Malmsten's integral , a short note

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ABSTRACT

We give some remarks on Malmsten's integral.

keywords: Malmsten's integral , number z

I. Introduction

Entry 1. (Malmsten's integral , 1842)

$$\int_0^1 \frac{\ln \ln (1/x)}{1+x^2} dx = \int_1^\infty \frac{\ln \ln x}{1+x^2} dx = \frac{1}{2}\int_0^\infty \frac{\ln x}{\cosh x} dx = \frac{\pi}{2} \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{1}$$

II. Formulas

Entry 2. (Trivial)

$$\int_0^\infty \frac{\ln x}{\cosh x} dx = \pi \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{2}$$

$$\int_0^1 \frac{\ln x}{\cosh x} dx < 0 \quad , \quad \int_1^\infty \frac{\ln x}{\cosh x} dx > 0 \tag{3}$$

Entry 3. $\exists\, z$, $0 < z < 1$, such that

$$\int_0^z \frac{\ln x}{\cosh x} dx = \pi \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) \tag{4}$$

$$\int_z^\infty \frac{\ln x}{\cosh x} dx = 0 \tag{5}$$

Entry 4. We have

$$z = 0.2009932337347154432298730 44 \ldots \tag{6}$$

Entry 5.

$$\frac{\pi \ln z}{2z} - \frac{2\ln z}{z} \tan^{-1}\left(\tanh\left(\frac{z}{2} \right) \right) + \int_1^\infty \frac{\ln x}{\cosh(z\,x)} dx = 0 \tag{7}$$

Entry 6.

$$\pi \ln\left(\frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{2\pi} \right) = \ln z \sum_{n=0}^\infty \frac{(-1)^n E_n z^{2n+1}}{(2n+1)!} - \sum_{n=0}^\infty \frac{(-1)^n E_n z^{2n+1}}{(2n+1)!\,(2n+1)} \tag{8}$$

where

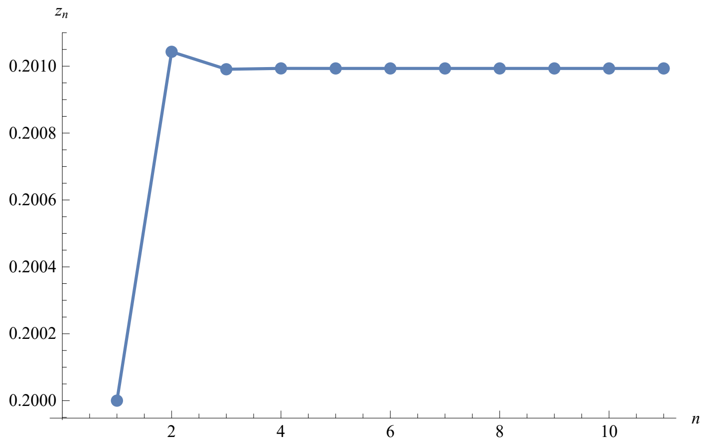
$$n \geq 0 \quad , \quad E_n = \{1,\ 1,\ 5,\ 61,\ 1385,\ 50\,521,\ \ldots\} \tag{9}$$

$$n \geq 0\ ,\ \ E_n = (-1)^n \sum_{k=0}^{2n} 2^{-k} \sum_{m=0}^{k} (-1)^m \binom{k}{m} (1+2m)^{2n} \tag{10}$$

remark: E_n , Euler numbers.

Entry 7.

$$z_1 = 1/5\ ,\ \ z_{n+1} = z_n - \frac{2}{3}\int_{z_n}^{\infty} \frac{\ln x}{\cosh x}\,dx \implies z_n \longrightarrow z \tag{11}$$



References

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