An Extension of Vajda's and D'Ocagne's Identities

Suaib Lateef

Abstract
In this paper, we present an identity which generalizes Vajda's and D'ocagne's identities involving Fibonacci sequence. Vajda's and D'ocagne's identities are known to contain 3 and 2 variables respectively, but the identity to be presented contains 7 variables. Binet's formula for generating the nth term of Fibonacci sequence will be used in proving the identity. Also, we prove another identity involving Fibonacci Numbers using some known basic identities

Keywords: Fibonacci sequence, Binet's formula, Vajda's and D'Ocagne's identities.

1 Introduction

It is widely known that Fibonacci sequence, along with Lucas sequence, are two of the most commonly used sequences for establishing recurrence relations. The recurrence relations concerning both Fibonacci and Lucas sequences give their next numbers as the sum of the preceeding two numbers. The classic Fibonacci sequence is denoted as

$$F_k = F_{k-1} + F_{k-2}, k \geq 2, F_0 = 0, F_1 = 1.$$

The first few Fibonacci numbers are given by:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...$$

Also, the classic Lucas sequence is denoted as

$$L_k = L_{k-1} + L_{k-2}, k \geq 2, L_0 = 2, L_1 = 1.$$

The first few Lucas numbers are given by:

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

Cennet Bolat, Ahmet Ipek, and Hasan Kose[4] noted that in recent years, many interesting properties of classic Fibonacci numbers, classic Lucas numbers and their generalizations have been shown by researchers and applied to almost every aspect of science and art. From the richness and related applications of these numbers, one can refer to the nature and different areas of science. Many identities which relate to both Fibonacci and Lucas sequences have been discovered over time. Francisco Regis Vieira Alves[5] noted that the role of the Fibonacci's sequence is usually discussed by most of mathematics history books. Despite its presentation in a form of mathematical problem,concerning the birth of rabbits' pairs, still occurs a powerful mathematical model that became the object of research, especially with the French mathematician François Édouard Anatole Lucas(1842–1891). From his work, a profusion of mathematical properties became known in the pure mathematical research, specially, from the sixties and the seventies. For instance, in 1680, Cassini discovered without proof that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1}$$

which was later independently proven by Robert Simson in 1753.
In 1879, Eugene Charles Catalan discovered and proved that

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2 \tag{2}$$

which generalizes cassini's identity. Another identity which generalizes Cassini's identity was discovered by Philbert Maurice D'Ocagne and the identity states that

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n} \tag{3}$$

In 1989, Steven Vajda[8] published a book on fibonacci numbers which contains the identity

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^nF_iF_j \tag{4}$$

which carries his name and thus generalizes both Cassini's and Catalan's Identities. Various methods have been employed to prove (1), (2), (3), and (4), some of which are proof by induction, Binet's formula for generating the nth term of Fibonacci sequence, proof by matrix method, combinatorial method.
V.C HARRIS [11], in his paper shows some Identities which are useful in proving the second Identity in this paper.

2 Main Results

Identity1 : If n, i, j, k, x, y, z are integers, then

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{5}$$

From (3), we can see that setting $x = y = z = k = 0$, we get

$$F_{n+i+x-z}F_{n+j+y+z} - F_{n+x+y-k}F_{n+i+j+k} = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x}$$
$$F_{n+i+0-0}F_{n+j+0+0} - F_{n+0+0-0}F_{n+i+j+0} = (-1)^{n+0-0-0}F_{i+0+0-0}F_{j+0+0-0}$$

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j \tag{6}$$

Now, we can see that (6) is equal to (4)

Also, we can see that setting $i = 1, j = m - n$ in (6), we get

$$F_{n+i}F_{n+j} - F_nF_{n+i+j} = (-1)^n F_iF_j$$

$$F_{n+1}F_{n+m-n} - F_nF_{n+1+m-n} = (-1)^n F_1F_{m-n}$$

$$F_{n+1}F_m - F_nF_{m+1} = (-1)^n F_{m-n} \tag{7}$$

Also, we can see that (7) is equal to (3)

Identity2 : If F_k is the kth Fibonacci Number, then

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{j}{F_k}^2\right)^3 = \left(\sum_{j=1}^{n}F_j\left(\sum_{k=1}^{j}{F_k}^2\right)\right)^2$$

3 Proof

Identity1 : We know from Binet's formula for generating nth Fibonacci number that if

$$\phi = \frac{1+\sqrt{5}}{2}, \varphi = \frac{1-\sqrt{5}}{2}$$

then

$$F_n = \frac{\phi^n - \varphi^n}{\sqrt{5}}$$

where F_n is the nth Fibonacci number

From (5), let

$$\alpha = F_{n+i+x-z}F_{n+j+y+z} \tag{8}$$

$$\beta = F_{n+x+y-k}F_{n+i+j+k} \tag{9}$$

$$\gamma = (-1)^{n+x+y-k}F_{i+k-y-z}F_{j+k+z-x} \tag{10}$$

such that

$$\alpha - \beta = \gamma \tag{11}$$

We can see that to prove (5), it suffices to show that (11) is true

Also, let

$$P_1 = \frac{\phi^{2n+i+x+j+y}}{5},$$
$$P_2 = \frac{\varphi^{2n+i+x+j+y}}{5},$$
$$P_3 = \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5},$$
$$P_4 = \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5},$$
$$P_5 = \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5},$$
$$P_6 = \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5}$$
$$P_7 = \phi^{i+k-y-z}\varphi^{j+z+k-x},$$
$$P_8 = \phi^{j+z+k-x}\varphi^{i+k-y-z},$$
$$P_9 = \phi^{2k+i+j-x-y},$$
$$P_{10} = \varphi^{2k+i+j-x-y}$$
$$P_{11} = F_{i+k-y-z}F_{j+k+z-x}$$

From (8), we see that

$$\alpha = F_{n+i+x-z}F_{n+j+y+z}$$
$$\alpha = (\frac{\phi^{n+i+x-z} - \varphi^{n+i+x-z}}{\sqrt{5}})(\frac{\phi^{n+j+y+z} - \varphi^{n+j+y+z}}{\sqrt{5}})$$
$$\alpha = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} - \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$
$$\alpha = P_1 - P_3 - P_4 + P_2 \tag{12}$$

From (9), we see that

$$\beta = F_{n+x+y-k}F_{n+i+j+k}$$
$$\beta = (\frac{\phi^{n+x+y-k} - \varphi^{n+x+y-k}}{\sqrt{5}})(\frac{\phi^{n+i+j+k} - \varphi^{n+i+j+k}}{\sqrt{5}})$$
$$\beta = \frac{\phi^{2n+i+x+j+y}}{5} - \frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} - \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5} + \frac{\varphi^{2n+i+x+j+y}}{5}$$

$$\beta = P_1 - P_5 - P_6 + P_2 \tag{13}$$

Deducting (13) from (12) gives

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4) \tag{14}$$

From (14), let

$$V_1 = -(P_3 + P_4)$$
$$V_2 = (P_5 + P_6)$$

then

$$V_1 = -(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5})$$
$$V_1 = -\frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}(\frac{\phi^{n+i+x-z}\varphi^{n+j+y+z}}{5} + \frac{\phi^{n+j+y+z}\varphi^{n+i+x-z}}{5})$$
$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\frac{\phi^{n+i+x-z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+j+y+z}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+j+y+z}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+x-z}}{\varphi^{n+x+y-k}})$$
$$V_1 = -\frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{i+k-y-z}\varphi^{j+z+k-x} + \phi^{j+z+k-x}\varphi^{i+k-y-z})$$

Note that

$$\phi\varphi = -1$$
$$V_1 = -\frac{1}{5}((-1)^{n+x+y-k})(P_7 + P_8)$$

Also,

$$V_2 = (\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5})$$
$$V_2 = \frac{(\phi\varphi)^{n+x+y-k}}{(\phi\varphi)^{n+x+y-k}}(\frac{\phi^{n+x+y-k}\varphi^{n+i+j+k}}{5} + \frac{\phi^{n+i+j+k}\varphi^{n+x+y-k}}{5})$$
$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\frac{\phi^{n+x+y-k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+i+j+k}}{\varphi^{n+x+y-k}} + \frac{\phi^{n+i+j+k}}{\phi^{n+x+y-k}}\frac{\varphi^{n+x+y-k}}{\varphi^{n+x+y-k}})$$
$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^0\varphi^{2k+i+j-x-y} + \phi^{2k+i+j-x-y}\varphi^0)$$
$$V_2 = \frac{1}{5}((\phi\varphi)^{n+x+y-k})(\phi^{2k+i+j-x-y} + \varphi^{2k+i+j-x-y})$$

Note that

$$\phi\varphi = -1$$

$$V_2 = \frac{1}{5}((-1)^{n+x+y-k})(P_9 + P_{10})$$

Now, we see from (14) that

$$\alpha - \beta = (P_5 + P_6) - (P_3 + P_4)$$

$$\alpha - \beta = V_1 + V_2$$

$$\alpha - \beta = \frac{1}{5}((-1)^{n+x+y-k})(P_9 - P_7 - P_8 + P_{10}) \tag{15}$$

From (10), we see that

$$\gamma = (-1)^{n+x+y-k} F_{i+k-y-z} F_{j+k+z-x}$$

$$\gamma = (-1)^{n+x+y-k}(P_{11}) \tag{16}$$

But

$$P_{11} = F_{i+k-y-z} F_{j+k+z-x}$$

$$P_{11} = \left(\frac{\phi^{i+k-y-z} - \varphi^{i+k-y-z}}{\sqrt{5}}\right)\left(\frac{\phi^{j+k+z-x} - \varphi^{j+k+z-x}}{\sqrt{5}}\right)$$

$$P_{11} = \frac{1}{5}(\phi^{i+k-y-z} - \varphi^{i+k-y-z})(\phi^{j+k+z-x} - \varphi^{j+k+z-x}) \tag{17}$$

We can see that the expansion of (17) gives

$$P_{11} = \frac{1}{5}(P_9 - P_7 - P_8 + P_{10}) \tag{18}$$

So, putting (18) in (16) gives

$$\gamma = \frac{1}{5}(-1)^{n+x+y-k}(P_9 - P_7 - P_8 + P_{10}) \tag{19}$$

Since (15) equals (19) then, (11) is true which completes the proof.

Identity2 :

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{j} {F_k}^2\right)^3 = \left(\sum_{j=1}^{n} F_j \left(\sum_{k=1}^{j} {F_k}^2\right)\right)^2 \tag{20}$$

We know that

$$\sum_{k=1}^{j} {F_k}^2 = F_j F_{j+1} \tag{21}$$

Putting (21) in (20), we have

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left(\sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2 \tag{22}$$

Also, we know that

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2 \tag{23}$$

Putting (23) in (22), we have

$$\frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2 = \left(\sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2$$

$$\left(\sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2 = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2$$

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2} \tag{24}$$

The next step is to use proof by induction to show that (24) holds for all positive integers n
From (24), for $n = 1$,

$$F_1^2 F_2 = \frac{1}{2} F_1 F_2 F_3$$

$$(1)^2(1) = \frac{1}{2}(1)(1)(2)$$

$$(1) = (1)$$

is true.
for $n = k$,

$$\sum_{j=1}^{k} F_j^2 F_{j+1} = \frac{1}{2} F_k F_{k+1} F_{k+2}$$

for $n = k + 1$

$$\sum_{j=1}^{k+1} F_j^2 F_{j+1} = F_{k+1}^2 F_{k+2} + \sum_{j=1}^{k} F_j^2 F_{j+1} = F_{k+1}^2 F_{k+2} + \frac{1}{2} F_k F_{k+1} F_{k+2}$$

So,

$$2 \sum_{j=1}^{k+1} F_j^2 F_{j+1} = 2F_{k+1}^2 F_{k+2} + F_k F_{k+1} F_{k+2}$$

$$2 \sum_{j=1}^{k+1} F_j^2 F_{j+1} = F_{k+1} F_{k+2} \left(2F_{k+1} + F_k \right) \tag{25}$$

We know that

$$2F_{k+1} + F_k = F_{n+3} \tag{26}$$

Putting (26) in (25), we have

$$2\sum_{j=1}^{k+1} F_j^2 F_{j+1} = F_{k+1}F_{k+2}F_{n+3}$$

$$\sum_{j=1}^{k+1} F_j^2 F_{j+1} = \frac{1}{2}F_{k+1}F_{k+2}F_{n+3}$$

which shows that (24) holds. This completes the proof.

4 Conclusion

In this paper, we have stated and proven an identity which generalizes both Vajda's and D'Ocagne's identities (involving Fibonacci sequence) using Binet's formula for generating nth Fibonacci number. Also, we have proven another identity using some previously known identities.

References

[1] A.F. Horadam, A generalized Fibonacci sequence,*Amer. Math. Monthly*, 68(1961), pp. 455–459.

[2] A.F. Horadam, Basic properties of a certain generalized sequence of numbers, *Fib. Quart*,3(3)(1965), pp. 161–176.

[3] A.T. Benjamin and J.J.Quinn, Recounting Fibonacci and Lucas identities, *CollegeMath. J.* , 30, No.5(1999), pp. 359-366.

[4] Cennet Bolat, Ahmet Ipek, Hasan Kose, On the sequence related to Lucas Numbers and its properties, *Mathematica Aeterna*, Vol. 2(2012), pp. 63-75.

[5] Francisco Regis Vieira Alves, The Quaterniontonic and Octoniontonic Fibonacci Cassini's Identity:An Historical Investigation with the Maple's Help, *International Electronics Journal of Mathemarics Education*, Vol. 13 (2018), pp. 125-138.

[6] Jeffrey R. Chasnov, Fibonacci Numbers and the Golden Ratio, *The Hong kong University of science and technology*, Hong Kong,(2016), 5.

[7] Johnson, R.C., Matrix methods for Fibonacci and related sequences, 2003, Available online *http://www.dur.ac.uk/bob.johnson/fibonacci/.*

[8] Keskin, R., B. Demirturk, Some new Fibonacci and Lucas identities by matrix methods, *Internat.J.Math.Ed.Sci.Tech.*, Vol. 41(2010), pp. 379–387.

[9] Steven Vajda, "Fibonacci and Lucas Numbers, and the Golden Section:Theory and Applications", *Dover Publications, Inc*, New York, 1989.

[10] T.Koshy, "Fibonacci and Lucas Numbers with Applications", *A Wiley-Interscience Publication*, New York, 2001.

[11] V.C. HARRIS, "On Identities Involving Fibonacci Numbers", *San Diego State College, San Diego*, California.

[12] Y.K. Panwar, Bijendra Singh, V.K. Gupta, "Generalized Fibonacci sequences and its properties", *Palestine journal of mathematics*, vol. 3(1)(2014), pp. 141-147.