Analytical expression of Complex Riemann Xi function ξ(s) and proof of Riemann Hypothesis

Debasis Biswas
Chakdaha
W.B – 741 235. India
Email – biswasdebasis38@gmail.com.

Abstract :

[In this paper explicit analytical expression for Riemann Xi function ξ(s) is worked out for complex values of s. From this expression Riemann Hypothesis is proved. Analytic Expression for non-trivial Zeros of Riemann Zeta function ζ(s) is also found. A second proof of Riemann Hypothesis is also given.]

Keywords : Riemann Zeta and Riemann Xi function, Riemann Hypothesis, Critical line, non-trivial zeros.

1. Introduction :

One of the most difficult problems today to mathematicians and physicists is Riemann Hypothesis. It is a conjecture proposed by Bernhard Riemann which says that all the complex zeros of Riemann Zeta function ζ(s) has real part σ = ½ . Till now this conjecture is neither proved nor disproved. In this paper an explicit analytical expression of Riemann Xi function ξ(s) is found. From this expression of ξ(s) expression for zeros of ξ(s) is derived. As it is known that all the zeros of ξ(s) are identical with nontrivial zeros of Riemann zeta function, the zeros of ξ(s) are identified as nontrivial zeros of Riemann zeta function ζ(s).

The paper is organized as follows. In section 2, we summarize the definition of Riemann Xi and Riemann zeta function, mention the connection between ξ(s) and ζ(s) . We also mention some important results related to zeros of ζ(s). Next in section 3, the expression for ξ(s) in terms of an arbitrary function is derived. In section 4, the analytical expression for ξ(s) is worked out using a theorem of analysis. In section 5, Riemann Hypothesis is proved and equation for nontrivial zeros of ζ(s) is derived. In Section 6, a second proof of Riemann Hypothesis is given. Section 7 contains conclusion and comments. An appendix is added after section 7.

2. Riemann zeta and Xi function and Riemann Hypothesis

For real s > 1, Riemann zeta function ζ(s) is defined as [1, 2, 3]

$$\zeta(s) = \sum_{\nu=1}^{\infty} \nu^{-s} \quad \ldots (2.1)$$

The definite integral equivalent of (2.1) is

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\,dt\ ; \quad \text{real } s > 1 \quad \ldots (2.2)$$

and Γ(s) ≡ Gamma function

ζ(s) can be analytically continued from (2.1) and can be defined for 0 < real s <1

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{s-1}\,dt\ ; \quad 0 < \text{real } s < 1 \quad \ldots (2.3)$$

Riemann zeta function satisfies a well-known functional equation [3] :

$$\pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad \ldots (2.4)$$

This functional equation (2.4) is also a definition of ζ(s) over whole complex s-plane except a singularity at s = 1.

A consequence of symmetry of (2.4) suggests that if there is a complex zero of ζ(s) for Re s = ½ + δ, then there must be another zero of ζ(s) for Re s = ½ − δ.

Riemann introduced another function known as Riemann Xi function ξ(s) which satisfies a functional equation :

$$\xi(s) = \xi(1 - s) \quad \ldots (2.5)$$

The Riemann Xi and Riemann zeta function are connected through the equation [1] :

$$\xi(s) = \frac{1}{2}\, s(s-1)\, \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) \quad \ldots (2.6)$$

The Riemann zeta function has real and complex zeros [2]. The real zeros of ζ(s) are known as trivial zeros and are given by

$$\zeta(-2m) = 0\ ; \quad m = 1, 2, 3, \ldots \quad \ldots (2.7)$$

The complex zeros of ζ(s) lie [1] within the strip 0 <Re s < 1. This strip is known as critical strip and zeros lying in critical strip are known as nontrivial zeros. And the straight line Re s = ½ is called critical line .

Now a consequence of (2.6) is that the nontrivial zeros of Riemann zeta function ζ(s) are identical with zeros of Riemann Xi function ξ(s) [3].

Riemann Hypothesis is a conjecture made by Riemann which says that the nontrivial zeros of Riemann zeta function ζ(s) has real part σ = ½ . This implies that the zeros of Riemann Xi function ξ(s) has also real part of σ = ½ . This Hypothesis has not yet been proved or disproved.

The literatures on ξ(s) and ζ(s) are extremely large. Only a few important results related to zeros of ζ(s) are mentioned. G. H. Hardy [4] has proved that an infinite number of zeros lie on the critical line σ = ½. J. B. Conrey [5] has shown that more than 40% zeros of ζ(s) lie on critical line. N. Levinson [6] has shown that more than one third zeros of ζ(s) lie on critical line.

3. Expression of Riemann Xi function ξ(s) in terms of an arbitrary function.

As the nontrivial zeros of ζ(s) are identical with zeros of ξ(s) , we will consider ξ(s) and find its analytical expression in terms of an arbitrary function. This is more convenient which will be clear later.

Firstly, we will consider solution of (2.5) which is satisfied by ξ(s) . The solution of (2.5) in terms of an arbitrary function was given by Hymers [7]. Following Hymers [7] the general solution of the equation (2.5) can be written as in terms of an arbitrary function.

Hymers considers the functional equation of more general form

$$\varphi(a+bs) = n\varphi(s)\,; \qquad n, a, b \text{ are constants} \qquad \ldots(3.1)$$

and gives the solution of (3.1) as

$$\varphi(s) = \left(C_o s - \frac{C_o a}{1-b}\right)^{\frac{\log n}{\log b}} \theta_o\left[\mathrm{Cos}\left\{\frac{2\pi}{\log b}\,.\,\log\left(C_o s - \frac{C_o a}{1-b}\right)\right\}\right] \qquad \ldots(3.2)$$

C_o ≡ arbitrary constant

θ_o ≡ arbitrary function

Now for a = 1, b = – 1 and n = 1 equation (3.1) reduces to

$$\varphi(1-s) = \varphi(s) \qquad \ldots(3.3)$$

Equations (2.5) and (3.3) are of the same form. Hence solutions of (2.5) and (3.3) will have identical form.

The solution of (3.3) in terms of an arbitrary function θ_0 follows from (3.2) :

$$\varphi(s) = \left(s - \frac{1}{2}\right)^{\frac{\log 1}{\log(-1)}} \theta_o\left[\mathrm{Cos}\left\{\frac{2\pi}{\log(-1)}\,.\,\log\left(s-\frac{1}{2}\right)\right\}\right]$$

taking $C_o = 1$ in (3.2)

$$\text{i.e., } \varphi(s) = \left(s - \frac{1}{2}\right)^{\frac{2n\pi i}{(2n+1)\pi i}} \theta_o\left[\mathrm{Cos}\left\{\frac{2\pi}{\log(-1)}\,.\,\log\left(s-\frac{1}{2}\right)\right\}\right]$$

$$= \left(s - \frac{1}{2}\right)^{\frac{2n}{(2n+1)}} \theta_o\left[\mathrm{Cos}\left\{\frac{2\pi}{(2n+1)\pi i}\,.\,\log\left(s-\frac{1}{2}\right)\right\}\right]$$

$$= \left(s - \frac{1}{2}\right)^{\frac{2n}{(2n+1)}} \theta_o\left[\mathrm{Cos}\left\{\frac{-2i}{(2n+1)}\,.\,\log\left(s-\frac{1}{2}\right)\right\}\right]$$

$$= \left(s - \frac{1}{2}\right)^{\frac{2n}{(2n+1)}} \theta_o\left[\mathrm{Cosh}\left\{\frac{2}{(2n+1)}\,.\,\log\left(s-\frac{1}{2}\right)\right\}\right] \qquad \ldots(3.4)$$

A glance at (3.4) suggests that φ(s) can be written more conveniently in terms of another arbitrary function.

Therefore $\varphi(s) = \psi_0(s - ½)$ …(3.5)

$\psi_0(s - ½)$ is another arbitrary function.

Now comparing (2.5), (3.3) and (3.5) we can write the solution of (2.5) in terms of an arbitrary function:

$$\xi(s) = \psi(s - ½) \qquad \ldots(3.6)$$

Equation (3.6) is the solution of (2.5) and represents the Riemann Xi function ξ(s) in terms of an arbitrary function.

4. Explicit analytical expression of Riemann Xi function ξ(s).

To derive the exact analytical expression of ξ(s) we state a result of analysis in the form of a theorem.

The theorem is due to J. Brill [8]

Theorem 1. (Due to J. Brill)

The theorem states that if α is a root of

$$A\alpha^2 + B\alpha + C = 0 \qquad \text{A, B, C are constants} \qquad \ldots(4.1)$$

Then φ (y + αx) can be expressed as

$$\varphi(y + \alpha x) = \eta + \alpha\theta \qquad \ldots(4.2)$$

where α is independent of x, y and η = η(x, y), θ = θ(x, y) satisfy

$$\frac{1}{A}\frac{\partial\theta}{\partial y} = \frac{1}{B}\left(\frac{\partial\eta}{\partial y} - \frac{\partial\theta}{\partial x}\right) = -\frac{1}{C}\frac{\partial\eta}{\partial x} \qquad \ldots(4.3)$$

We will use the results from (4.1) to (4.3) of above theorem to find analytic expression of Riemann Xi function ξ(s), where s is complex. Conventionally s is written as

$$s = \sigma + it \ ; \qquad i = \sqrt{-1} \qquad \ldots(4.4)$$

σ, t being real and imaginary parts of s respectively.

Now from (4.3) we find

$$\frac{1}{A}\frac{\partial\theta}{\partial y} + \frac{1}{C}\frac{\partial\eta}{\partial x} = 0 \qquad \ldots(4.5)$$

$$\frac{1}{A}\frac{\partial\theta}{\partial y} - \frac{1}{B}\frac{\partial\eta}{\partial y} + \frac{1}{B}\frac{\partial\theta}{\partial x} = 0 \qquad \ldots(4.6)$$

$$\frac{1}{C}\frac{\partial\eta}{\partial x} + \frac{1}{B}\frac{\partial\eta}{\partial y} - \frac{1}{B}\frac{\partial\theta}{\partial x} = 0 \qquad \ldots(4.7)$$

From (4.7) one finds

$$\frac{1}{C}\,\frac{\partial^2\eta}{\partial y\,\partial x} + \frac{1}{B}\,\frac{\partial^2\eta}{\partial y^2} - \frac{1}{B}\,\frac{\partial^2\theta}{\partial y\,\partial x} = 0 \qquad \ldots(4.8)$$

and from (4.5)

$$\frac{1}{A}\,\frac{\partial^2\theta}{\partial x\,\partial y} + \frac{1}{C}\,\frac{\partial^2\eta}{\partial x^2} = 0 \qquad \ldots(4.9)$$

Now from (4.8) and (4.9) eliminating $\frac{\partial^2\theta}{\partial x\,\partial y}$ we get assuming partial derivatives are continuous

$$\frac{\partial^2\eta}{\partial x^2} + \frac{B}{A}\,\frac{\partial^2\eta}{\partial x\,\partial y} + \frac{C}{A}\,\frac{\partial^2\eta}{\partial y^2} = 0 \qquad \ldots(4.10)$$

Likewise from (4.5)

$$\frac{\partial^2\theta}{\partial y^2} + \frac{A}{C}\,\frac{\partial^2\eta}{\partial x\,\partial y} = 0 \qquad \ldots(4.11)$$

And from (4.6)

$$\frac{1}{A}\,\frac{\partial^2\theta}{\partial x\,\partial y} - \frac{1}{B}\,\frac{\partial^2\eta}{\partial x\,\partial y} + \frac{1}{B}\,\frac{\partial^2\theta}{\partial x^2} = 0 \qquad \ldots(4.12)$$

Again assuming partial derivatives are continuous, we eliminate $\frac{\partial^2\eta}{\partial x\,\partial y}$ from (4.11) and (4.12). The result is:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{B}{A}\,\frac{\partial^2\theta}{\partial x\,\partial y} + \frac{C}{A}\,\frac{\partial^2\theta}{\partial y^2} = 0 \qquad \ldots(4.13)$$

A comparison of (4.10) and (4.13) shows that we can choose

$$\eta\,(x, y) = \theta\,(x, y) \qquad \ldots(4.14)$$

Now in (4.1) we take

$$\alpha = 1 \qquad \ldots(4.15)$$

And in (4.2) we take

$$\left.\begin{array}{l} y = (\sigma - \frac{1}{2}) \\ x = i\,t \end{array}\right] \qquad \ldots(4.16)$$

Then in view of (4.15) we find from (4.1)

$$A + B + C = 0 \qquad \ldots(4.17)$$

and from (4.2), (4.14), (4.15) and (4.16)

$$\varphi(\sigma - \tfrac{1}{2} + i\,t) = \eta(\sigma - \tfrac{1}{2}, i\,t) + \theta(\sigma - \tfrac{1}{2}, i\,t) \qquad \ldots(4.18)$$

Using (4.16), (4.10) can be rewritten as

$$-\frac{\partial^2\eta}{\partial t^2} - \frac{iB}{A}\,\frac{\partial^2\eta}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} + \frac{C}{A}\,\frac{\partial^2\eta}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0$$

i.e., $$\frac{\partial^2\eta}{\partial t^2} + \frac{iB}{A}\frac{\partial^2\eta}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} - \frac{C}{A}\frac{\partial^2\eta}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0$$

i.e., $$\frac{\partial^2\eta}{\partial t^2} + \frac{iB}{A}\frac{\partial^2\eta}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} + \frac{A+B}{A}\frac{\partial^2\eta}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0,$$ using (4.17)

Therefore, $$\frac{\partial^2\eta}{\partial t^2} + ik\frac{\partial^2\eta}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} + (1+k)\frac{\partial^2\eta}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0\ ; \quad k = \frac{B}{A} = \text{constant} \qquad \ldots(4.19)$$

Likewise (4.13) reduces to

$$\frac{\partial^2\theta}{\partial t^2} + ik\frac{\partial^2\theta}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} + (1+k)\frac{\partial^2\theta}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0 \qquad \ldots(4.20)$$

Equation (4.2), (4.14), (4.18), (4.19) and (4.20) imply that (3.6) can be written as

$$\begin{aligned} \xi(s) &= \psi(s - \tfrac{1}{2}) \\ &= \psi(\sigma - \tfrac{1}{2} + i\,t) \\ &= \eta_0\,(\sigma - \tfrac{1}{2}, i\,t) + \theta_0\,(\sigma - \tfrac{1}{2}, i\,t) \qquad \ldots(4.21) \end{aligned}$$

Where η_0 and θ_0 satisfy (4.19) and (4.20) and

$$\eta_0 = \theta_0 \qquad \ldots(4.22)$$

Hence from (4.21) and (4.22) the expression for $\xi(s)$ can be written as

$$\xi(s) = 2\eta_0\,(\sigma - \tfrac{1}{2}, i\,t) \qquad \ldots(4.23)$$

where η_0 satisfies

$$\frac{\partial^2\eta_0}{\partial t^2} + ik\frac{\partial^2\eta_0}{\partial t.\partial\left(\sigma - \frac{1}{2}\right)} + (1+k)\frac{\partial^2\eta_0}{\partial\left(\sigma - \frac{1}{2}\right)^2} = 0 \qquad \ldots(4.24)$$

There exist methods for solution of (4.24). However we will use the method due to Forsyth [9].

We use a trial solution for (4.24) :

$$\eta_0 = A_0\, e^{il_1 t + il_2\left(\sigma - \frac{1}{2}\right)} \qquad \ldots(4.25)$$

A_0 = constant (real) ; and l_1 real

Then from (4.24) , using (4.25) one finds

$$l_1^2 + ikl_1l_2 + (1+k)l_2^2 = 0 \qquad \ldots(4.26)$$

Treating (4.26) as a quadratic in l_1

We get $l_1 = i\,l_2$

i.e., $l_2 = -\,i\,l_1$ …(4.27)

and $l_1 = -\,i\,(k+1)\,l_2$

i, e., $l_2 = \frac{il_1}{(k+1)}$ … (4.28)

Thus we have two solutions of (4.25)

$$\eta_{0,1} = A_0\, e^{il_1 t + l_1\left(\sigma - \frac{1}{2}\right)} \quad \text{[Using (4.27)]} \qquad \ldots(4.29)$$

$$= A_0\, e^{l_1\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos} l_1 t + i\,\mathrm{Sin} l_1 t\right] \qquad \ldots(4.29A)$$

$$\eta_{0,2} = A_0\, e^{il_1 t - \frac{l_1}{k+1}\left(\sigma - \frac{1}{2}\right)} \quad \text{[Using (4.28)]} \qquad \ldots(4.30)$$

$$= A_0\, e^{-\frac{l_1}{k+1}\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos} l_1 t + i\,\mathrm{Sin}\, l_1 t\right] \qquad \ldots(4.30A)$$

Now it is known that ξ(s) is an entire function i.e., analytic in the whole complex plane.
Here $\eta_{0,1}$ is an analytic function because $\eta_{0,1}$ satisfies Cauchy-Riemann equations.
But $\eta_{0,2}$ is not analytic for arbitrary values of k. For $k = -2$, $\eta_{0,2}$ is only analytic.
So we choose $k = -2$ in (4.30) and (4.30A).
But though the choice $k = -2$ makes $\eta_{0,2}$ analytic, $\eta_{0,2}$ becomes identical with $\eta_{0,1}$ for $k = -2$
So we choose $k = -2$, but the ansatz for η_0 in (4.25) leads to one solution of (4.24) which is given by (4.29) or (4.29A).

Now we take another ansatz for $\eta_0 = A_0\, e^{-il_1 t - il_2\left(\sigma - \frac{1}{2}\right)}$ …(4.31)

This ansatz when plugged into (4.24) gives once again

$$l_1^2 + ikl_1 l_2 + (1+k) l_2^2 = 0 \qquad \ldots(4.32)$$

Here also we get like previous case

$$l_2 = -\,i\, l_1 \qquad \ldots(4.33)$$

$$\text{and} \quad l_2 = \frac{il_1}{(k+1)} \qquad \ldots(4.34)$$

with these values we find from (4.31)

$$\eta_{0,3} = A_0\, e^{-il_1 t - l_1\left(\sigma - \frac{1}{2}\right)} \qquad \ldots(4.35)$$

$$= A_0\, e^{-l_1\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos}\, l_1\ t - i\,\mathrm{Sin}\, l_1\ t\right] \qquad \ldots(4.35A)$$

$$\text{and} \quad \eta_{0,4} = A_0\, e^{-il_1 t + \frac{l_1}{k+1}\left(\sigma - \frac{1}{2}\right)}$$

$$= A_0\, e^{\frac{l_1}{k+1}\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos}\, l_1\ t - i\,\mathrm{Sin}\, l_1\ t\right] \qquad \ldots(4.36)$$

$\eta_{0,3}$ turns out to be analytic whereas $\eta_{0,4}$ is analytic only for k = – 2 and for k = – 2 , $\eta_{0,4}$ becomes identical with $\eta_{0,3}$.

So the ansatz (4.31) leads to one more solution like previous analysis. And this solution is $\eta_{0,3}$ given by (4.35) and (4.35A).

It can be checked that $\eta_{0,1}$ and $\eta_{0,3}$ are linearly independent solutions of (4.24) as Wronskian of $\eta_{0,1}$ and $\eta_{0,3}$ is nonzero.

The Wronskian of $\eta_{0,1}$ and $\eta_{0,3}$ is [from (4.29) and (4.35)]

$$W = \begin{vmatrix} A_0 e^{il_1 t + l_1\left(\sigma - \frac{1}{2}\right)} & A_0 e^{-il_1 t - l_1\left(\sigma - \frac{1}{2}\right)} \\ \left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\left(\sigma - \frac{1}{2}\right)}\right\} A_0 e^{il_1 t + l_1\left(\sigma - \frac{1}{2}\right)} & \left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial\left(\sigma - \frac{1}{2}\right)}\right\} A_0 e^{-il_1 t - l_1\left(\sigma - \frac{1}{2}\right)} \end{vmatrix}$$

$$= \begin{vmatrix} A_0 e^{il_1 t + l_1\left(\sigma - \frac{1}{2}\right)} & A_0 e^{-il_1 t - l_1\left(\sigma - \frac{1}{2}\right)} \\ (il_1 + l_1) A_0 e^{il_1 t + l_1\left(\sigma - \frac{1}{2}\right)} & -(il_1 + l_1)\, A_0 e^{-il_1 t - l_1\left(\sigma - \frac{1}{2}\right)} \end{vmatrix}$$

$$= -\, A_0^2\ (l_1 + i\, l_1) - A_0^2\ (l_1 + i\, l_1)$$

$$= -2\, A_0^2\ (l_1 + i\, l_1) \neq 0$$

The sum of two independent solutions is also a solution of (4.24) [9].

Hence we write the solution of (4.24) as [from (4.29A) and (4.35A)]

$$\eta_0 = \eta_{0,1} + \eta_{0,3}$$

$$= A_0 e^{l_1\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos}\ l_1\ t + i\ \mathrm{Sin}\ l_1\ t\right] + A_0\, e^{-l_1\left(\sigma - \frac{1}{2}\right)} \left[\mathrm{Cos}\ l_1\ t - i\ \mathrm{Sin}\ l_1\ t\right]$$

$$= 2A_0\ \mathrm{Cos}\ l_1 t \left\{\frac{e^{l_1(\sigma - ½)} + e^{-l_1\left(\sigma - \frac{1}{2}\right)}}{2}\right\} + i2A_0\ \mathrm{Sin}\ l_1 t \left\{\frac{e^{l_1\left(\sigma - \frac{1}{2}\right)} - e^{-l_1\left(\sigma - \frac{1}{2}\right)}}{2}\right\}$$

$$= A_1\ \mathrm{Cos}\ l_1 t\ \mathrm{Cos\,h}\ l_1\left(\sigma - \frac{1}{2}\right) + i\ A_1\ \mathrm{Sin}\ l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \frac{1}{2}\right); \quad 2\,A_0 = A_1$$

$$= A_1 \left[\mathrm{Cos}\ l_1 t\ \mathrm{Cos\,h}\ l_1\left(\sigma - \frac{1}{2}\right) + i\ \mathrm{Sin}\ l_1\ t\ \mathrm{Sin\,h} l_1\left(\sigma - \frac{1}{2}\right)\right] \qquad \ldots(4.37)$$

The constant A_1 can be looked upon [9] as a function of the parameter l_1

So we write from (4.37)

$$\eta_0 = A_1(l_1) \left[\mathrm{Cos}\ l_1 t\ \mathrm{Cos\,h}\ l_1\left(\sigma - \frac{1}{2}\right) + i\ \mathrm{Sin}\ l_1 t\ \mathrm{Sin\,h} l_1\left(\sigma - \frac{1}{2}\right)\right] \qquad \ldots(4.38)$$

Now, the solution of (4.24) should contain [9] two constants of the form $A_1(l_1)$; so the final expression of η_0 is of the form

$$\eta_0 = A_2(l_1) + A_1(l_1)\,[\mathrm{Cos}\, l_1 t\, \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\, \mathrm{Sin}\, l_1 t\, \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)] \qquad ..(4.39)$$

Using (4.23) and (4.39) we can now write the analytic expression of Riemann X_i function $\xi(s)$ as

$$\xi(s) = \xi(\sigma + it).$$
$$= 2\eta_0$$
$$= 2\, A_2(l_1) + 2\, A_1(l_1)\,[\mathrm{Cos}\, l_1 t\, \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\, \mathrm{Sin}\, l_1 t\, \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)]$$

i.e., $$\xi(s) = F_2(l_1) + F_1(l_1)\,[\mathrm{Cos}\, l_1 t\, \mathrm{Cos\,h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\, \mathrm{Sin}\, l_1 t\, \mathrm{Sin\,h} l_1\left(\sigma - \tfrac{1}{2}\right)] \qquad \ldots(4.40)$$

where $F_2(l_1) = 2\, A_2(l_1)$ and $F_1(l_1) = 2\, A_1(l_1)$

$F_2(l_1)$ and $F_1(l_1)$ both being real

Equation (4.40) is the final analytic expression for Riemann X_i function $\xi(s)$. This solution is analytic and satisfies the equation (2.5) i.e., $\xi(s) = \xi(1 - s)$. The two constants $F_2(l_1)$ and $F_1(l_1)$ in (4.40) are functions of the parameter l_1.

The solution (4.40) has certain advantage. It contains no arbitrary functions of the independent variables ; instead it contains two constants ; the constants being arbitrary functions of the parameter l_1.

Now for t = 0, we have from (4.40)

$$\xi(\sigma) = F_2(l_1) + F_1(l_1)\, \mathrm{Cos\,h} l_1 \left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(4.41)$$

Now it is known that

$$\left.\begin{aligned} &\xi(0) = \xi(1) = \tfrac{1}{2} \\ &\xi(2) = \frac{\pi}{6} \approx 0.52 \\ &\xi(3) \approx 0.57 \\ &\xi(4) = \frac{\pi^2}{15} \approx 0.65 \end{aligned}\right] \qquad \ldots(4.42)$$

Using (4.42) we find from (4.41)

$$F_2(l_1) + F_1(l_1)\, \mathrm{Cosh}\, \frac{l_1}{2} = 0.50 \qquad \ldots(4.43)$$

$$F_2(l_1) + F_1(l_1)\, \mathrm{Cosh}\, \frac{3l_1}{2} = 0.52 \qquad \ldots(4.44)$$

$$F_2(l_1) + F_1(l_1)\, \mathrm{Cosh}\, \frac{5l_1}{2} = 0.57 \qquad \ldots(4.45)$$

$$F_2(l_1) + F_1(l_1)\, \mathrm{Cosh}\, \frac{7l_1}{2} = 0.65 \qquad \ldots(4.46)$$

It is an easy task [Appendix] to check that the parameter l_1 and the arbitrary function $F_1(l_1)$, $F_2(l_1)$ cannot be determined uniquely from the equations (4.43) to (4.46) or other equations formed from equation (4.41) with known values of $\xi(\sigma)$.

Thus l_1, $F_1(l_1)$, $F_2(l_1)$ in (4.40), (4.41) and in (4.43) to (4.46) are undetermined. However the non- existence of unique solution of the system (4.43) to (4.46) cannot prevent proving Riemann Hypothesis.

5. Proof of Riemann Hypothesis and analytical expression for zeros of ξ(s) (i.e., nontrivial zeros of ζ(s))

The proof of Riemann Hypothesis is concerned with the form of nontrivial zeros of Riemann Zeta function $\zeta(s)$ i.e., the zeros of Riemann Xi function $\xi(s)$.

The zeros of $\xi(s)$ imply that both real and imaginary parts of equation (4.40) are zero. Therefore zero of $\xi(s)$ imply

$$\text{Real } \xi(s) = R_E = F_2(l_1) + F_1(l_1) \text{ Cos } l_1 t \text{ Cos } hl_1 \left(\sigma - \frac{1}{2}\right) = 0 \qquad \ldots(5.1)$$

and $$\text{Imaginary } \xi(s) = I_M = F_1(l_1) \text{ Sin } l_1 t \text{ Sin } hl_1 (\sigma - ½) = 0 \qquad \ldots (5.2)$$

Firstly we ignore considering $R_E = 0$. Because $R_E = 0$ implies $F_2(l_1) = 0$ along with either $F_1(l_1)$ or Cos l_1t equal to zero as Cos $hl_1 \left(\sigma - \frac{1}{2}\right)$ is always defined to be positive. But $F_2(l_1)$ and $F_1(l_1)$ are both non zero. Hence this conclusion.

Now as $F_2(l_1) \neq 0$ and $F_1(l_1) \neq 0$, as a second possibility, $I_M = 0$ requires to find a value of σ $(0 < \sigma < 1)$ for which Sinh $l_1 \left(\sigma - \frac{1}{2}\right) = 0$. And for Sin h $l_1 \left(\sigma - \frac{1}{2}\right) = 0$, we can find the condition for $R_E = 0$ from equation (5.1).

As a third possibility when Sinh $l_1 \left(\sigma - \frac{1}{2}\right) \neq 0$, $I_M = 0$ implies Sin $l_1t = 0$ and $R_E = 0$ would imply Cos $hl_1 \left(\sigma - \frac{1}{2}\right) = \pm \frac{F_2(l_1)}{F_1(l_1)}$, as Cos $l_1t = \pm 1$ when Sin $l_1t = 0$. This possibility is ruled out because $\pm \frac{F_2(l_1)}{F_1(l_1)}$ is independent of σ.

Now considering the second possibility a look into equation (5.1) and (5.2) assert that only for $\sigma = \frac{1}{2}$, $\operatorname{Sinh} l_1 \left(\sigma - \frac{1}{2}\right) = 0$ and so $I_M = 0$. And condition for $R_E = 0$ follows from (5.1) with $\sigma = \frac{1}{2}$:

$$\operatorname{Cos} l_1 t = -\frac{F_{2\,(l_1)}}{F_{1\,(l_1)}} \qquad \ldots(5.3)$$

Therefore the nontrivial zeros of ζ(s) turn out to be of the form $\left(\frac{1}{2} + it\right)$ where t is given by (5.3) or more explicitly

$$t = \frac{1}{l_1} \operatorname{Cos}^{-1} \left[-\frac{F_{2\,(l_1)}}{F_{1\,(l_1)}}\right] \qquad \ldots(5.4)$$

As l_1, $F_2(l_1)$, $F_1(l_1)$ are unknown, we cannot compute the value of t from (5.4).
Thus the proof of Riemann Hypothesis is established.

6. A second proof of Riemann Hypothesis :

The second of proof of Riemann Hypothesis is proof of a result which is equivalent to proof of Riemann Hypothesis. This Riemann Hypothesis equivalent is due to Brian Conrey [10]. Conrey has shown that the truth of Riemann Hypothesis is equivalent to proving that zeros of derivatives of all orders of Riemann Xi functions ξ(s) has real part σ = ½. We will prove this in the following .

From equation (4.40) we write once again the expression for Riemann Xi functions ξ(s) :

$$\xi(s) = \xi(\sigma + it)$$
$$= \left[F_2(l_1) + F_1(l_1) \operatorname{Cos} l_1 t \operatorname{Cos} h\, l_1\left(\sigma - \tfrac{1}{2}\right)\right] + i\left[F_1(l_1) \operatorname{Sin} l_1 t \operatorname{Sin} h l_1\left(\sigma - \tfrac{1}{2}\right)\right] \ldots(6.1)$$

Therefore $\frac{d}{ds}\xi(s)$

$$= \xi^{(1)}(s)$$
$$= \frac{\partial}{\partial\sigma}\left[F_2(l_1) + F_1(l_1) \operatorname{Cos} l_1 t \operatorname{Cos} h\, l_1\left(\sigma - \tfrac{1}{2}\right)\right] + i\,\frac{\partial}{\partial\sigma}\left[F_1(l_1) \operatorname{Sin} l_1 t \operatorname{Sin} h l_1\left(\sigma - \tfrac{1}{2}\right)\right]$$
$$= l_1\, F_1(l_1) \operatorname{Cos} l_1 t \operatorname{Sin} h\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\, l_1\, F_1(l_1) \operatorname{Sin} l_1 t \operatorname{Cos} h\, l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(6.2)$$

Likewise $\frac{d^2}{ds^2}\xi(s)$

$$= \xi^{(2)}(s)$$
$$= l_1^{\,2}\, F_1(l_1) \operatorname{Cos} l_1 t \operatorname{Cos} h\, l_1\left(\sigma - \tfrac{1}{2}\right) + i l_1^{\,2}\, F_1(l_1) \operatorname{Sin} l_1 t \operatorname{Sin} h l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(6.3)$$

$$\frac{d^{2m}}{ds^{2m}}\xi(s)$$
$$= \xi^{(2m)}(s)$$
$$= {l_1}^{2m} F_1(l_1) \text{ Cos } l_1 t \text{ Cos h } l_1\left(\sigma - \tfrac{1}{2}\right) + i{l_1}^{2m} F_1(l_1) \text{ Sin } l_1 t \text{ Sin h} l_1\left(\sigma - \tfrac{1}{2}\right) \quad \ldots(6.4)$$

And $\frac{d^{2m+1}}{ds^{2m+1}}\xi(s)$

$$= \xi^{(2m+1)}(s)$$
$$= {l_1}^{2m+1} F_1(l_1) \text{ Cos } l_1 t \text{ Sin h } l_1\left(\sigma - \tfrac{1}{2}\right) + i{l_1}^{2m+1} F_1(l_1) \text{ Sin } l_1 t \text{ Cos h} l_1\left(\sigma - \tfrac{1}{2}\right) \quad \ldots(6.5)$$

Now we first consider the zeros of $\frac{d^{2m}}{ds^{2m}}\xi(s)$.

From (6.4) zeros of $\frac{d^{2m}}{ds^{2m}}\xi(s)$ imply

$${l_1}^{2m} F_1(l_1) \text{ Cos } l_1 t \text{ Cos h } l_1\left(\sigma - \tfrac{1}{2}\right) = 0 \quad \ldots(6.6)$$

And $${l_1}^{2m} F_1(l_1) \text{ Sin } l_1 t \text{ Sin h} l_1\left(\sigma - \tfrac{1}{2}\right) = 0 \quad \ldots(6.7)$$

Now suppose $\sigma \neq ½$ then Cos h $l_1\left(\sigma - \tfrac{1}{2}\right) \neq 0$ and Sin h$l_1\left(\sigma - \tfrac{1}{2}\right) \neq 0$ then (6.6) and (6.7) imply both Cos $l_1 t = 0$ as well as Sin $l_1 t = 0$ which is impossible.

On the other hand if $\sigma = ½$ equation (6.7) is satisfied because Sin h0 = 0 and (6.6) implies Cos $l_1 t = 0$ (as Cos h0 $\neq 0$)

i.e., $l_1 t = (2n + 1)\frac{\pi}{2}$

i.e., $$t = \frac{1}{l_1}(2n + 1)\frac{\pi}{2} \quad \ldots(6.8)$$

Thus it turns out that zeros of $\xi^{(2m)}(s)$ are of the form $\frac{1}{2} + i\frac{1}{l_1}(2n + 1)\frac{\pi}{2}$ where l_1 is an undetermined parameter.

We next consider the zeros of $\xi^{(2m+1)}(s)$

From (6.5) zeros of $\xi^{(2m+1)}(s)$ imply

$${l_1}^{2m+1} F_1(l_1) \text{ Cos } l_1 t \text{ Sin h } l_1\left(\sigma - \tfrac{1}{2}\right) = 0 \quad \ldots(6.9)$$

$${l_1}^{2m+1} F_1(l_1) \text{ Sin } l_1 t \text{ Cos h} l_1\left(\sigma - \tfrac{1}{2}\right) = 0 \quad \ldots(6.10)$$

A similar argument as above reveals that zeros of $\xi^{(2m+1)}(s)$ has real part $\sigma = ½$ and imaginary part is given by

Sin $l_1 t = 0$

i.e., $l_1 t = n\pi$

i.e., $t = \frac{1}{l_1} n\pi$

Therefore zeros of $\xi^{(2m+1)}(s)$ are of the form $\frac{1}{2} + i\frac{1}{l_1} n\pi$

Thus the second proof of Riemann Hypothesis is established.

It is interesting to observe from above analysis that zeros of all even order derivates of ξ(s) are identical and zeros of all odd order derivatives of ξ(s) are also identical.

7. Conclusion :

In this paper two proofs of Riemann Hypothesis are given. The first proof may be called a direct proof. It is not a proof of any Riemann Hypothesis equivalent [11, 12]. It fails to identify the position of nontrivial zeros of ζ(s) on critical axis. However it confirms that all the nontrivial zeros of ζ(s) lie on critical axis. The second proof is a proof of Riemann Hypothesis equivalent. We can now safely conclude that Riemann Hypothesis is not at all trifling or baffling. It is perfectly true.

Appendix

We write equations (4.43) to (4.46) once again

$$F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\frac{l_1}{2} = 0.50 \qquad \ldots(A1)$$

$$F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\frac{3l_1}{2} = 0.52 \qquad \ldots(A2)$$

$$F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\frac{5l_1}{2} = 0.57 \qquad \ldots(A3)$$

$$F_2(l_1) + F_1(l_1)\,\mathrm{Cosh}\,\frac{7l_1}{2} = 0.65 \qquad \ldots(A4)$$

Now (A2) – (A1) gives

$$F_1(l_1)\left[\mathrm{Cosh}\,\frac{3l_1}{2} - \mathrm{Cosh}\,\frac{l_1}{2}\right] = 0.02$$

Therefore

$$F_1(l_1).\ 2.\ \mathrm{Sinh}l_1.\ \mathrm{Sinh}\,\frac{l_1}{2} = 0.02 \qquad \ldots(A5)$$

(A3) – (A2) gives

$$F_1(l_1)\left[\mathrm{Cosh}\,\frac{5l_1}{2} - \mathrm{Cosh}\,\frac{3l_1}{2}\right] = 0.05$$

Therefore

$$F_1(l_1).\ 2.\ \mathrm{Sinh}2l_1.\ \mathrm{Sinh}\,\frac{l_1}{2} = 0.05 \qquad \ldots(A6)$$

(A4) – (A3) gives

$$F_1(l_1)\left[\mathrm{Cosh}\,\frac{7l_1}{2} - \mathrm{Cosh}\,\frac{5l_1}{2}\right] = 0.08$$

Therefore $F_1(l_1).\ 2.\ \text{Sinh}3l_1.\ \text{Sinh}\,\frac{l_1}{2} = 0.08$ …(A7)

(A4) – (A2) gives

$$F_1(l_1)\left[\text{Cosh}\,\frac{7l_1}{2} - \text{Cosh}\,\frac{3l_1}{2}\right] = 0.13$$

Therefore $F_1(l_1).\ 2.\ \text{Sinh}5l_1.\ \text{Sinh}\ l_1 = 0.13$ …(A8)

Next (A6) ÷ (A5) gives

$$\frac{\text{Sinh}2l_1}{\text{Sinh}l_1} = \frac{0.05}{0.02} = \frac{5}{2}$$

Therefore $\text{Cos h}\ l_1 = \frac{5}{4} = 1.25$ …(A9)

Again (A7) ÷ (A6) gives

$$\frac{\text{Sinh}3l_1}{\text{Sinh}l_1} = \frac{0.08}{0.05} = \frac{8}{5}$$

i.e., $$\frac{4\text{Sinh}^3 l_1 + 3\text{Sinh}\ l_1}{2\text{Sinh}l_1\ .\text{Cosh}l_1} = \frac{8}{5}$$

i,e, $$\frac{4\text{Sinh}^2 l_1 + 3}{\text{Cosh}l_1} = \frac{16}{5}$$

i.e., $$4(\text{Cosh}^2\ l_1 - 1) + 3 = \frac{16}{5}\ \text{Cosh}\ l_1$$

i.e., $$4\text{Cosh}^2\ l_1 - \frac{16}{5}\ \text{Cosh}\ l_1 - 1 = 0$$

Therefore $$\text{Cosh}\ l_1 = \frac{\frac{16}{5} \pm \sqrt{\left(\frac{16}{5}\right)^2 + 16}}{8}$$

$$= \frac{3.2 \pm \sqrt{(3.2)^2 + 16}}{8}$$

$$= \frac{3.2 \pm \sqrt{26.24}}{8}$$

As Cosh l_1 is defined always to be positive, hence we get from above

$$\text{Cosh } l_1 = \frac{3.2+\sqrt{26.24}}{8} = \frac{3.2+5.12}{8} = \frac{8.32}{8} = 1.04 \qquad \ldots(A10)$$

A comparison of (A9) and (A10) shows that l_1 cannot be uniquely determined and consequently $F_1(l_1)$ and $F_2(l_1)$ remain undetermined uniquely.

REFERENCES :

1;. E. C. Titchmarsh : The theory of Riemann Zeta function. Second edition. O. U. P. 1986

2. H. M. Edwards : Riemann's Zeta function. Academic Press, New York. 1974

3. Aleksandar Ivic : The Riemann Zeta function. Theory and applications. Dover Publications. 2003. N. Y.

4. G. H. Hardy : Compt. Rend. Acad. Sci. 158 (1914), 1012 – 1014.

5. J. B. Conrey : J. Reine Angew. Math. 399 (1989). 1 – 26.

6. N. Levinson : Adv. in Math. 13 : 383 – 436. (1974).

7. J. Hymers : A Treatise on Differential Equations and on the Calculus of Finite Differences. Second Edition. PP. 94 (2nd half). Longman, Brown, Green, Longmans and Roberts. 1858.

8. J. Brill : Messenger of Mathematics. Vol. 23 (1894) PP. 108 – 111 and PP. 185 – 194.

9. A. R. Forsyth : A TREATISE ON DIFFERENTIAL EQUATIONS. Sixth edition, Dover. New York. PP. 516 – 517.

10. B. Conrey : J. Number theory 16(1983), 49-74

11. Kevin Broughan : Equivalents of Riemann Hypothesis.
Vol. One ; Arithmetic Equivalents.
Vol. Two ; Analytic Equivalents.
C. U. P. 2017.

12. Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller : The Riemann Hypothesis
A Resource for the Afficionado and Virtuoso Alike.
Springer, 2007