x + V^c \cdot F^b + V^z \cdot F^a - V^d \cdot F^t)$$

$$\frac{dp^d}{dt} = -e \cdot (V^a \cdot F^d + V^t \cdot F^z + V^x \cdot F^c + V^b \cdot F^y -$$

$$- V^y \cdot F^b - V^c \cdot F^y + V^z \cdot F^t + V^d \cdot F^a)$$

$$V^{\mu} = \frac{dq^{\mu}}{dt} \qquad V^{t} = 1$$

To move from the octadic electrodynamics to the classical one, you should put:

$$F^{a} = 0 \quad F^{t} = 0 \quad F^{x} = -H^{x} \quad F^{b} = E^{x}$$

$$F^{y} = -H^{y} \quad F^{c} = E^{y} \quad F^{z} = -H^{z} \quad F^{d} = E^{z}$$

And we get the following 8 equations:

$$\frac{dp^{a}}{dt} = -\, e \cdot (\vec{v}, \vec{H})$$

$$\frac{dp^{t}}{dt} = \; e \cdot (\vec{v}, \vec{E})$$

$$\frac{dp^{x}}{dt} = \; e \cdot (\vec{E} + \left[\vec{v} \times \vec{H}\right])^{x}$$

$$\frac{dp^{b}}{dt} = \; e \cdot (\vec{H} - \left[\vec{v} \times \vec{E}\right])^{x}$$

$$\frac{dp^{y}}{dt} = \; e \cdot (\vec{E} + \left[\vec{v} \times \vec{H}\right])^{y}$$

$$\frac{dp^{c}}{dt} = \; e \cdot (\vec{H} - \left[\vec{v} \times \vec{E}\right])^{y}$$

$$\frac{dp^{z}}{dt} = \; e \cdot (\vec{E} + \left[\vec{v} \times \vec{H}\right])^{z}$$

$$\frac{dp^{d}}{dt} = \; e \cdot (\vec{H} - \left[\vec{v} \times \vec{E}\right])^{z}$$

And we see that the formula gives the correct Lorentz force:

$$\frac{d\vec{p}}{dt} = e \cdot (\vec{E} + \left[\vec{v} \times \vec{H}\right])$$

As well as the correct change in the energy of the particle in time:

$$\frac{dp^t}{dt} = e \cdot (\vec{v}, \vec{E})$$

So, we can assume that the correct index-free formula for a particle in the field F is:

$$\frac{dp}{dt} = -\, e \cdot V \cdot F \qquad V = \frac{dq}{dt}$$

A completely index-free formula is obtained by replacing dt with ds:

$$\frac{dp}{ds} = -\, e \cdot U \cdot F \qquad U = \frac{dq}{ds}$$

15.Density Λ for the third octad action function

$${}_3S = \frac{1}{8\pi} \cdot \int \overline{F} \cdot F \cdot d\Omega \qquad F = n_{\mu} \cdot F^{\mu}$$

$${}_3S = \int \Lambda \cdot d\Omega$$

$$d\Omega = da \cdot dt \cdot dx \cdot db \cdot dy \cdot dc \cdot dz \cdot dd$$

$$\Lambda = \frac{1}{8\pi} \cdot \overline{F} \cdot F = n_\lambda \cdot \Lambda^\lambda \qquad \overline{n_\mu} = n_\rho \cdot c^{\rho}{}_{\mu}$$

$$\Lambda^\lambda = \frac{1}{8\pi} \cdot f^{\lambda}{}_{\rho\nu} \cdot c^{\rho}{}_{\mu} \cdot F^\mu \cdot F^\nu$$

For octadic electrodynamics, we obtain:

$$\Lambda^a = \frac{1}{8\pi} \cdot [F^a \cdot F^a - F^t \cdot F^t + F^x \cdot F^x - F^b \cdot F^b +$$

$$+ F^y \cdot F^y - F^c \cdot F^c + F^z \cdot F^z - F^d \cdot F^d]$$

$$\Lambda^t = \frac{1}{4\pi} \cdot [F^a \cdot F^t + F^x \cdot F^b + F^y \cdot F^c + F^z \cdot F^d]$$

$$\Lambda^x = \Lambda^b = \Lambda^y = \Lambda^c = \Lambda^z = \Lambda^d = 0$$

To move from the octadic electrodynamics to the classical one, you should put:

$$F^a = 0 \quad F^t = 0 \quad F^x = -H^x \quad F^{\,b} = E^x$$

$$F^{\,y} = -H^y \quad F^c = E^{\,y} \quad F^z = -H^z \quad F^d = E^z$$

And we get the following 8 equations:

$$\Lambda^a = \frac{1}{8\pi} \cdot [H^2 - E^2] \qquad \Lambda^t = -\frac{1}{4\pi} \cdot (\vec{H}, \vec{E})$$

$$\Lambda^x = \Lambda^b = \Lambda^y = \Lambda^c = \Lambda^z = \Lambda^d = 0$$

For Q2 electrodynamics, we take

$$F = m_{\,\mu} \cdot F^{\mu} \qquad \overline{m_\mu} = m_\rho \cdot C^{\rho}{}_{\mu}$$

Also, for Q2 electrodynamics, we should use the algebraic tensor $F^{\,\lambda}{}_{\nu\,\mu}$.

16.Results

From mathematics, we took the language of quaternions and constructed the octads. And in their language, we wrote down three *indexless* action functions – for a free particle, for a free field, and for their interaction. And from them we got two *index-free* formulas:

$$\overline{D} \cdot F = -4\,\pi \cdot j$$

$$\frac{dp}{dt} = -\,e \cdot V \cdot F$$

From the first formula, we can find the fields F (a generalization of the electric and magnetic fields). From the second formula, we can find the trajectory of a charged massive particle in the fields F. And all this simultaneously in two electrodynamics – in eight-dimensional octadic electrodynamics, and in sixteen-dimensional Q2 electrodynamics. Due to the large length of the index formulas in Q2 electrodynamics, we have not given these formulas here.

The unsolved problem is the derivation of the energy-momentum tensor for the particle, the field F, and their interaction in these two electrodynamics. But to solve this problem, we must consider curved space-time. And this is troublesome. For now, we can take comfort in the thought that our three *hypercomplex indexless* action functions give the correct Maxwell equations and the Lorentz force, and are therefore correct themselves.

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