The Center and the Barycenter

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In the first part we deal with the question which points we have to connect to generate a non self-intersectioning polygon. Afterwards we introduce *polyholes*, which is a generalization of polygons. Roughly spoken a polyhole is a big polygon, where we cut out a finite number of small polygons.
In the second part we present two 'centers', which we call *center* and *barycenter*. In the case that both centers coincide, we call these polygons as *nice*. We show that if a polygon has two symmetry axes, it is nice. We yield examples of polygons with a single symmetry axis which are nice and which are not nice.
In a third part we introduce the *Spieker center* and the *Point center* for polygons. We define *beautiful* polygons and *perfect* polygons. We show that all symmetry axes intersect in a single point.

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1 Introduction

We look for a criterion to generate a simple polygon.
Let us assume a set of $k+1$ points called $Points \subset \mathbb{R}^2$, $Points := \{(x_1, y_1), (x_2, y_2), \ldots (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1})\}$. We joint the possible edges. We define the subset $Union$ of $\mathbb{R}^2$, $Union := \bigcup \{[(x_i, y_i), (x_{i+1}, y_{i+1})]\}$ for $i \in \{1, 2, \ldots k-1, k\}$. With the expression '$[a, b]$' we mean all points between a and b and the boundaries a and b. We say that $Union$ is *suitable* if and only if $Union$ is homeomorphic to the circle $\{x^2 + y^2 = 1 \mid x, y \in \mathbb{R}\}$.

Definition 1.1. We presume $k+1$ points (x_i, y_i) of $\mathbb{R}^2$ where $1 \leq i \leq k+1$ and $k > 2$. We name $Union$ as a *polygon* if and only if it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$. The element (x_i, y_i) is called a *vertex*. We call a polygon such that $Union$ is suitable as *simple polygon.* If we have a simple polygon we include its interior.

2 Star-Shaped Polygons

We can start also with a circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, We presume a finite set called A of k different points on the circle, where $k > 2$ and $A := \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_{k-1}, \vec{a}_k\}$ is in counterclockwise

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order. We change it into a set $H := \{\vec{h}_1, \vec{h}_2, \dots \vec{h}_{k-1}, \vec{h}_k\}$ such that the three points $\vec{h}_i, \vec{a}_i$ and $(0,0)$ are collinear and $(0,0)$ is not between $\vec{a}_i$ and $\vec{h}_i$. We keep the order as in A. We move all points with the same vector $\vec{m}$, i.e. $Z := \{\vec{z}_1, \vec{z}_2 \dots z_i \dots \vec{z}_{k-1}, \vec{z}_k\} \subset \mathbb{R}^2$ where $1 \leq i \leq k$ and $\vec{z}_i := \vec{h}_i + \vec{m}$. We keep the order. We call a polygon with a set of vertices $Z := \{\vec{z}_1, \vec{z}_2, \dots \vec{z}_{k-1}, \vec{z}_k\}$ as a *star-shaped polygon* if and only if Z provided with an appropriate order can be constructed as it is just described. We add $\vec{z}_{k+1} := \vec{z}_1$. Please see the following Proposition 2.2.

Question 2.1. *Is there an alternative description of star-shaped polygons? Is every convex simple polygon a star-shaped polygon?*

Proposition 2.2. *We get a simple polygon P if there is a finite set of points $Z := \{\vec{z}_1, \vec{z}_2, \dots \vec{z}_{k-1}, \vec{z}_k, \vec{z}_{k+1}\} \subset \mathbb{R}^2$ constructed as above where $k > 2$ and*

$P := \bigcap \{\ W \subset \mathbb{R}^2 \ \mid Z \subset W,$ *where W is homeomorphic to the circle area $\{x^2 + y^2 \leq 1\}$ and the points between $\vec{z}_i$ and $\vec{z}_{i+1}$, $1 \leq i \leq k-1$, including $\vec{z}_i$ and $\vec{z}_{i+1}$ are a subset of W and also the points between $\vec{z}_k$ and $\vec{z}_1$ belong to W* $\}$

Proof. The claim of the proposition is trivial, since *Union* is a suitable set. □

It follows that a star-shaded polygon is a compact set, homeomorphic to any circle, and for all $1 \leq i \leq k$ the vertex $\vec{z}_i$ is a boundary point.
We get that a triangle is a star-shaped polygon, too.

We assume a set of points $\{(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3),\ \dots (x_{k-1}, y_{k-1}),\ (x_k, y_k),\ (x_{k+1}, y_{k+1})\}$ of $\mathbb{R}^2$. We call this set *Points*. We demand that in *Points* three successive elements are distinct and not collinear. We assume $k > 2$. We connect the points of *Points* by the given order and we call this set *Union*. In the case that *Union* is homeomorphiv to a circle we get a simple polygon. For $k = 3$ the polygon is a triangle.

Proposition 2.3. If we have a set of $k+1$ points $Points := \{(x_1, y_1), (x_2, y_2), \dots (x_k, y_k), (x_{k+1}, y_{k+1})\}$ with $(x_{k+1}, y_{k+1}) = (x_1, y_1)$ and $k > 2$, we get a simple polygon if and only if the set *Union* is suitable.

Proof. Trivial. □

3 Polyholes

We define a subset of $\mathbb{R}^2$, which we will call a *polyhole*. This geometric structure consists of a finite number of simple polygons $P, P_1, P_2, P_3, \dots P_{m-1}, P_m$. From the polygon P we cut out polygons $P_1, P_2, P_3, \dots P_{m-1}, P_m$.

Definition 3.1. Let $\{P, P_1, P_2, P_3, \dots P_{m-1}, P_m\}$ be a set of simple polygons. A *polyhole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \dots \cup P_{m-1} \cup P_m$

A corresponding definition is possible for polytops.

Definition 3.2. Let $\{P, P_1, P_2, P_3, \dots P_{m-1}, P_m\}$ be a set of polytops in $\mathbb{R}^n$. A *polytophole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \dots \cup P_{m-1} \cup P_m$.

Question 3.3. *What is the barycenter of a polyhole, if it is realized with homogeneous material of constant thickness? What is the barycenter of a polytophole in $\mathbb{R}^3$, if it is realized with homogeneous material?*

4 Nice Polygons

We define two 'centers', where the center *Cent* is just the arithmetic means of the first and second coordinates of the generating points, respectively.
We got the following formulas for the *barycenter* $B = (B_x, B_y)$ of a simple polygon from [1] or [2]. Please see also [3] and [4]. Area is the area of a simple polygon. Note that $\mathsf{Area} \neq 0$, and that in [1] and [3] the barycenter is called a Centroid, and further that B is the center of gravity of the polygon, if it is realized with homogeneous material of constant thickness. Note that the order in the polygon is counterclockwise. We write

$$D_i = x_i \cdot y_{i+1} - x_{i+1} \cdot y_i, \text{ where } 1 \leq i \leq k \tag{4.1}$$

$$\mathsf{Area} = \frac{1}{2} \cdot \sum_{i=1}^{k} D_i \tag{4.2}$$

$$B_x = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (x_i + x_{i+1}) \cdot D_i, \quad B_y = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (y_i + y_{i+1}) \cdot D_i \tag{4.3}$$

$$Cent = \frac{1}{k} \cdot \left(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i \right) \tag{4.4}$$

Definition 4.1. Let us presume a simple polygon P. We call P *nice* if and only if it holds $B = Cent$.

Remark 4.2. When we use the term *symmetry axis of a polygon* P we mean a line segment s in the convex hull of P of maximal length, i.e. it holds for a symmetry axis t in the convex hull of P with more than one common point with s that $t \subset s$.

Proposition 4.3. *If a simple polygon has two different symmetry axes, it is nice*

Proof. The proposition is an easy consequence of the following important two lemmas.

Lemma 4.4. *Let P be a polygon. The following two operations yield a polygon again. The property of being nice or being not nice remain under these operations.*

- *Revolving P by an arbitrary angle around any point*
- *Shifting P by an arbitrary vector*

Proof. We assume $B \neq Cent$. Let us revolve P by an arbitrary angle around any point. There is a positive distance d between B and $Cent$. It will be kept, since a rotation is a distance preserving map. Hence the distance between the images points of B and $Cent$ is also d. After the rotation still P is not nice.
In the case $B = Cent$ the claim of the lemma is trivial. □

Lemma 4.5. *Both operations which we have mentioned above in Lemma 4.4 are distance preserving operations. Therefore the shape of a polygon is kept after these operations.*

Proof. Trivial. □

In a polygon we fix four real numbers.

Definition 4.6. Let $Points$ be the set of vertices of a polygon P. We define
min_x := minimum of the set of the first coordinates of the set of the vertices $Points$ of P.
min_y := minimum of the second coordinates of $Points$,
max_x := maximum of the first coordinates of $Points$,
max_y := maximum of the second coordinates of $Points$.
We define a rectangle called $Rectangle(P)$ by four vertices
$(max_x, max_y), (min_x, max_y), (min_x, min_y), (max_x, min_y)$.

Remark 4.7. In a polygon P it holds that both P and the convex hull of $Points$ is in $Rectangle(P)$.

Definition 4.8. Let $s = \{\vec{a} + r \cdot \vec{d} \mid r \in [m, n]$ for fixed real numbers $m, n\}$ be a symmetry axis of a polygon. We define $l(s)$ as the line $\{\vec{a} + r \cdot \vec{d} \mid r \in \mathbb{R}\}$.

Remark 4.9. It holds that s is a subset of $l(s)$.

Lemma 4.10. *Let s be a symmetry axis of a simple polygon P. Both B and $Cent$ are on the line $l(s) \cap Rectangle(P)$.*

Proof. We assume a simple polygon P with a symmetry axis s and centers B and $Cent$. Note that B is the center of gravity of P. Hence B must be on $l(s)$, since s is a symmetry axis of P. For the same reason B is in $Rectangle(P)$.
We use Lemma 4.5. We map P by a rotation and a shift parallel the vertical y axis into a second polygon P' with a symmetry axis s' and centers B' and $Cent'$ such that s' is on the x axis. Assume a vertex (x', y') of P'. Since s' is a symmetry axis and it is on the x axis either $y' = 0$ or there is a second vertex $(x', -y')$ of P'. If we add all vertices together we get $Cent' = (c', 0)$ with any real number c', i.e. $Cent'$ is on the x axis. This means that $Cent'$ is on $l(s')$. Since P' has the same shape as P we get that $Cent$ is on $l(s)$.
It is easy to show that $Cent$ is a point in $Rectangle(P)$: It holds

$$min_x = \frac{1}{k} \cdot \sum_{i=1}^{k} min_x \leq \frac{1}{k} \sum_{i=1}^{k} x_i \leq \frac{1}{k} \sum_{i=1}^{k} max_x = max_x \tag{4.5}$$

If we consider correspondingly the second coordinate of $Cent$ we get $min_y \leq \frac{1}{k} \sum_{i=1}^{k} y_i \leq max_y$, and it follows that $Cent$ is in $Rectangle(P)$.
We get that $Cent$ is in $l(s) \cap Rectangle(P)$. Lemma 4.10 has been proved. □

Two symmetry axes intersect in a single point. It is both B and $Cent$. The proof of Proposition 4.3 is finished. □

Corollary 4.11. In a simple polygon all symmetry axes intersect in a single point. It is both B and $Cent$. It follows that a simple polygon with more than one symmetry axis is nice.

Note that a single symmetry axis is not sufficient, as the kite defined by $(0,0),(1,-1),(3,0),(1,1)$ shows, since $\frac{5}{4} \neq \frac{4}{3}$. It is not nice.

It follows an example of a polygon with a single symmetry axis which is nice.
Take the 5-gon with vertices

$$(0,0),(1,0),(1,1),\left(\frac{1}{2},1+\frac{1}{2}\cdot\sqrt{6}\right),(0,1). \text{ We get } B = Cent = \left(\frac{1}{2},\frac{1}{10}\cdot\left(6+\sqrt{6}\right)\right) \approx (0.50, 0.85) \tag{4.6}$$

The last example proves that the conjecture that besides triangles only polygons with two or more symmetry axes are nice is wrong.

5 Spieker Center and Point Center

In a triangle the Spieker center is well-known. We have got the formulas of the Spieker center from [4]. Please see also [5]. The Spieker center is the barycenter of a triangle $A = (x_A, y_A), B = (x_B, y_B), C = (x_C, y_C)$ without the interior, which is formed by a wire of constant thickness. The barycenter is outside the wire. The sidelengths of the triangle are l_1, l_2 and l_3, where sides with lengths l_2 and l_3 intersect in A, while sides with lengths l_1 and l_3 intersect in B. The coordinates of the Spieker center $S = (spieker_x, spieker_y)$ are

$$spieker_x = \frac{(l_2+l_3)\cdot x_A + (l_1+l_3)\cdot x_B + (l_1+l_2)\cdot x_C}{2\cdot(l_1+l_2+l_3)} \quad \text{and} \tag{5.1}$$

$$spieker_y = \frac{(l_2+l_3)\cdot y_A + (l_1+l_3)\cdot y_B + (l_1+l_2)\cdot y_C}{2\cdot(l_1+l_2+l_3)} \tag{5.2}$$

The concept of the *Spieker center* can easily be generalized on polygons. We imagine the polygon is made from a wire of constant diameter. We look for its center of gravity; it is generally outside the wire. We consider a new polygon, constructed by k mass centers. Therefore it also has k vertices. We compute the *Point center* of the new polygon. The Point center of a r-gon is defined by the imagination that the masses are in the vertices of the polygon. Let $m_1, m_2, \ldots m_{r-1}, m_r$ be r masses. The polygon has the Point center $Point = (point_x, pointy)$.

$$point_x = \frac{1}{M}\cdot\sum_{i=1}^{r} m_i \cdot a_i \tag{5.3}$$

$$point_y = \frac{1}{M}\cdot\sum_{i=1}^{r} m_i \cdot b_i, \text{ and} \tag{5.4}$$

$$M = \sum_{i=1}^{r} m_i \text{ is the sum of the masses and} \tag{5.5}$$

$$(a_1,b_1),(a_2,b_2),\ldots(a_{r-1},b_{r-1}),(a_r,b_r) \text{ are different vertices of the polygon.} \tag{5.6}$$

To calculate the Spieker center of a given simple polygon we have to consider a new polygon, constructed by k mass centers of the k edges. Therefore it also has k vertices. We assume that in the new polygon the masses are on these k vertices. The Spieker center of the given polygon is the Point center of the new polygon. The formulas are

$$spieker_x = \frac{1}{U} \cdot \sum_{i=1}^{k} l_i \cdot \left(\frac{1}{2} \cdot (x_i + x_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot x_i \tag{5.7}$$

$$spieker_y = \frac{1}{U} \cdot \sum_{i=1}^{k} l_i \cdot \left(\frac{1}{2} \cdot (y_i + y_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot y_i \tag{5.8}$$

$$\text{where } l_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \quad \text{and} \quad U = \sum_{i=1}^{k} l_i \tag{5.9}$$

We define $l_{k+1} := l_1$. Note the indices in the formulas! Note that it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$. The variable '$l_i$' means the length of one edge of the polygon. Every edge $[(x_i, y_i), (x_{i+1}, y_{i+1})]$ has a center of gravity $\frac{1}{2} \cdot ((x_i, y_i) + (x_{i+1}, y_{i+1}))$. U is the perimeter of the polygon.
As an example we take the 5-gon of above. Its Spieker center is about $(0.50, 0.93)$. In exact coordinates it is

$$\left(\frac{1}{2}, \frac{1}{6 + 2 \cdot \sqrt{7}} \cdot \left(2 + 2 \cdot \sqrt{7} + \frac{1}{2} \cdot \sqrt{42} \right) \right) = \left(\frac{1}{2}, \sqrt{7} - 2 + \frac{1}{8} \cdot \left(3 \cdot \sqrt{42} - \sqrt{294} \right) \right). \tag{5.10}$$

Definition 5.1. Let us presume a simple polygon P. We call P *beautiful* if and only if it holds that B equals the Spieker center. We call P *perfect* if and only if all three centers are the same, i.e. it holds that B equals both $Cent$ and the Spieker center.

Lemma 5.2. *Let s be a symmetry axis of a simple polygon P. The Spieker center is on the line $l(s) \cap Rectangle(P)$.*

Proof. The segment s is a symmetry axis both for the entire polygon and for its contour. Therefore the Spieker center is on s. Because the Spieker center is the barycenter of the contour it has to be in $Rectangle(P)$. □

Proposition 5.3. *Let a simple polygon has two or more symmetry axes. Then it is perfect and all symmetry axis intersect in a single point.*

Proof. The three points $B, Cent$ and the Spieker center all are on the line determined by a symmetry axis. There is only a single possibility that all points are on every symmetry axis. □

Conjecture 5.4. *In a simple polygon which is not a triangle we have that $B = Cent$ if and only if $B = Cent =$ Spieker center.*

Conjecture 5.5. *A triangle is perfect if and only if it is an equilateral triangle.*

Conjecture 5.6. *A r-gon is perfect if and only if it is an regular r-gon.*

Conjecture 5.7. *A simple polygon is beautiful if and only if it has more than one symmetry axis. In other words it holds that we have $B =$ Spieker center if and only if $B =$ Spieker center $= Cent$.*

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