Integrals for $\pi/\sqrt{\phi}$

Edgar Valdebenito

March 1 , 2021

ABSTRACT. We recall some integrals for $\pi/\sqrt{\phi}$.
Keywords. number Pi , integrals , golden mean .

I. Introduction .

Recall that:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots \tag{1}$$

$$\phi = \frac{1+\sqrt{5}}{2} \tag{2}$$

In this note we recall some integrals for $\pi/\sqrt{\phi}$.

II. Integrals .

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\infty}\tan^{-1}\left(\frac{2}{1+x^2}\right)dx \tag{3}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\infty}\frac{4x^2}{5+2x^2+x^4}dx \tag{4}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,\frac{1}{1+(1-x)^2}dx \tag{5}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{1}{1+(1+x)^2}dx \tag{6}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\pi/2}\frac{4\cos^2 x}{1+4\sin^4 x}dx \tag{7}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\pi/2}\frac{4\sin^2 x}{1+4\cos^4 x}dx \tag{8}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\tan^{-1}2}\sqrt{2\cot x-1}\,dx \tag{9}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_1^\infty \frac{2\sqrt{x-1}}{4+x^2}\,dx \tag{10}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^2 \frac{\sqrt{2-x}}{\sqrt{x}\,(1+x^2)}\,dx \tag{11}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{\pi/2} \frac{1}{\cos^2 x}\tan^{-1}(2\cos^2 x)\,dx \tag{12}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \cosh x \tan^{-1}\left(\frac{2}{\cosh^2 x}\right)dx \tag{13}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{2\sinh x \sinh(2x)}{4+\cosh^4 x}\,dx \tag{14}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \sin^{-1}\left(\frac{2}{\sqrt{4+(1+x^2)^2}}\right)dx \tag{15}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \left(\frac{\pi}{4} + \tan^{-1}\left(\frac{1-x^2}{3+x^2}\right)\right)dx \tag{16}$$

$$\frac{\pi}{\sqrt{\phi}} = 4\int_0^1 \sqrt{\frac{1-x}{x+\sqrt{5x-4x^2}}}\,dx \tag{17}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^2 \frac{\tan^{-1} x}{x\sqrt{x(2-x)}}\,dx \tag{18}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^1 \left(\frac{4}{1+2x^2+5x^4} + \frac{4x^2}{5+2x^2+x^4}\right)dx \tag{19}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{4}{1+2x^2+5x^4}\,dx \tag{20}$$

$$\frac{\pi}{\sqrt{\phi}} = 4\int_0^1 \tan^{-1}\left(\sqrt{\frac{1-2x+\sqrt{1+16x-16x^2}}{10x}}\right)dx \tag{21}$$

$$\frac{\pi}{\sqrt{\phi}} = 4\int_0^1 \sin^{-1}\left(\sqrt{\frac{2-2x}{1+\sqrt{1+16x-16x^2}}}\right)dx \tag{22}$$

$$\frac{\pi}{\sqrt{\phi}} = 4 \int_{2+2\sqrt{5}}^{\infty} \frac{\sqrt{x-2-2\sqrt{5}}}{x^2}\, dx \tag{23}$$

$$\frac{\pi}{\sqrt{\phi}} = 4 \int_{0}^{\infty} \frac{\sqrt{x}}{\left(x+2+2\sqrt{5}\right)^2}\, dx \tag{24}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{\pi/2} \frac{4}{\left(2+2\sqrt{5}\right)\cos^2 x + \sin^2 x}\, dx \tag{25}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{\pi/2} \frac{4}{1+\left(1+2\sqrt{5}\right)\cos^2 x}\, dx \tag{26}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{\pi/2} \frac{4}{\left(2+2\sqrt{5}\right)-\left(1+2\sqrt{5}\right)\sin^2 x}\, dx \tag{27}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{1/\phi} \left(\sqrt{\frac{2-x+2\sqrt{1-x-x^2}}{x}} - \sqrt{\frac{2-x-2\sqrt{1-x-x^2}}{x}} \right) dx \tag{28}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{1} \frac{2}{\sqrt{x}} \left(\frac{1}{2+2\sqrt{5}+x} + \frac{1}{1+\left(2+2\sqrt{5}\right)x} \right) dx \tag{29}$$

$$\frac{\pi}{\sqrt{\phi}} = 2\pi - 4 \int_{0}^{1} \sin^{-1}\left(\sqrt{\frac{10x}{1+8x+\sqrt{1+16x-16x^2}}} \right) dx \tag{30}$$

$$\frac{\pi}{\sqrt{\phi}} = 2\pi - 4 \int_{1/\left(2+2\sqrt{5}\right)}^{1} \tan^{-1}\left(\sqrt{\frac{\left(2+2\sqrt{5}\right)x-1}{1-x}} \right) dx \tag{31}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{1} \frac{4}{\left(2+2\sqrt{5}\right)x^2+(1-x)^2}\, dx \tag{32}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{0}^{1} \frac{4}{\left(2+2\sqrt{5}\right)(1-x)^2+x^2}\, dx \tag{33}$$

$$\frac{\pi}{\sqrt{\phi}} = -\frac{\sqrt{2}\,\ln\left(2+2\sqrt{5}\right)}{\sqrt{1+\sqrt{5}}} + 4 \int_{0}^{\infty} \frac{x \ln\left(x+\sqrt{2+2\sqrt{5}+x^2}\right)}{\left(2+2\sqrt{5}+x^2\right)^{3/2}}\, dx \tag{34}$$

$$\frac{\pi}{\sqrt{\phi}} = 4\int_0^\infty \frac{\left(x^2+2x-2-2\sqrt{5}\right)}{\left(2+2\sqrt{5}+x^2\right)^2}\ln(1+x)\,dx \tag{35}$$

$$\frac{\pi}{\sqrt{\phi}} = 4\int_0^\infty \frac{\left(x^2-2-2\sqrt{5}\right)\ln x}{\left(2+2\sqrt{5}+x^2\right)^2}\,dx \tag{36}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \ln\left(\frac{\sqrt{5}+2+4x^2}{\sqrt{5}-2+4x^2}\right)dx \tag{37}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{1}{x^2}\ln\left(\frac{4+\left(\sqrt{5}+2\right)x^2}{4+\left(\sqrt{5}-2\right)x^2}\right)dx \tag{38}$$

$$\frac{\pi}{\sqrt{\phi}} = \frac{1}{2}\int_0^{2\ln\left(2+\sqrt{5}\right)} \sqrt{2\left(\frac{e^x+1}{e^x-1}\right)-\sqrt{5}}\,dx \tag{39}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{1}{\sqrt{x}\,(x+\phi)}\,dx \tag{40}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{2x^2}{(x^2-1)^2+\phi x^2}\,dx \tag{41}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{-\infty}^\infty \frac{e^{-x/2}}{\phi+e^{-x}}\,dx \tag{42}$$

$$\frac{\pi}{\sqrt{\phi}} = \frac{2}{\sqrt{\phi}}\tan^{-1}\left(\frac{1}{\sqrt{\phi}}\right)+\int_0^1 \frac{2x}{\sqrt{1+\phi x^2}\,\sqrt{1-x^2}}\,dx \tag{43}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{-\infty}^\infty \frac{1}{\phi\cosh x-\sinh x}\,dx \tag{44}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^{2/\phi} \cosh^{-1}\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}-\frac{1}{\phi}}\right)dx \tag{45}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \frac{\ln((e-\phi)^2+\phi x^2)}{\phi+x^2}\,dx \tag{46}$$

$$\frac{\pi}{\sqrt{\phi}} = \frac{2}{\sqrt{\phi}}\tan^{-1}\left(\sqrt{\frac{e}{\phi}-1}\right)+\int_1^\infty \frac{1}{\sqrt{e^x-\phi}}\,dx \tag{47}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^1 \frac{4+6x^2}{(4+x^4)\sqrt{1-x^2}}\,dx \tag{48}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_0^\infty \left(\sqrt[3]{\sqrt{\left(\frac{\phi}{3}\right)^3 + \frac{1}{4x^2}} + \frac{1}{2x}} - \sqrt[3]{\sqrt{\left(\frac{\phi}{3}\right)^3 + \frac{1}{4x^2}} - \frac{1}{2x}} \right)^2 dx \tag{49}$$

$$\frac{\pi}{\sqrt{\phi}} = 2\int_0^\infty \left(\sqrt[3]{\frac{1}{2x^2} + \frac{(2+2\sqrt{5})^3}{27} + \sqrt{\frac{1}{4x^4} + \frac{(2+2\sqrt{5})^3}{27x^2}}} + \right.$$
$$\left. \sqrt[3]{\frac{1}{2x^2} + \frac{(2+2\sqrt{5})^3}{27} - \sqrt{\frac{1}{4x^4} + \frac{(2+2\sqrt{5})^3}{27x^2}}} - \frac{2(2+2\sqrt{5})}{3} \right) dx \tag{50}$$

$$\frac{\pi}{\sqrt{\phi}} = \int_{-1}^{1} \frac{x}{(1+a^2-2ax)\sqrt{1-x^2}} dx \tag{51}$$

where

$$a = \sqrt[3]{\sqrt{\phi} + \sqrt[3]{\sqrt{\phi} + \sqrt[3]{\sqrt{\phi} + \ldots}}} \tag{52}$$

III. References .

[1] Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag , 2001.

[2] Berndt, B.: Ramanujan's Notebooks. Part I , Springer - Verlag , 1985.

[3] Blatner, D.: The Joy of π . Walker Publ., 1999.

[4] Boros, G., and Moll, V.: Irresistible Integrals , Cambridge University Press, 2004.

[5] Castellanos, D.: The ubiquitous Pi. Part I. Math. Magazine, 61 , 1988, 67-98.

[6] Castellanos, D.: The ubiquitous Pi. Part II. Math. Magazine 61, 1988, 148-163.

[7] Glasser, M.L.: A remarkable property of definite integrals. Math. Comp., 40 , 1983, 561-563.