The Vortegy Concept in the Viscous Incompressible Fluid.

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ABSTRACT. In this paper is shown that the quantity $V = \int_{\mathbb{R}^n} |\boldsymbol{\omega}(x,t)|^2 dx$, n = 2 or 3, called here the vortegy, is a globally controlled scalar measure of the fluid vorticity degree. In the incompressible fluid, the physical properties of the vortegy are like the properties of energy $E = \frac{\rho}{2} \int_{\mathbb{R}^n} |\mathbf{u}(x,t)|^2 dx$. In the inviscid fluid, the law of vortegy conservation operates, in the viscous fluid, vortegy is subject to dissipation, the law of vortegy dissipation is established. However, in contrast to the supercritical energy E (for $n = 3$), the vortegy V is subcritical. It is also shown that when vortegy dissipation is considered, the system of generalized Helmholtz equations expresses the law of its conservation. The supercriticality paradox of the 3D Navier-Stokes equations is resolved, the impossibility of a blowup scenario for their solutions and the inevitability of such a scenario for 3D solutions of the Euler equations are shown.

The paper [1] analyzes possible strategies for solving the problem of global regularity of 3D Navier-Stokes equations. Three possible strategies are highlighted, and the main obstacle is indicated. Such an obstacle is the supercriticality of the Navier-Stokes equations with respect to scaling. According to the author in [1]: "… all of the globally controlled quantities for Navier-Stokes evolution which we are aware of (and we are not aware of very many) are either supercritical with respect to scaling, which means that they are much weaker at controlling fine-scale behaviour than controlling coarse-scale behaviour, or they are non-coercive, which means that they do not really control the solution at all, either at coarse scales or at fine".

Since as follows from the above, the supercriticality problem of the Navier-Stokes equations is of extra importance, we will briefly touch upon this problem following [1].

The Navier-Stokes equations obey scale invariance. Let some smooth velocity field $\mathrm{u}(x,t)$ and the pressure field $p(x,t)$ satisfy the Navier-Stokes equations in $\mathbb{R}^n$ (n=2 or 3) during the half-time interval $0 \le t < T$. Then, for any scaling parameter $\lambda > 0$, a new velocity field and a new pressure field can be formed:

$$\mathrm{u}^{(\lambda)}(x,t) = \frac{1}{\lambda}\mathrm{u}\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right),$$

$$p^{(\lambda)}(x,t) = \frac{1}{\lambda^2}p\left(\frac{x}{\lambda},\frac{t}{\lambda^2}\right),$$

which will also be a solution to the Navier-Stokes equation for $0 \leq t < \lambda^2 T$. Further in [1], two known (for today) globally controlled quantities are considered:

- the maximum kinetic energy:

$$\sup_{0\leq t<T} \frac{1}{2}\int\limits_{\mathbb{R}^n} |\mathrm{u}(x,t)|^2\, dx,$$

- the cumulative energy dissipation:

$$\frac{1}{2}\int\limits_0^T \int\limits_{\mathbb{R}^n} |\nabla \mathrm{u}(x,t)|^2\, dxdt,$$

(since in this case we are interested in purely mathematical aspects of the problem, in the formulas above, the constants ρ and ν are omitted).

Both quantities are bonded even for very large T, since the energy conservation law implies that these values are always less than the initial energy value E_0.

From the formulas above it follows that when scaling the energy (both), its scaled value will be equal to λE for $n = 3$ and E for $n = 2$. Hence, in the three-dimensional case, the control of these two key quantities worsened with the appearance of the coefficient λ when going to small scales (small values of λ). Because of this worsening, these values are called supercritical. They become more and more useless as you move to smaller scales. In the two-dimensional case, the energy E is invariant to scaling and the quantities under consideration are critical. The critical quantities control all scales equally well (or equally poorly). One can also assume the existence of subcritical quantities, they will strengthen control on small scales and weaken on large ones.

Further, in [1], by scaling the original solution, it is shown that for supercritical quantities there is no way to exclude the blowup scenario of the evolution of an initially smooth solution; in a finite time, singularities can appear in it. This is the main problem of the Navier-Stokes equations. To resolve it, the author in [1] sees only three possible strategies. One of the strategies is to discover a fundamentally new method that will provide smooth solutions. However, any new method that claims to solve this problem must necessarily resolve the supercriticality paradox of these equations. And this, according to the author in [1], can be done only using the two remaining strategies:

1. Solve the Navier-Stokes equation exactly and explicitly (or at least transform this equation exactly and explicitly to a simpler equation).

2. Discover a new globally controlled quantity which is both coercive and either critical or subcritical.

The first way is probably impossible for obvious reasons, only the second remains.

Generalized Helmholtz equations are the direct consequence of the Navier-Stokes equations for a viscous incompressible fluid:

$$\frac{\partial \omega_x}{\partial t} + u_x \frac{\partial \omega_x}{\partial x} + u_y \frac{\partial \omega_x}{\partial y} + u_z \frac{\partial \omega_x}{\partial z} = \omega_x \frac{\partial u_x}{\partial x} + \omega_y \frac{\partial u_x}{\partial y} + \omega_z \frac{\partial u_x}{\partial z} + \nu \Delta \omega_x$$

$$\frac{\partial \omega_y}{\partial t} + u_x \frac{\partial \omega_y}{\partial x} + u_y \frac{\partial \omega_y}{\partial y} + u_z \frac{\partial \omega_y}{\partial z} = \omega_x \frac{\partial u_y}{\partial x} + \omega_y \frac{\partial u_y}{\partial y} + \omega_z \frac{\partial u_y}{\partial z} + \nu \Delta \omega_y \qquad (1)$$

$$\frac{\partial \omega_z}{\partial t} + u_x \frac{\partial \omega_z}{\partial x} + u_y \frac{\partial \omega_z}{\partial y} + u_z \frac{\partial \omega_z}{\partial z} = \omega_x \frac{\partial u_z}{\partial x} + \omega_y \frac{\partial u_z}{\partial y} + \omega_z \frac{\partial u_z}{\partial z} + \nu \Delta \omega_z.$$

It is well known that different groups of terms in these equations describe different physical mechanisms through which vorticity propagates in the fluid, see for example [2]. These mechanisms are vorticity convection, vorticity diffusion, and vortex stretching. In accordance with the belonging to these mechanisms, each of the equations of system (1) can be conditionally divided into three equations:

- vorticity convection equation

$$\frac{\partial \omega_i}{\partial t} + u_x \frac{\partial \omega_i}{\partial x} + u_y \frac{\partial \omega_i}{\partial y} + u_z \frac{\partial \omega_i}{\partial z} = 0, \qquad (2)$$

- vorticity diffusion equation

$$\frac{\partial \omega_i}{\partial t} = \nu \Delta \omega_i, \qquad (3)$$

- vortex stretching equation

$$\frac{\partial \omega_i}{\partial t} = \omega_x \frac{\partial u_i}{\partial x} + \omega_y \frac{\partial u_i}{\partial y} + \omega_z \frac{\partial u_i}{\partial z}. \qquad (4)$$

Each of these equations has a clear physical meaning, understanding of which will allow to understand the essence of what is happening.

Vorticity convection is a process of vorticity transfer by a flowing fluid (vorticity transfer by velocity). From equation (2) it follows that the vorticity value ω at some point M during the time dt will be transferred to the neighboring point M' lying in the direction of the velocity vector $\mathbf{u}$ at a distance udt.

Vorticity diffusion is a process of vorticity propagation caused by its uneven distribution in a fluid. All diffusion processes in nature are described by Fick's laws. Equation (3) is an expression of Fick's second law for vorticity, from which, in particular, it follows that the diffusion mechanism always tends to equalize (to smooth) the vorticity values at adjacent points.

Vortex stretching is a set of processes that occur when the vortex tube is stretched. The author of this paper considers the term "vortex stretching" unsuccessful; this term rather speaks of the cause, rather than the essence of the occurring physical phenomena. When the vortex is axially stretched, its diameter decreases due to the incompressibility of the fluid. This, in accordance with the law of conservation of angular momentum, leads to an increase in vorticity in the vortex. Outwardly, for the point under consideration, this process looks like the emergence of vorticity directly in the mass of the fluid and is mathematically described by the divergence of the field $u_i\boldsymbol{\omega}$, equation (4) can be written differently,

$$\frac{\partial \omega_i}{\partial t} = \text{div}\,(u_i\boldsymbol{\omega}). \tag{5}$$

Therefore, further, another term will be used - vorticity divergence. The process of vorticity divergence is reversible, since the vortex can not only stretch, but also contract, while the vorticity will be absorbed. Next, we need to derive one formula.

For an arbitrary point M in a fluid, consider a rectangular coordinate system ξ, η, ζ such that the axis ξ passing through this point has a direction that coincides with the direction of the vorticity vector $\boldsymbol{\omega}$ at this point. Therefore, at the point under consideration, the vector $\boldsymbol{\omega}$ has only one nonzero component ω_ξ. In a small neighborhood of the point M, we also consider the velocity field $\mathbf{u}$; the projection of the velocity vector $\mathbf{u}$ onto the ξ axis is denoted by u_ξ. Let's calculate the divergence of the field $u_\xi\boldsymbol{\omega}$ at point M, taking into account the identity $\partial\omega_\xi/\partial\xi + \partial\omega_\eta/\partial\eta + \partial\omega_\zeta/\partial\zeta = 0$, we obtain,

$$\text{div}\,(u_\xi\boldsymbol{\omega}) = \frac{\partial(u_\xi\omega_\xi)}{\partial\xi} + \frac{\partial(u_\xi\omega_\eta)}{\partial\eta} + \frac{\partial(u_\xi\omega_\zeta)}{\partial\zeta} = \omega_\xi\frac{\partial u_\xi}{\partial\xi} = |\boldsymbol{\omega}|\frac{\partial u_\xi}{\partial\xi}. \tag{6}$$

Divergence is an invariant to the choice of a coordinate system at a given point. If its value is known in some coordinate system, then this value will remain the same in any other coordinate system. Then we can write

$$\operatorname{div}\left(u_\xi \boldsymbol{\omega}\right) = \omega_x \frac{\partial u_\xi}{\partial x} + \omega_y \frac{\partial u_\xi}{\partial y} + \omega_z \frac{\partial u_\xi}{\partial z}. \tag{7}$$

Let γ_x, γ_y, γ_z be the angles between the ξ axis and the X, Y and Z axes, respectively. Then the value of the function u_ξ in a neighborhood of the point M can be represented as follows

$$u_\xi = u_x \cos\gamma_x + u_y \cos\gamma_y + u_z \cos\gamma_z. \tag{8}$$

Now, using this formula, one can calculate the derivatives $\partial u_\xi/\partial x_i$, while differentiating the angles γ_x, γ_y, γ_z are assumed to be constant, from formula (6) it follows that to calculate the value $\operatorname{div}\left(u_\xi \boldsymbol{\omega}\right)$ it is not required to know how orientation of the vector $\boldsymbol{\omega}$ in space changes at points adjacent to M. Calculating the derivatives $\partial u_\xi/\partial x_i$, putting them in formula (7) and transforming it, we get:

$$\operatorname{div}\left(u_\xi \boldsymbol{\omega}\right) = \operatorname{div}\left(u_x \boldsymbol{\omega}\right)\cos\gamma_x + \operatorname{div}\left(u_y \boldsymbol{\omega}\right)\cos\gamma_y + \operatorname{div}\left(u_z \boldsymbol{\omega}\right)\cos\gamma_z. \tag{9}$$

This formula has a simple physical meaning, which becomes obvious if we notice that this formula is a complete analogue of formula (8). The quantities $\operatorname{div}\left(u_i \boldsymbol{\omega}\right)\partial t$ are the increments of the vorticity components $\partial\omega_i$. When recalculated to the direction ξ, these quantities are summed up as vectors, since the vorticity is of a vector nature.

To simplify the writing of formulas further, introduce the notation,

$$\omega = |\boldsymbol{\omega}|.$$

Now formula (9) can be transformed by replacing the cosines with their values, $\cos\gamma_i = \omega_i/\omega$, we get:

$$\omega\operatorname{div}\left(u_\xi \boldsymbol{\omega}\right) = \omega_x \operatorname{div}\left(u_x \boldsymbol{\omega}\right) + \omega_y \operatorname{div}\left(u_y \boldsymbol{\omega}\right) + \omega_z \operatorname{div}\left(u_z \boldsymbol{\omega}\right). \tag{10}$$

This formula was the ultimate goal. If we talk about what equations (4) describe in the whole space, then this will be an overly complex process, the essence of which can be understood using the following simple example. Suppose we are considering the process of axial stretching (along ξ axis) of a vortex with an initial radius r_0 and an initial vorticity ω_0. Due to the incompressibility of the fluid, its current radius r under tension will decrease, and the vorticity will increase in accordance with the law of conservation of angular momentum, $\omega_\xi = \omega_0 r_0^2/r^2$. If the vortex radius r is strictly constant along the vortex axis ξ, then there will be no axial vorticity gradient in the vortex, $\partial\omega_\xi/\partial\xi = 0$. Let us now assume that a constriction has appeared on the vortex - a small local decrease in the radius r. In this case, the vorticity unevenness will appear in the constricted area, i.e. axial vorticity gradient, $\partial\omega_\xi/\partial\xi \neq 0$. But then in accordance with the identity

$$\partial\omega_\xi/\partial\xi + \partial\omega_\eta/\partial\eta + \partial\omega_\zeta/\partial\zeta = 0,$$

a vorticity gradient must necessarily appear in the direction perpendicular to the ξ axis, the vortex will begin to deform, which will lead to interaction between different parts of the vortex. The result of this process is illustrated by this video (available on the Internet in open sources).

Considering the mechanisms of vorticity propagation, important conclusions can be drawn. To do this, it is enough to trace a small area of a moving fluid during its movement. Then, from equations (2), (3), (4) it follows that the convection mechanism cannot change the value of vorticity in this region at all. The diffusion mechanism always tends to smooth out the vorticity value in this area with the level of the surrounding background. The divergence mechanism can increase the vorticity value, and this is the only mechanism that can lead to its unlimited growth. The action of the mechanisms of divergence and diffusion of vorticity is always in different directions, they always work against each other. All the variety of phenomena occurring in a viscous fluid: the birth and decay of vortices, their interaction, the formation of vortex cascades, etc. - this is always a manifestation of only convection, diffusion and divergence of vorticity, there are no other mechanisms, this follows from the form of equations (1).

Consider the system of equations (1). Multiplying the first equation by $2\omega_x$, the second by $2\omega_y$ and the third by $2\omega_z$, and add these three equations, then taking into account that $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ and using formulas (6), (10) we get:

$$\frac{\partial(\omega^2)}{\partial t} + u_x\frac{\partial(\omega^2)}{\partial x} + u_y\frac{\partial(\omega^2)}{\partial y} + u_z\frac{\partial(\omega^2)}{\partial z} = 2\omega^2\,\frac{\partial u_\xi}{\partial\xi} + \qquad (11)$$

$$+2\nu\left(\omega_x\Delta\omega_x + \omega_y\Delta\omega_y + \omega_z\Delta\omega_z\right).$$

In this equation, the quantity $\partial u_\xi/\partial\xi$ is a function of coordinates and time. The structure of this equation is like the structure of the equations of system (1). This similarity suggests that the scalar quantity ω^2 has propagation mechanisms like vorticity, and equation (11) is the law of its evolution.

Let us introduce the concept of vortegy V by determining its bulk density,

$$\text{vortegy bulk density} = \omega^2. \qquad (12)$$

From this definition, it follows that the vortegy contained in the volume of the fluid W is equal to:

$$V = \int_W \omega^2 dW.$$

Regarding the vortegy, let us put forward two hypotheses with the aim to see which equation of evolution these hypotheses will lead to.

So, the first hypothesis asserts that convection, diffusion, and divergence are the mechanisms of vortegy propagation, there are no other mechanisms.

The second hypothesis is the statement about the persistence of the vortegy. To formulate the second hypothesis, we first select in the space of fluid motion a simply connected volume W bounded by a closed surface S. Now the second hypothesis can be formulated as follows:

The change of vortegy in the volume W during the time ∂t is equal to the vortegy that entered this volume during this time through the surrounding surface S, plus the vortegy formed (absorbed) in this volume due to divergence, minus the vortegy that left this volume through the same surface during this same time.

This formulation at $W \to 0$ corresponds to the following equation,

$$\frac{\partial(\omega^2)}{\partial t}+\frac{\partial j_x}{\partial x}+\frac{\partial j_y}{\partial y}+\frac{\partial j_z}{\partial z}=\mathrm{Div}, \tag{13}$$

where j_x, j_y, j_z are the vortegy flux densities along the corresponding axes, and the term Div describes the vortegy divergence.

For the convection mechanism (transfer of vortegy by velocity), write down the vortegy flux density in the direction of i^{-d} coordinate,

$$j_i = u_i \omega^2.$$

Then, from (13) and the condition of fluid incompressibility $\mathrm{div}\,\mathbf{u}=0$ the equation of vortegy convection follows,

$$\frac{\partial(\omega^2)}{\partial t}+u_x\frac{\partial(\omega^2)}{\partial x}+u_y\frac{\partial(\omega^2)}{\partial y}+u_z\frac{\partial(\omega^2)}{\partial z}=0. \tag{14}$$

For the mechanism of vortegy diffusion, write down Fick's first law in the direction of the i^{-d} coordinate,

$$j_i = -\nu\frac{\partial(\omega^2)}{\partial x_i}.$$

Then, from (13) follows the equation of vortegy diffusion (Fick's second law)

$$\frac{\partial(\omega^2)}{\partial t}=\nu\left(\frac{\partial^2(\omega^2)}{\partial x^2}+\frac{\partial^2(\omega^2)}{\partial y^2}+\frac{\partial^2(\omega^2)}{\partial z^2}\right)\equiv \nu\Delta\omega^2. \tag{15}$$

To describe the process of vortegy divergence, let us turn to definition (12). It can be seen from it that the formation (absorption) of vortegy will always occur with any change in the vorticity ω. Then, if there is a small change in the vorticity $\partial\omega$, then the change in the value of ω^2 will obviously be equal to $2\omega\partial\omega$. Considering formulas (5), (6), we can write,

$$\frac{\partial(\omega^2)}{\partial t} = 2\omega\ \mathrm{div}\left(u_\xi \boldsymbol{\omega}\right) = 2\omega^2\ \frac{\partial u_\xi}{\partial \xi}. \tag{16}$$

It is also necessary to show that the process of vortegy divergence in an infinite volume of fluid under the conditions of the Cauchy problem does not lead to the emergence or loss of its quantities. In other words, it is necessary to show that the integral

$$\iiint\limits_{-\infty}^{+\infty} \omega\ \mathrm{div}\left(u_\xi \boldsymbol{\omega}\right) dxdydz, \tag{17}$$

at all times will be identically equal to zero.

As applied to the Cauchy problem, only the problem with a finite velocity u and finite total energy of the initial state, $E_0 < \infty$, is of interest. This condition corresponds to the velocity field $\mathbf{u}$, which decreases at infinity faster than $r^{-3/2}$. In this case, the decrease order of the vorticity field $\boldsymbol{\omega}$ will be faster than $r^{-5/2}$.

Formula *(16)* describes the divergence of the vortegy; therefore, the quantity $2\omega\ \mathrm{div}\left(u_\xi \boldsymbol{\omega}\right)$ can be represented as the divergence of some abstract vector field $\boldsymbol{\psi}$, i.e. can be written as

$$\mathrm{div}(\boldsymbol{\psi}) = 2\omega\ \mathrm{div}\left(u_\xi \boldsymbol{\omega}\right).$$

This condition does not uniquely determine the field $\boldsymbol{\psi}$ itself; however, it uniquely determines the decrease order of this field at infinity - faster than $r^{-13/2}$. Now the integral (17) can be written like this,

$$\iiint\limits_{-\infty}^{+\infty} \omega\ \mathrm{div}\left(u_\xi \boldsymbol{\omega}\right) dxdydz = \frac{1}{2} \iiint\limits_{-\infty}^{+\infty} \mathrm{div}(\boldsymbol{\psi})\ dxdydz.$$

We apply to the second integral the Gauss-Ostrogradsky formula (divergence theorem), which expresses the integral over the volume in terms of the integral over the surface covering this volume. Consider a sphere of infinitely large radius as such a surface,

$$\iiint\limits_{-\infty}^{\infty} \mathrm{div}(\boldsymbol{\psi})\ dxdydz = \iint\limits_{S} \psi_n dS,$$

here S is the surface of the sphere, ψ_n is the normal component of the field $\boldsymbol{\psi}$. From this it is clearly seen that for the field $\boldsymbol{\psi}$ decreasing at infinity faster than $r^{-13/2}$, this integral will be always identically equal to zero. This means that the process of vortegy divergence does not lead to the emergence (loss) of its quantities but is associated only with its redistribution in space.

Collecting all three equations (14), (15), (16) together we obtain

$$\frac{\partial(\omega^2)}{\partial t} + u_x \frac{\partial(\omega^2)}{\partial x} + u_y \frac{\partial(\omega^2)}{\partial y} + u_z \frac{\partial(\omega^2)}{\partial z} = 2\omega^2 \frac{\partial u_\xi}{\partial \xi} + \nu\Delta\omega^2. \qquad (18)$$

Expanding the Laplace operator and using the formula $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, we get

$$\frac{\partial(\omega^2)}{\partial t} + u_x \frac{\partial(\omega^2)}{\partial x} + u_y \frac{\partial(\omega^2)}{\partial y} + u_z \frac{\partial(\omega^2)}{\partial z} = 2\omega^2 \frac{\partial u_\xi}{\partial \xi} + \qquad (19)$$

$$+2\nu\left(\omega_x\Delta\omega_x + \omega_y\Delta\omega_y + \omega_z\Delta\omega_z\right) + \mathrm{Dis},$$

where,

$$\mathrm{Dis} = 2\nu\left(\left(\frac{\partial\omega_x}{\partial x}\right)^2 + \left(\frac{\partial\omega_x}{\partial y}\right)^2 + \left(\frac{\partial\omega_x}{\partial z}\right)^2\right) + 2\nu\left(\left(\frac{\partial\omega_y}{\partial x}\right)^2 + \left(\frac{\partial\omega_y}{\partial y}\right)^2 + \left(\frac{\partial\omega_y}{\partial z}\right)^2\right) +$$

$$+2\nu\left(\left(\frac{\partial\omega_z}{\partial x}\right)^2 + \left(\frac{\partial\omega_z}{\partial y}\right)^2 + \left(\frac{\partial\omega_z}{\partial z}\right)^2\right). \qquad (20)$$

It is clearly seen that the value of Dis does not depend on the sign of the derivatives $\partial\omega_i/\partial x_j$; this sign is always only positive. Therefore, the value Dis, regardless of the sign of the quantities $\partial\omega_i/\partial x_j$, always only increases the value of the derivative $\partial(\omega^2)/\partial t$ in equation (19). Comparing equation (11), obtained from the Helmholtz equations, with equation (19), which expresses the conservation law, we see that they differ only in the term Dis. Hence, it becomes clear that the term Dis describes the irreversible loss of vortegy - its dissipation, and in the solutions of the Helmholtz equations (1) at $\nu \neq 0$, vortegy will not be conserved.

In an inviscid fluid ($\nu = 0$), Eq. (11) and Eq. (19) coincide, hence the vortegy will be conserved. However, here it is necessary to make a reservation, since it is possible to correctly speak about the conservation of vortegy only for those moments of time for which the solutions of these equations remain regular.

So, the first hypothesis has been completely confirmed, the vortegy really has only three propagation mechanisms: convection, diffusion, and divergence. The second hypothesis in the formulation given above was not confirmed: in a viscous fluid, vortegy is not conserved, it is subject to dissipation. However, it is still possible to talk about the law of vortegy conservation if we modify the second hypothesis and formulate this law as follows:

The change of vortegy in the volume W during the time ∂t is equal to the vortegy that has entered this volume during this time through the surrounding surface S, plus the vortegy formed (absorbed) in the considered volume due to its divergence, minus the vortegy absorbed in the considered volume of radiation. for its dissipation, minus the vortegy that left this volume through the same surface during the same time.

This formulation corresponds equation (11) obtained from the system of Helmholtz equations (1).

The physical properties of the vortegy V are terribly similar the properties of energy E, this similarity explains its name, the vortegy - vortex&energy. Kinetic energy in a viscous fluid is lost, it transforms into another form of energy - heat, which is not considered by the Navier-Stokes (Helmholtz) equations. This process is known as energy dissipation. However, if in the Cauchy problem we add the residual kinetic energy of the entire volume of the fluid and the accumulated energy of dissipation, then for any moment of time this value will be the same - E_0, it is set by the initial conditions, the law of energy conservation is in effect. Something similar happens with vortegy; it is lost in a viscous fluid, but no new forms of vortegy arise in this case. If we add up the residual vortegy of the entire mass of the fluid and the accumulated vortegy of dissipation, a time-independent quantity V_0 will also be obtained, it is also determined only by the initial conditions, the law of vortegy conservation is in effect.

So, by analogy with energy, we have in $\mathbb{R}^n$ (n = 2 or 3) two new globally controlled quantities:

- the maximum value of vortegy:

$$\sup_{0 \le t < T} \int_{\mathbb{R}^n} |\boldsymbol{\omega}(x,t)|^2 \, dx,$$

- the cumulative vortegy dissipation (the term Dis is defined by (20)):

$$\int_0^T \int_{\mathbb{R}^n} \mathrm{Dis}(x,t) \, dx dt.$$

Both quantities are bonded for any values of time T, since the vortegy conservation law implies that these quantities are always less than its initial value V_0, therefore, both quantities are coercive. However, in contrast to the supercritical energy E, both values are subcritical. It is easy to verify that the formulas above give a scaled vortegy value equal to V/λ for $n = 3$ and V/λ^2 for $n = 2$.

An example of vortegy calculating can be given, for example, for the toroidal vortex shown in the figure below. For simplicity, let us assume that for the initial moment of time t_0, all the vortegy (and vorticity ω_0) of this vortex was concentrated in the volume of the torus W, which is characterized by dimensions d and r. Then the initial vortex vortegy V_0 will be determined by the formula

$$V_0 = W\omega_0^2 = \pi^2 d^2 r \omega_0^2 / 2.$$

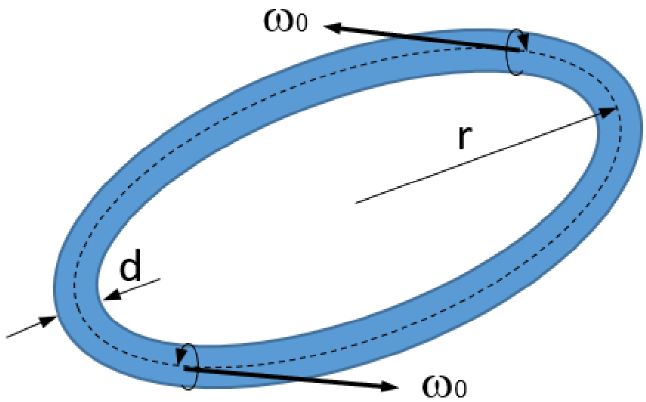

Figure 1. Toroidal vortex.

In a viscous fluid at any subsequent time instant, i.e., for $t > t_0$, all the parameters of this vortex - d, r, ω_0, will lose their initial meaning. However, it can be argued that the vortex vortegy will monotonically decrease over time $V(t)_{t_0<t\leq\infty} < V_0$, this is a manifestation of the process of vortegy dissipation. During evolution, this vortex can disintegrate into a system of vortices; the total vortegy of this vortices system will also be less than the value V_0. This video (available on the Internet in open sources) shows the process of interaction of two toroidal vortices, initially lying in the in parallel planes. As a result of this interaction, a system of vortices is formed that no longer lie in these planes. And in this case, the total vortegy of the system of vortices will be a monotonically decreasing function of time, and at any subsequent moment of time it will always be less than the initial vortegy value of the two vortices.

This could be the end of this paper. The next logical step would be to try to apply the already existing techniques of rigorous proofs of global regularity, which turned out to be ineffective when using supercritical energy E. Considering what was said in [1], there is every reason to expect that replacing supercritical energy E with subcritical vortegy V will lead to success ...

The desire of the author of this paper to "look beyond the horizon here and now" is understandable. It justifies the use of less rigorous, but simpler and faster methods for obtaining preliminary results of solving the problem.

It is easy to see that equation (11) is invariant to a change in the coordinate system. In any other system of rectangular coordinates $\varphi\eta\zeta$, shifted and rotated relative to the XYZ coordinate system, the equation will not change its form and will look exactly the same,

$$\frac{\partial(\omega^2)}{\partial t} + u_\varphi \frac{\partial(\omega^2)}{\partial \varphi} + u_\eta \frac{\partial(\omega^2)}{\partial \eta} + u_\zeta \frac{\partial(\omega^2)}{\partial \zeta} = 2\omega^2 \frac{\partial u_\xi}{\partial \xi} + \qquad (21)$$

$$+2\nu\left(\omega_\varphi \Delta\omega_\varphi + \omega_\eta \Delta\omega_\eta + \omega_\zeta \Delta\omega_\zeta\right).$$

Let us assume that in a fluid we observe a certain small region that moves with the fluid and in which conditions for the growth of vorticity have been created (vorticity divergence occurs - the values of $\partial u_\xi/\partial\xi$ are very large and continue to increase). We are interested in the answer to the question, can the growth of vorticity in this region be unlimited?

We place the origin of the coordinate system $\varphi\eta\zeta$ at the point of vorticity maximum and direct the φ axis along the direction of the vector $\boldsymbol{\omega}$ at this point. The coordinate system $\xi\eta\zeta$, in which the ξ axis coincides with the direction of the vorticity vector $\boldsymbol{\omega}$, has already been considered. Now the coordinate system $\xi\eta\zeta$ moves together with the fluid, all the time coinciding with its origin with the vorticity maximum, the direction of the ξ axis in this case all the time coincides with the direction of the vector $\boldsymbol{\omega}$, i.e. the coordinate system $\xi\eta\zeta$ can also rotate in space (XYZ) about some axis orthogonal to the ξ axis. Let us return to the consideration of equation (21). Here, however, the question arises about the legitimacy of using the non-inertial frame of reference $\xi\eta\zeta$ to write equation (21). The answer to this question is quite obvious, equation (21) does not operate with the concept of mass. All quantities included in this equation contain only the dimensions of length L and time T and do not contain the dimensions of mass M, which means that the concept of forces (inertia) cannot arise, so the question disappears by itself.

Since the coordinate system $\xi\eta\zeta$ moves with the fluid, convective terms will drop out of equation (21), now $u_\xi = u_\eta = u_\zeta = 0$. Since the ξ axis coincides with the direction of the vector $\boldsymbol{\omega}$, then $\omega_\eta = \omega_\zeta = 0$, and the equation takes the form,

$$\frac{\partial(\omega)^2}{\partial t} = 2\omega^2\frac{\partial u_\xi}{\partial\xi} + 2\nu\omega\Delta\omega. \tag{22}$$

In a small area of space in the vicinity of the vorticity maximum, all fluid particles will move along trajectories close to circular (more precisely, spiral-like). The appearance of such cylindrical symmetry is due to the vorticity is a vector quantity, it is characterized by the direction relative to which the cylindrical symmetry is formed. Moreover, this cylindrical symmetry will become more expressed, the more expressed the maximum is in comparison with the background vorticity values, and the faster it grows. If vorticity increases, this means that vorticity divergence occurs, since divergence is the only possible mechanism for its growth. Physically, this means that a small rotating cylinder on the axis of the vortex undergoes deformation of radial compression (axial tension - vortex stretching). The strain rate tensor of this state, if the strain rate ε_ξ is taken as a unit, will look like this: $\varepsilon_\xi = 1$, $\varepsilon_\eta = -1/2$, $\varepsilon_\zeta = -1/2$, $\gamma_{ij} = 0$. With this symmetry, the following relationships will hold:

$$\frac{\partial^2\omega}{\partial\eta^2} = \frac{\partial^2\omega}{\partial\zeta^2} = \frac{\partial^2\omega}{\partial r^2}.$$

Let us consider the case in which the changes in vorticity along the ξ axis is small compared to changes along the radius (it will become clear later that this case is conservative), i.e. the quantities $\partial\omega/\partial\xi$ and $\partial^2\omega/\partial\xi^2$ will be small compared to $\partial\omega/\partial r$ and $\partial^2\omega/\partial r^2$. Then the quantity $\partial^2\omega/\partial\xi^2$ in $\Delta\omega$ can be neglected and the equation (22) can be written as follows:

$$\frac{\partial(\omega)^2}{\partial t} = 2\omega^2\frac{\partial u_\xi}{\partial\xi} + 4\nu\omega\frac{\partial^2\omega}{\partial r^2}. \tag{23}$$

Let us continue to observe the small fluid cylinder on the vortex axis ξ. Let at time t_0 the initial radius of this cylinder be r_0. Further, assume that the radial compression of the cylinder occurs at a constant velocity ε, and the radius of the cylinder changes according to the law,

$$r = r_0 - \varepsilon t.$$

Then, from the condition of incompressibility of the fluid, we can determine the value $\partial u_\xi/\partial\xi$

$$\frac{\partial u_\xi}{\partial\xi} = \frac{2\varepsilon}{(r_0 - \varepsilon t)}. \tag{24}$$

As can be seen from this formula, the value of ε controls the intensity of the process of vortegy divergence (the vorticity increase rate ω), the values of this parameter will then vary within the widest range $0 \le \varepsilon \le \infty$.

Let at time t_0 the vorticity value on the vortex axis ω_m be equal to ω_0, i.e. at $t = t_0$ it was $\omega_m = \omega_0$. Since the value of ω_m is very large (and continues to increase) compared to the background values (at the outer edge, at $r = r_0 - \varepsilon t$), these values can be neglected. Then we can write

$$\frac{\partial\omega}{\partial r}_{r=0} = 0 \quad ; \quad \frac{\partial\omega}{\partial r}_{r=r_0-\varepsilon t} \approx -\frac{\omega_m}{(r_0 - \varepsilon t)} \quad ;$$

$$\frac{\partial^2\omega}{\partial r^2}_{r=0} \approx -\frac{\omega_m}{(r_0 - \varepsilon t)^2}. \tag{25}$$

Substituting formulas (24) and (25) into equation (23), we obtain:

$$\frac{d\omega_m^2}{dt} = \frac{4\varepsilon\omega_m^2}{(r_0 - \varepsilon t)} - \frac{4\nu\omega_m^2}{(r_0 - \varepsilon t)^2}.$$

The solution to this equation satisfying the initial conditions is the function,

$$\omega_m = \omega_0 r_0^2 e^{\frac{2\nu}{\varepsilon r_0}} \frac{e^{-\frac{2\nu}{\varepsilon(r_0-\varepsilon t)}}}{(r_0-\varepsilon t)^2}. \tag{26}$$

This formula (up to coefficients) had already been obtained by the author of this paper, but then the method for obtaining it did not track changes in all three vorticity components simultaneously. The graph of the function $\omega_m(t)$ for different values of the parameter ε is shown in the figure below.

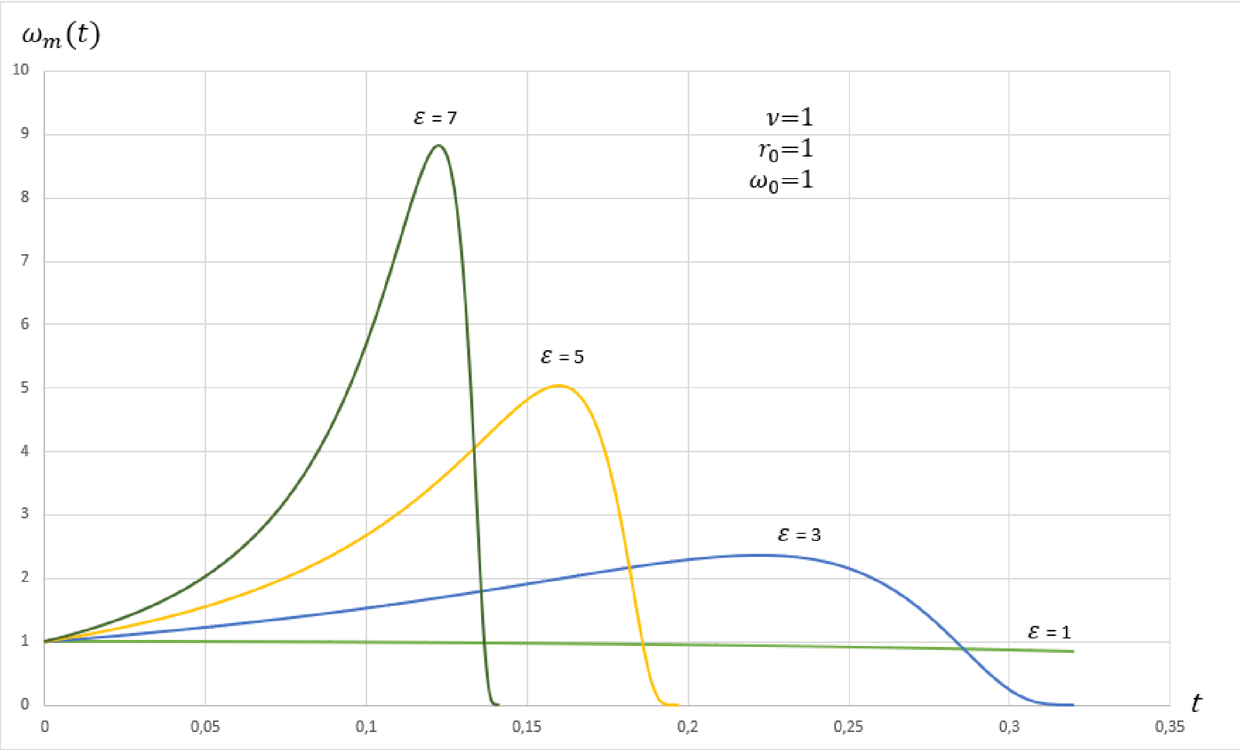

Figure 2. Dependence $\omega_m(t)$.

At

$$t = (r_0/\varepsilon - \nu/\varepsilon^2), \tag{27}$$

the function $\omega_m(t)$ has a maximum

$$\omega_{max} = \omega_0 r_0^2 \frac{e^{((2\nu/\varepsilon r_0)-2)}}{\nu^2} \varepsilon^2. \tag{28}$$

If we introduce into consideration the dimensionless parameter

$$R = \varepsilon r_0/\nu,$$

let's carefully call it an analogue of the Reynolds number, then formula (28) can be written as

$$\omega_{max} = \omega_0 e^{((2/R)-2)} R^2.$$

Hence, it is clear that at $R \leq 1$ the maximum of function (26) disappears. It can be noted here that the solvability of the global problem considered here for small Reynolds numbers is proved. For large numbers $R > 1$, function (26) has a maximum, the value of which increases rapidly with increasing number R and is very sensitive to its changes. But for finite values of the number R (for finite values of ε), the values of the maximum ω_{max} remain finite.

From all that has been said above, the conclusion inevitably follows that the process of vorticity diffusion dominates over the process of its divergence. This conclusion will only be reinforced if we recall the assumption that was made when deriving formula (23). There it was assumed that the axial vorticity gradient is negligible compared to the radial one, i.e. $\partial\omega/\partial\xi \ll \partial\omega/\partial r$. If we assume the opposite, then the intensity of the diffusion process will only increase, i.e. this assumption is conservative. And only in one case the vorticity divergence process can dominate, this is the case of an infinitely large ε. However, this case is not of interest, since in this case an infinitely large energy of the initial state of the fluid is required.

Formula (27) explains another well-known fact: small vortices are more susceptible to dissipation than large ones. This formula determines the time t that passes from the initial state of the vortex, characterized by the radius r_0, until the vortex reaches the vorticity maximum. If we put $t = 0$ in (27) and express r_0, the minimum size of vortices will be obtained,

$$r_{min} = \nu/\varepsilon, \qquad (29)$$

for vortices of this size, further vorticity growth is impossible ($t = 0$!). The evolution of such vortices passes into the stage of their degradation - a smooth decrease in vorticity to the level of the surrounding background. Consequently, in the cascade scheme of vortex development, with the decay of vortices along a chain from large to small, there is a lower limit on the size of vortices. Upon reaching this limiting size, the cascade breaks off, the existence of smaller vortices is impossible. This result is in full agreement with the phenomenological model of turbulence known as the "spectrum law -5/3". In this model, the development of a vortex cascade also ends with the smallest possible vortex scale - the Kolmogorov scale. This scale is characterized by a velocity scale u, a size scale η, and a time scale τ, and

$$u = (\nu\epsilon)^{1/4}; \qquad \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}; \qquad \tau = \left(\frac{\nu}{\epsilon}\right)^{1/2},$$

where ϵ is the rate of energy dissipation (dissipated energy per unit mass and unit time). From the first and second formulas, by eliminating the quantity ϵ, the follows formula,

$$\eta = \nu / u, \tag{30}$$

completely equivalent to formula (29). The only question is how the quantities ε and u in formulas (29) and (30) are related to the quantity u_0, where $u_0 = \sup|\mathbf{u}(x,0))|_{x\in\mathbb{R}^3}$ - the maximum value of the velocity in its initial distribution in the Cauchy problem. However, only the Cauchy problem with an arbitrarily large but finite energy of the initial state E_0 is of interest, which means that the values ε and u are finite, then there is a minimum vortex size r_{min}. In this case, formula (29) fully explains the supercriticality paradox of the Navier-Stokes equations, since it is described in [1], in terms of the energy E. The concentration of energy on scales less than r_{min} (29) becomes impossible since it is impossible to scale solutions to scales less than r_{min}. This means that a blowup scenario of the evolution of these solutions becomes impossible. Moreover, this restriction arises from the Navier-Stokes equations themselves, or rather the Stokes viscous fluid model, which is embedded in their basis.

If formula (29) is applied to an inviscid fluid ($\nu = 0$), then there is no restriction on the minimum size of vortices. In this case, the supercriticality of the Euler equations will manifest itself in full force and blowup scenario of the evolution of an initially smooth solution is inevitable. The same conclusion can be reached from other considerations. In an inviscid fluid, the vorticity diffusion mechanism does not work, but the mechanism of its divergence does. Since these two mechanisms are always oppositely directed, the divergence mechanism will be unbalanced, which will inevitably lead to an unlimited increase in vorticity. It is possible to estimate the time τ over which singularities appear in the initially smooth vorticity field,

$$\tau \sim r_{0m}/u_0,$$

where r_{0m} is the minimum vortex size in the initial velocity distribution.

As a result of the analysis, the following preliminary conclusions can be drawn:

- the vorticity field in the solutions of the 3D Navier-Stokes equations for the Cauchy problem can have extremely large and sharp maxima, but it always remains smooth.

- the derivatives $\partial\omega_i/\partial x_j$ always exist everywhere and are continuous functions of coordinates, therefore the velocity and pressure fields will have the same properties.

REFERENCES:

1. T. Tao. Why Global Regularity for Navier-Stokes is Hard. https://terrytao.wordpress.com/2007/03/18/why-global-regularity-for-navier-stokes-is-hard/
2. A. Bertozzi and A. Majda. Vorticity and Incompressible Flows, Cambridge U. Press, Cambridge, 2002.

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