Proving $\zeta(n \geq 2)$ is Irrational Using Decimal Sets

Timothy W. Jones

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Abstract

We prove that partial sums of $\zeta(n) - 1 = z_n$ are not given by any single decimal in a number base given by a denominator of their terms. These sets of single decimals we call decimal sets. This result, applied to all partials, shows that partials are excluded from an ever greater number of rational, possible convergence points, elements of these decimal sets. The limit of the partials is z_n and the limit of the exclusions leaves only irrational numbers. Thus z_n is proven to be irrational.

1 Introduction

Apery's $\zeta(3)$ proof and its simplifications are the only proofs that a specific odd argument for $\zeta(n)$ is irrational [1, 4, 6, 9, 11]. The irrationality of even arguments of zeta are a natural consequence of Euler's formula [2] for $\zeta(2n)$:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1}}{(2n!)} B_{2n} \pi^{2n}. \tag{1}$$

Apery also showed $\zeta(2)$ is irrational, and Beukers, based on the work (tangentially) of Apery, simplified both proofs [3] a lot; see Poorten [12] for the history of Apery's proof and Havil [8] for an approachable introduction to Apery's original proof. Beukers's proofs replace Apery's mysterious recursive relationships with multiple integrals and are easier to understand; see Huylebrouck [9] for an historical context for Beukers's proofs. Papers by Poorten and Beukers are in *Pi: A Source Book* [4] and Eymard and LaFon *The Number* π [6] gives Beukers's proofs and related material. Both the proofs of Apery and Beukers require the prime number theorem and subtle $\epsilon - \delta$ reasoning.

Thus we have the irrationality of all evens immediate from a classic formula and one odd hard to prove; whereas you would think that both evens and odds could be proven in the same way. Attempts to generalize the techniques of the one odd success seem to be hopelessly elusive. It is not for a lack of trying. Apery's and other ideas can be seen in the long and difficult results of Rivoal and Zudilin [13, 16]. Their results, that there are an infinite number of odd n such that $\zeta(n)$ is irrational and at least one of the cases 5,7,9, 11 likewise irrational, seem less than encouraging.

In this paper we explore a different direction. We claim all $\zeta(n \geq 2)$ can be proven to be irrational by using what we call decimal sets and well known and relatively simple properties of decimal bases: [7, Chapter 9]. We still need the lesser cousin of the prime number theorem, Bertrand's postulate, and some new, but straight forward epsilon reasoning.

2 Motivation

We use the following symbols:

$$z_n = \zeta(n) - 1 = \sum_{j=2}^{\infty} \frac{1}{j^n} \text{ and } s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}.$$

Any convergent series can be expressed using any number base. As the upper index of a partial sum for the series grows, decimal digits in base b, say, become fixed. We confine ourselves to single decimals and note all rational numbers a/b in $(0, 1)$ can be given as a single decimal: $.a$ in base b.

Definition 1. *Let $.(x)_y$ denote the single digit x in base y and $.(x)_y^z$ denote a single digit x in base y occurring at partial with upper index z in a series. Let K_b be the least upper index of partials such that the first decimal is fixed. We indicate this with $.(x)_y^{z+}$ where $z = K_y$.*

Example 1. Using a spreadsheet we can form partial sums for

$$z_2 = \frac{\pi^2}{6} - 1 = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots.$$

The first partial sum is .1 base 4 or $.(1)_4^1$.

In Example 1, we used the first term's denominator, 4, as a base. With each new partial we can continue this pattern and develop a system of inequalities. So for upper index 4 we derive inequalities for bases 4 and 9:

$$(.1)_4^3 < (.3)_9^3 < s_3^2 = .(13)_{36}^3 < .(4)_9^3 < .(2)_4^3; \tag{2}$$

Note calculators, like a TI-89, can give values for finite series and then give reduced fractions. These reduced fractions give a decimal digit in a decimal base. Using a calculator $s_4^2 = 1/4 + 1/9 + 1/16 = 61/144 = .(61)_{144}^4$. And we generate another system of inequalities:

$$.(1)_4^4 < .(3)_9^4 < .(6)_{16}^4 < s_4^2 = .(61)_{144} < .(7)_{16}^4 < .(4)_9^4 < .(2)_4^4. \tag{3}$$

The inequalities in (2) and (3) nest. If it were the case that this nesting continued indefinitely, then we could exclude every more rational values as possible convergence points. Consider

$$.(x-1)_4^{k+} < .(x-1)_9^{k+} < .(x-1)_{16}^{k+} < \cdots < .(x)_{16}^{k+} < .(x)_9^{k+} < .(x)_4^{k+},$$

where the left and right decimal digits and k values are indexed with the base; all monotonically increasing; and the ... symbolize a continuation of this pattern. As the bases increase the intervals squeeze to zero and z_2, the limit of partial is not equal to a single digit in any of these bases. As all rational numbers in $(0, 1)$ can be given as a single decimal in these bases (Lemma 1), we can conclude that z_2 is irrational, as $z_2 \in (0, 1)$.

z_2 intervals do not continue to nest. Continuing with just the bases 4, 9, and 16, we observe

$$.(1)_4^5 < .(7)_{16}^5 < .(4)_9^5 < s_5^2 = .(1669)_{3600} < .(8)_{16}^5 = .(2)_4^5 < .(5)_9^5.$$

Base 16 and base 9 have been transposed and, on the right, base 16 and base 4 endpoints collide (i.e. are equal). The next two iterations are

$$.(1)_4^6 < .(7)_{16}^6 < .(4)_9^6 < s_6^2 = .(1769)_{3600} < .(8)_{16}^6 = .(2)_4^6 < .(5)_9^6$$

and

$$.(4)_9^7 < .(8)_{16}^7 = .(2)_4^{7+} < s_7^2 = .(90281)_{176400} < .(5)_9^7 < .(9)_{16}^7 < .(3)_4^{7+}.$$

The right digit for base 4 has migrated to $.2_4$. As $.2_4 < z_2 < .3_4$ these left and right values for base 4 are fixed for $k \geq 7$. That is $K_4 = 7$.

z_2 and generally z_n don't consistently nest, but $e-2$ gives an easy case:

$$e - 2 = \sum_{j=2}^{\infty} \frac{1}{j!} = \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \ldots;$$

and the K_b values simply increment with each added term:

$$.(1)_2^{1+} < .(4)_6^{2+} < .(17)_{24}^{3+} < \cdots < .(18)_{24}^{3+} < .(5)_6^{2+} < (1)_2^{1+}. \tag{4}$$

As the endpoints *cover* possible convergence points,

$$\frac{p}{q} = \frac{p(q-1)!}{q!},$$

we have a proof (assuming the pattern of (4) is correct) that $e-2$, hence e is irrational; see [15] for a complete proof.

Looking at the inequalities for z_2, the bases for partial sums exceed those of the terms used. We will show that s_k^n is not an element of sets of single decimals in the bases of its terms, their denominators (Corollary 1); nota bene general n. We claim that these properties of partials *escaping* terms and terms covering rationals are enough to show the irrationality of all z_n. We use these properties to show partials get arbitrarily close to numbers of ever greater precision, Theorem 2; this implies irrationality.

3 Terms cover rationals

First two definitions.

Definition 2. *Let*

$$d_{j^n} = \{1/j^n, \ldots, (j^n - 1)/j^n\} = \{.1, \ldots, .(j^n - 1)\} \textit{ base } j^n.$$

That is d_{j^n} consists of all single decimals greater than 0 and less than 1 *in base j^n. The decimal set for j^n is*

$$D_{j^n} = d_{j^n} \setminus \bigcup_{k=2}^{j-1} d_{k^n}.$$

The set subtraction removes duplicate values.

Definition 3.

$$\bigcup_{j=2}^{k} D_{j^n} = \Xi_k^n$$

We next show this union of decimal sets give all rational numbers in $(0,1)$.

Lemma 1.

$$\lim_{k\to\infty} \Xi_k^n = \bigcup_{j=2}^{\infty} D_{j^n} = \mathbb{Q}(0,1),$$

where $\mathbb{Q}(0,1)$ *designates all rational numbers in the interval* $(0,1)$*.*

Proof. Every rational $a/b \in (0,1)$ is included in a D_{j^n}. This follows as $ab^{n-1}/b^n = a/b$ and as $a < b$, per $a/b \in (0,1)$, $ab^{n-1} < b^n$ and so $a/b \in D_{b^n}$. □

As $0 < z_n < 1$ for $n \geq 2$, Lemma 1 shows, for large enough k, Ξ_k^n will contain any possible rational convergence point for any given z_n.

4 Partials escape terms

We show partial sums of z_n can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. A reduced fraction can't be expressed as a single digit decimal in a base less than its denominator. We just need to show, then, that the reduced denominator of s_n^k exceeds k^n.

Table 1 suggests we attempt to prove the reduced forms of partial sums of z_n are divisible by powers of 2 and some relatively large prime. As twice something greater than half is bigger than the whole, this is a good starting observation. Apostol's *Introduction to Analytic Number Theory* (Chapter 2, problem 21), solutions in [10], gives the general technique used in this section.

Lemma 2. *If* $s_k^n = r/s$ *with* r/s *a reduced fraction, then* 2^n *divides* s*.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a

k	s_k^2	Prime factorization
3	$.(13)_{36}$	$36 = 2^2 3^2$
4	$.(61)_{144}$	$144 = 2^4 3^2$
5	$.(1669)_{3600}$	$3600 = 2^4 3^2 5^2$
6	$.(1769)_{3600}$	$3600 = 2^4 3^2 5^2$
7	$.(90281)_{176400}$	$176400 = 2^4 3^2 5^2 7^2$

Table 1: The reduced fractions (given as decimals) are divisible by powers of 2 and a prime greater than $k/2$.

powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{5}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (5) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 3. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n}\sum_{j=2}^{k}\frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{6}$$

As $(k,p) = 1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p < k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 4. *For any $k \geq 2$, there exists a prime p such that $k < p < 2k$.*

Proof. This is Bertrand's postulate. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 4, for even k, we are assured that there exists a prime p such that $k > p > k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 4, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Corollary 1.

$$s_k^n \notin \Xi_k^n \text{ or } s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n$$

where $\mathbb{R}(0,1)$ is the set of real numbers in $(0,1)$.

Proof. This is a restatement of Theorem 1. □

5 Towards Greater Precision

Progress has been made. Consider the following heuristic.

Using Lemma 1,

$$\lim_{k\to\infty} \Xi_k^n = \mathbb{Q}(0,1),$$

with Corollary 1

$$\lim_{k\to\infty} \mathbb{R}(0,1) \setminus \Xi_k^n = \mathbb{R}(0,1) \setminus \mathbb{Q}(0,1) = \mathbb{H}(0,1), \tag{7}$$

where $\mathbb{H}(0,1)$ is the set of irrational numbers in $(0,1)$.

We have then

$$\lim_{k\to\infty} s_k^n \in \mathbb{R}(0,1) \setminus \Xi_k^n \implies z_n \in \mathbb{H}(0,1),$$

using $s_k^n \to z_n$, (7) and Corollary 1. That is z_n is irrational.

It seems reasonble that if s_k^n's require and our close to numbers requiring larger bases than those contained in $\{2^n, 3^n, \ldots, k^n\}$ then the numbers close to these partials are not single decimals with these bases, so too for z_n. That is the partials s_k^n and hence z_n are getting arbitrarily close to numbers requiring ever greater bases. We now give a formal proof.

Definition 4. *Let $D_{j^n}^{\epsilon_j}$ be the set of all D_{j^n} decimal sets having an element within ϵ_j of s_j^n.*

Lemma 5. *If for every monotonically decreasing sequence ϵ_j such that*

$$\lim_{j\to\infty} \epsilon_j = 0,$$

we have

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset, \tag{8}$$

then z_n is irrational

Proof. We use proof by contraposition: $p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$. Suppose z_n is rational then $z_n \in D_{j^n}^*$, using Lemma 1. Define

$$\epsilon_j^* = z_n - s_j^n \text{ for } j \geq 2$$

and set

$$\epsilon_j = 2\epsilon_j^*.$$

Then

$$D_{j^n}^* \subset \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j},$$

so the intersection is not empty. □

Example 2. .5, a single decimal, is a limit point of $.4\overline{9}_n$, where the subscript indicates the repetition of 9's. Ordering the convergence point base and partial bases for this example, one has $10^*, 10, 10^2, 10^3, \ldots$, where the superscript asterisk indicates the convergence point base. A repeating decimal, say $.\overline{123}$ base 10 moves towards $.(123)_{999}$ or $.(41)_{333}$: $[(10^3) - 1]^*, 10^3, 10^6, 10^9, \ldots$ – again the based approached is to the left.

Example 3. An irrational, like $\sqrt{2}$ expressed as a series in base 10, has decimal digits that approach rational repeating digits until the non-repeating pattern breaks the pattern. At which time rational numbers of greater precision start to be approached until once again the repeating pattern is broken. This situation can be depicted with terms and their migrating partials always moving to the left: $10, 10^2, 10^*$; $10, 10^2, 10^3, \ldots 10^*$. The movement is migrating and to the left. The partials themselves are never convergence points.

These examples give the ideas for Theorem 2.

Lemma 6. *If* $.(a)_b \in (0, 1)$ *and* $.(a)_b \notin D_{j^n}$ *then*

$$.(a)_b \in (.(x-1), .(x))_{j^n},$$

where $(.(x-1), .(x))_{j^n}$ *is the open set with end points* $.(x-1)_{j^n}$ *and* $.(x)_{j^n}$. *Further for any given* $\epsilon > 0$,

$$|.(a)_b - .(x-1)_{j^n}| < \frac{1}{j^n} < \epsilon, \tag{9}$$

for large enough j.

Proof. D_{j^n} partitions the interval $(0, 1)$ forcing $.(a)_b$ into such an interval. The maximum distance between a point in such an open interval and the endpoints of the interval is $1/j^n$. The right hand inequality in (9) follows from the Archimedean property of the reals [14]. □

Lemma 7. *For* z_n *there exists a sequence* ϵ_j *such that*

$$\bigcap_{j=2}^{\infty} D_j^{\epsilon_j} = \emptyset.$$

Proof. We need to define a sequence ϵ_j. Let

$$\epsilon_j^* = \min\{|x - s_j^n| : x \in \Xi_j^n \}.$$

We know by Corollary 1 that $\epsilon_j^* > 0$. We proceed inductively. For the first iteration, let ϵ_3 be a number such that $\epsilon_3 < \epsilon_3^*$. This excludes the decimal sets of Ξ_3^n at this our first iteration. Assume we can generally do this for the jth iteration. For the $j+1$st iteration, using Lemma 6, there exists a base in Ξ_{j+r}^n, for some r such that $\epsilon_{j+r}^* < \epsilon_j/2$. Set $\epsilon_{j+1} = \epsilon_{j+r}^*$. The procedure gives ϵ values that cumulatively exclude ever more decimal sets from $D_{j^n}^{\epsilon_j}$. Regroup the series. By Lemma 1, the exclusions are exhaustive, so

$$\bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j} = \emptyset,$$

as needed. □

Theorem 2. *z_n is irrational.*

Proof. Let the sequence given in Lemma 7 be given by ϵ_{j_1} and let a general sequence needed for Lemma 5 be given by ϵ_j. Suppose

$$\frac{p}{q} \in \bigcap_{j=2}^{\infty} D_{j^n}^{\epsilon_j}. \tag{10}$$

That is suppose the intersection in (10) is not empty. As $\epsilon_{j_1} \to 0$ and $\epsilon_j \to 0$, for any fixed ϵ_{j_1} that excludes p/q there will be an ϵ_j such that $\epsilon_j < \epsilon_{j_1}$. This implies that p/q will be excluded using ϵ_j, contradicting (10). □

6 Conclusion

How does this proof compare to the work of Beukers? Why do we get a general result here and not with his techniques?

Beukers uses double integrals that evaluate to numbers involving partials for $\zeta(2)$. He uses

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy}\, dxdy = \text{various expressions related to } \zeta(2)$$

and uses this to calculate

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy}\, dxdy,$$

where $P_n(x)$ is the nth derivative of an integral polynomial.

These calculations yield integers A_n and B_n in

$$0 < |A_n + B_n\zeta(2)|d_n^2 < \left\{\frac{\sqrt{5}-1}{2}\right\}^{5n}\zeta(2) < \left\{\frac{5}{6}\right\}^{n}, \tag{11}$$

where d_n designates the least common multiple of the set of integers $\{1,\ldots,n\}$. This last, assuming $\zeta(2)$ is rational, forces an integer between 0 and 1, giving a contradiction. An upper limit for d_n requires the prime number theorem.

These themes repeat for $\zeta(3)$ with the complexity of the expressions at least doubling.

We don't use integrals to generate in effect an interval, a trap, like (11), but the relationships between terms and partials to generate partitions of $(0,1)$ narrowing and leaving only irrational numbers. We use inherent and simple properties z_n's partials and terms, Corollary 1, to avoid intractable complexity.

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Berggren, L., Borwein, J., Borwein, P. (2004). *Pi: A Source Book*, 3rd ed. New York: Springer.

[5] Erdös, P. (1932). Beweiss eines Satzes von Tschebyschef. *Acta Litt. Sci. Reg. Univ. Hungar, Fr.-Jos., Sect. Sci. Math.* 5: 194-198.

[6] Eymard, P., Lafon, J.-P. (2004). *The Number π*. Providence, RI: American Mathematical Society.

[7] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[8] J. Havil (2012). *The Irrationals.* Princeton, NJ: Princeton Univ. Press.

[9] Huylebrouck, D. (2001). Similarities in irrationality proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$, *Amer. Math. Monthly* 108(10): 222–231.

[10] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[11] Nesterenko, Y. V. (1996). A few remarks on $\zeta(3)$, *Math. Zametki* 59(6): 865–880.

[12] van der Poorten, A. (1978/9). A proof that Euler missed... Apery's proof of the irrationality of $\zeta(3)$, an informal report. *Math. Intelligencer* 1(4): 195–203.

[13] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[14] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.

[15] Sondow, J. (2006). A geometric proof that e is irrational and a new measure of its irrationality. *Amer. Math. Monthly* 113(7): 637–641.

[16] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.